INTERMEDIATE MATHEMATICS
(ANALYSIS)

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9676

LONDON
MACMILLAN & CO LTD
NEW YORK · ST MARTIN'S PRESS
1954

MACMILLAN AND COMPANY LIMITED
London Bombay Calcutta Madras Melbourne

THE MACMILLAN COMPANY OF CANADA LIMITED
Toronto

ST MARTIN'S PRESS INC
New York

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First Edition 1925
Reprinted 1928, 1933, 1937, 1942, 1943, 1946, 1948, 1949, 1954

Cat
510
U3I

5/3

PRINTED IN GREAT BRITAIN

PREFACE

THIS book is intended for students who have passed the Matriculation Examination, and for others who are reading mathematics of the Intermediate Standard for a degree in Arts, Science, or Engineering. All the branches of mathematical analysis usually included in an intermediate syllabus are here grouped together in one volume and made, so far as possible, mutually dependent. The order adopted is suitable not only for students who are reading Intermediate Mathematics with the view of proceeding to a pass degree, but also for those who will ultimately take mathematics either as a chief or as a subsidiary subject in a final honours examination. At the same time the index and system of cross-references make it possible to vary the order.

Some elementary work of matriculation standard has been included for revision purposes, while the difficult parts, which most teachers would agree to postpone for a second reading, have been starred. But as the scope of the intermediate syllabus varies with the University, the range of the book is wide, and it should be possible for every student to make a selection suitable for his requirements.

Acknowledgments are due to the Universities of Birmingham, Durham, Leeds, London, Manchester and South Wales, for permission to print questions from papers set at their intermediate examinations, and to the Institutions of Civil and Mechanical Engineers for similar permission. The following abbreviations have been used to indicate the sources of such questions :—

[B.U.] for University of Birmingham,
[L.U.] for University of London,
[Man.] for University of Manchester,
[S.W.] for the University of Wales,
[I.C.E.] for the Institution of Civil Engineers,
and [I.M.E.] for the Institution of Mechanical Engineers.

The Authors wish to express their gratitude to Sir Richard Gregory for his invaluable help in seeing the work through the press; to Mr. S. T. Shovelton, M.A., formerly Fellow of Merton College, Oxford, for his indefatigable assistance and many valuable suggestions; and also to Mr. H. F. Purday, B.Sc., Mr. H. V. Styler, M.A., and Mr. A. J. V. Gale, M.A., for their great care in the thankless task of reading proofs.

T. S. USHERWOOD.
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CHRIST'S HOSPITAL,
March, 1925.

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CHAPTER I.

EQUATIONS.

Simultaneous Equations.

Ex. 1. *The sum of two numbers is 16. Twice the greater exceeds the smaller by 15. What are the numbers?*

Let x, y be the larger and smaller numbers respectively.

Since their sum is 16, $x + y = 16.$ ....(1)

Since twice the greater exceeds the smaller by 15,

$$2x - y = 15. \quad \text{..............................(2)}$$

Equations (1) and (2) each involve *two unknown quantities*. They must be used together if x and y are both to be found, and consequently are called **Simultaneous Equations.**

From (1), $x = 16 - y.$

Substituting in (2) for x, we have

$$2(16 - y) - y = 15;$$

$$\therefore\ 32 - 2y - y = 15.$$

Adding $3y$ to and subtracting 15 from each side, we have

$$17 = 3y;$$

$$\therefore\ y = \frac{17}{3}.$$

Using (1), $x + \dfrac{17}{3} = 16;$

$$\therefore\ x = 10\tfrac{1}{3}.$$

The numbers are therefore $10\frac{1}{3}$ and $5\frac{2}{3}$.

Verification. $10\frac{1}{3} + 5\frac{2}{3} = 16$, and $2(10\frac{1}{3}) - 5\frac{2}{3}$ is 15.

When verifying, it is important to substitute in the *original* equations.

Alternative Solution. Adding equations (1) and (2), we have

$$3x=31\;;$$
$$\therefore\; x=10\tfrac{1}{3}.$$

Using (1), we have $10\frac{1}{3}+y=16$;

$$\therefore\; y=16-10\tfrac{1}{3}$$
$$=5\tfrac{2}{3} \text{ as before.}$$

Ex. 2. *Solve the equations*

$$3x-16y=3, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$2x+5y=49. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1), $3x=3+16y$;

$$\therefore\; x=1+5\tfrac{1}{3}y.$$

Using (2), $2(1+5\frac{1}{3}y)+5y=49$;

$$\therefore\; 2+10\tfrac{2}{3}y+5y=49\;;$$
$$\therefore\; 15\tfrac{2}{3}y=47\;;$$
$$\therefore\; y=3 \text{ (since } \tfrac{47}{3}=15\tfrac{2}{3}).$$

Using (1), $3x-48=3$;

$$\therefore\; 3x=51\;;$$
$$\therefore\; x=17.$$

The required solution is $\left.\begin{aligned}x&=17,\\ y&=3.\end{aligned}\right\}$

Alternative Solution. Note that if twice the first equation be subtracted from three times the second, x disappears.

Multiply (1) by 2, and we have $6x-32y=6.$ ....(3)

Multiply (2) by 3, and we have $6x+15y=147$ ....(4)

Subtract (3) from (4). Then $15y+32y=147-6,$

$$i.e.\; 47y=141\;;$$
$$\therefore\; y=3, \text{ as before.}$$

Using (1) as before, we find $x=17$.

Ex. 3. *If apples cost 7d. per dozen and pears 9d., a man would pay 142 pence for the fruit he intends to buy; whereas if apples cost 8d. per dozen and pears 10d., he would pay 160 pence. How many apples and how many pears does he intend to buy?* (B.E.)

Let him intend to buy x doz. apples and y doz. pears. Under the first conditions, he spends $(7x+9y)$ pence ;

$$\therefore\; 7x+9y=142. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Under the second conditions, he spends $(8x+10y)$ pence ;

$$\therefore\ 8x+10y=160. \qquad \dots\dots\dots\dots(2)$$

Dividing (2) throughout by 2, we have

$$4x+5y=80. \qquad \dots\dots\dots\dots(3)$$

Multiplying (1) by 4, we have

$$28x+36y=568. \qquad \dots\dots\dots\dots(4)$$

Multiplying (3) by 7, we have

$$28x+35y=560. \qquad \dots\dots\dots\dots(5)$$

Subtracting (5) from (4), we have

$$y=8.$$

Using (3), $\quad 4x+40=80$;

$$\therefore\ x=10.$$

He therefore intended to buy 10 doz. apples and 8 doz. pears

Ex. 4. *It is found that the equation*

$$C=a+bW$$

connects the cost C of an article in pence with the weight W of the article in ounces. C is 6 when W is 4, and C is 9 when W is 8. Find the values of a and b and the cost of an article weighing 12 ounces.

Using the data, we have

$$6=a+b\,.\,4, \qquad \dots\dots\dots\dots(1)$$

$$9=a+b\,.\,8. \qquad \dots\dots\dots\dots(2)$$

Subtracting (1) from (2), $\quad 3=4b$;

$$\therefore\ b=0{\cdot}75.$$

Using this value of b in (1), we have

$$6=a+3\ ;$$

$$\therefore\ a=3.$$

The formula connecting C and W is consequently

$$C=3+0{\cdot}75W.$$

Hence, if $W=12$, $\quad C=3+0{\cdot}75\times 12$

$$=3+9.$$

Therefore the cost of an article weighing 12 ounces is 1*s*.

Simultaneous Equations with three Unknowns.

Ex. 1. *Solve the equations*

$$x+4y+4z=7, \quad \text{.........................(1)}$$
$$3x+2y+2z=6, \quad \text{.........................(2)}$$
$$9x+6y+2z=14. \quad \text{.........................(3)}$$

[L.U.]

Multiplying (2) by 2, $6x+4y+4z=12$.........................(4)

Subtract (1) from (4), and we have

$$5x=5;$$
$$\therefore\ x=1.$$

Substituting $x=1$ in (2), $2y+2z=3$. ....(5)

Substituting $x=1$ in (3), $6y+2z=5$. ....(6)

Subtract (5) from (6), and we have $4y=2$;

$$\therefore\ y=0{\cdot}5.$$

Substituting $y=0{\cdot}5$ in (5), $2z=2$;

$$\therefore\ z=1.$$

Thus $x=1$, $y=0{\cdot}5$ and $z=1$ satisfy equations (1), (2) and (3).

Verification. Substituting $x=1$, $y=0{\cdot}5$ and $z=1$ in (1), (2), and (3), we have

$$1+4(0{\cdot}5)+4=7,$$
$$3+2(0{\cdot}5)+2=6,$$
$$9+6(0{\cdot}5)+2=14.$$

Ex. 2. *Solve the equations*

$$2x+y+z=4, \quad \text{.........................(1)}$$
$$3x-2y+2z=1, \quad \text{.........................(2)}$$
$$x+3y-3z=-7. \quad \text{.........................(3)}$$

Eliminate y between (1) and (2) as follows:

Multiplying (1) by 2, $4x+2y+2z=8$. ....(4)

Add (2), and we have $7x+4z=9$. ....(5)

Eliminate y between (1) and (3) as follows:

Multiplying (1) by 3, $6x+3y+3z=12$. ....(6)

Subtract (3), and we have $5x+6z=19$. ....(7)

Notice that (5) and (7) are two simultaneous equations in two unknowns.

We find x by eliminating z.

Multiplying (5) by 6,

$$42x+24z=54. \qquad \text{(8)}$$

Multiplying (7) by 4,

$$20x+24z=76. \qquad \text{(9)}$$

Subtract (9) from (8), and we have

$$22x=-22;$$
$$\therefore\ x=-1.$$

Substituting $x=-1$ in (5),

$$-7+4z=9;$$
$$\therefore\ z=4.$$

Substituting $x=-1$ and $z=4$ in (1),

$$-2+y+4=4;$$
$$\therefore\ y=2.$$

Thus $x=-1$, $y=2$ and $z=4$ satisfy the given equations.

Verification. Substituting for x, y and z in (2) and (3), we have

$$-3-2(2)+2(4)=1,$$
$$-1+3(2)-3(4)=-7.$$

Ex. 3. *The distance, x yards, traversed in t seconds by a rifle bullet is given by the series of observations:*

x	100	200	300	400	500	600
t	0·128	0·268	0·423	0·596	0·790	1·008

Shew that they may be connected by the relation $t=ax+bx^2+cx^3$, where a, b, c are constants; and that, with this law, the time of traversing 1000 *yards is* 2·18 *seconds.* [L.U.]

Taking the first three values of x and t, we have

$$0{\cdot}128=100a+100^2b+100^3c, \qquad \text{(1)}$$
$$0{\cdot}268=2\,.\,100a+4\,.\,100^2b+8\,.\,100^3c, \qquad \text{(2)}$$
$$0{\cdot}423=3\,.\,100a+9\,.\,100^2b+27\,.\,100^3c. \qquad \text{(3)}$$

Multiplying (1) by 2 and 3 in succession,

$$0{\cdot}256=2\,.\,100a+2\,.\,100^2b+2\,.\,100^3c, \qquad \text{(4)}$$

and

$$0{\cdot}384=3\,.\,100a+3\,.\,100^2b+3\,.\,100^3c, \qquad \text{(5)}$$

Subtracting (4) from (2),

$$0{\cdot}012 = 2 \,.\, 100^2 b + 6 \,.\, 100^3 c. \quad \text{.................(6)}$$

Subtracting (5) from (3),

$$0{\cdot}039 = 6 \,.\, 100^2 b + 24 \,.\, 100^3 c \quad \text{.................(7)}$$

Multiplying (6) by 3 and subtracting from (7),

$$0{\cdot}003 = 6 \,.\, 100^3 c\,;$$

$$\therefore\ 100^3 c = 0{\cdot}0005. \quad \text{.........................(8)}$$

And, using (8) in (6), $0{\cdot}009 = 2 \,.\, 100^2 b$;

$$\therefore\ 100^2 b = 0{\cdot}0045. \quad \text{.......................(9)}$$

Finally, using (8) and (9) in (1),

$$0{\cdot}128 = 100a + 0{\cdot}0045 + 0{\cdot}0005\,;$$

$$\therefore\ 100a = 0{\cdot}123. \quad \text{.......................(10)}$$

Equations (8), (9) and (10) shew that x and t are connected by the relation

$$t = 0{\cdot}123\,\frac{x}{100} + 0{\cdot}0045\left(\frac{x}{100}\right)^2 + 0{\cdot}0005\left(\frac{x}{100}\right)^3.$$

If $x = 400$, we find

$$\begin{aligned} t &= 0{\cdot}123\,(4) + 0{\cdot}0045\,(16) + 0{\cdot}0005\,(64) \\ &= 0{\cdot}492 + 0{\cdot}072 + 0{\cdot}032 \\ &= 0{\cdot}596, \text{ as given in the table.} \end{aligned}$$

In the same way it will be found that if $x = 500$, $t = 0{\cdot}790$, and if $x = 600$, $t = 1{\cdot}008$.

If $x = 1000$, $t = 0{\cdot}123(10) + 0{\cdot}0045\,(10)^2 + 0{\cdot}0005\,(10)^3$;

$$\therefore\ t = 2{\cdot}18.$$

N.B.—Any *three* corresponding values of x and t may be selected to give three equations, and from these the same values of a, b, c will be found.

From the preceding examples, it will be noticed that two equations are required to find two unknowns and three equations to find three unknowns. In general, if there are n unknowns, n equations are required for their determination.

A valuable method of solving simultaneous equations in which the coefficients are literal involves the use of **Determinants**, a brief explanation of which will be found in Chapter VIII. and in the authors' *Practical Mathematics for Technical Students*, Part II., Chap. III. Determinants are of great value in Higher Mathematics.

EXERCISES I.

Solve the equations

1. $\left.\begin{aligned} 3x + y &= 9, \\ 4x - y &= 5. \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x + y &= 13, \\ 3x - 2y &= 7. \end{aligned}\right\}$

3. $\left.\begin{aligned} 3x - 5y &= 6, \\ 4x + 3y &= 37. \end{aligned}\right\}$

4. $\left.\begin{aligned} \frac{x}{3} + \frac{y}{2} &= 4, \\ \frac{2x}{3} - \frac{y}{4} &= 3. \end{aligned}\right\}$

5. $2x - 15y = 3x - 24y = 1.$

6. $\left.\begin{aligned} \frac{x+2}{3} + 8y &= 17, \\ \frac{3y+2}{4} &= 3 - x. \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{11x - 6y}{x - 1} &= 3, \\ \frac{8y - 5x}{x + 1} &= 5. \end{aligned}\right\}$ [I.C.E.]

Solve the following problems :

8. Find two numbers such that their sum is 19 and their difference 3.

9. The sum of two numbers is 45·7 and their difference is 14·5. What are the numbers ?

10. Two angles of a triangle are equal and the remaining angle is 100° more than their sum. What are the angles ?

11. In 4 hours *A* walks 7 miles more than *B* walks in 3 hours ; in 3 hours *A* walks twice as far as *B* does in 2 hours. What is the rate of walking of each ?

12. A person buys 10 lb. of each of two kinds of tea and finds that he pays 5*s*. more for one kind than the other. The total cost was £2. 5*s*. 0*d*. What was the price per lb. of each kind of tea ?

13. Find two numbers such that if 4 times the first be added to 2·5 times the second, the sum is 17·3 ; and if 3 times the second is subtracted from twice the first, the difference is 1·2. [B.E.]

14. The ages of a man and his wife added together amount to 72·36 years. Fifteen years ago the man's age was 2·3 times that of his wife. What are their ages now ? [B.E.]

15. If C in. is the diameter of the bolt circle in a flange coupling and d in. is the diameter of the bolts, it is found that

$$C = xd + y, \quad C = 10 \text{ when } d = 3 \quad \text{and} \quad C = 6 \text{ when } d = 1{\cdot}5.$$

Find the values of x and y and the diameter of the bolt circle for 2 in. bolts.

16. The inclusive cost £C of compound tandem steam engines is connected with the horse-power H of the engine by the formula

$C=a+bH$. For an engine of 200 horse-power the cost is £495; for an engine of 1000 horse-power the cost is £3190. Find the values of a and b and the approximate horse-power of the engine which can be bought for £2640.

17. When an effort E lb. is applied to a certain machine it is found that resistance R lb. can be overcome, and that E and R are connected by the formula $E=a+bR$. An effort of 3·5 lb. overcomes a resistance of 5 lb., while an effort of 5·3 lb. overcomes a resistance of 8 lb. Find a and b and the effort required to overcome a resistance of 10 lb.

Solve the equations

18. $$\left.\begin{aligned} x+2y+3z&=5,\\ 3x+y+2z&=6,\\ 2x+3y+z&=1. \end{aligned}\right\}$$

19. $$\left.\begin{aligned} 2x+y-z-1&=0,\\ x+2y+2z+3&=0,\\ 3x+2y-3z-5&=0. \end{aligned}\right\}$$

20. $$\left.\begin{aligned} 2x+y+z&=5,\\ 3x-4y-2z&=1,\\ x+3y+4z&=9. \end{aligned}\right\}$$

21. $$\left.\begin{aligned} 5x-2y-3z&=1,\\ 3x+3y+5z&=8,\\ -4x+y+6z&=7. \end{aligned}\right\}$$

22. $$\left.\begin{aligned} 3x-4y+7z&=-6,\\ x+6y-z&=12,\\ 5x+y+3z&=16. \end{aligned}\right\}$$

23. $$\left.\begin{aligned} x+2y+3z&=5,\\ 3x+y+2z&=6,\\ 2x+3y+z&=1. \end{aligned}\right\}$$

24. $$\left.\begin{aligned} 0{\cdot}3x+0{\cdot}2y+0{\cdot}5z&=3{\cdot}2,\\ 0{\cdot}2x+0{\cdot}3y+0{\cdot}4z&=2{\cdot}9,\\ 0{\cdot}4x+0{\cdot}3y+0{\cdot}5z&=3{\cdot}7. \end{aligned}\right\}$$

25. $$\left.\begin{aligned} x+4y+4z&=7,\\ 3x+2y+2z&=6,\\ 9x+6y+2z&=14. \end{aligned}\right\}$$ [L.U.]

26. Eliminate x and y from the equations

$$\begin{aligned} a_1x+b_1y&=c_1,\\ a_2x+b_2y&=c_2,\\ a_3x+b_3y&=c_3. \end{aligned}$$

27. Shew that the following equations are not independent:

$$x+y-z=0, \quad x-y+z=2, \quad y+z-x=4, \quad x+y+z=6.$$

28. In dynamo design the following equations occur:

$$W=FV+HV+ZV^2, \qquad W'=FV'+HV'+ZV'^2.$$

Find Z in terms of W, W', V and V'.

*29. The following values of x and y are connected by a relation of the form $y=a+bx+cx^2$:

x	1·5	2·3	4·6
y	2·313	2·193	3·630

Find a, b and c.

***30.** If θ and ρ are connected by a formula of the form

$$\rho = 1{\cdot}0362(1 + a\theta + b\theta^2),$$

find the values of a and b, given

(i)

θ	15·65	20·11
ρ	1·0333	1·0327

(ii)

θ	32·76	63·23
ρ	1·0205	1·0152

***31.** The speed of the rim of a centrifugal pump or fan being V feet per second, the head H feet produced when Q cubic feet of fluid per second are passing seems from theory to be such that

$$\frac{H}{V^2} = a + b\frac{Q}{V} - c\frac{Q^2}{V^2}.$$

The following observations were made on a Rateau fan :

Q	V	H
1243	96·5	206·5
825	112	371
46	96	144

Find the values of a, b and c. [B.E.]

Quadratic Equations. It is assumed that the student is already familiar with the method of solving a quadratic equation with numerical coefficients.

Any equation of the form

$$ax^2 + bx + c = 0, \quad \dots\dots\dots\dots\dots(1)$$

in which a, b and c are known numbers and the highest power of the unknown number x is the square, is called a **quadratic equation.**

The roots of equation (1) are found as follows :

Divide throughout by a, obtaining

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0;$$

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}. \quad \dots\dots\dots\dots\dots(2)$$

Adding $\frac{b^2}{4a^2}$ to both sides of (2),

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2}. \quad \text{......................(3)}$$

Now $p^2+2pq+q^2=(p+q)^2$; $\therefore$ (3) can be written as

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}. \quad \text{......................(4)}$$

Taking the square root of both sides of (4),

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

(read 'plus or minus* the square root of b^2-4ac divided by $2a$');

$$\therefore\ x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

The required roots are therefore

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad\text{and}\quad \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

It is useful to be able to write down the roots of any quadratic equation, and the formulae for the two roots of the general quadratic should be remembered.

Ex. *The diagonal of a rectangle is 1·7 in. long, and one side is 0·4 in. longer than the other. Find the lengths of the sides to the nearest hundredth of an inch.*

Let x in. be the length of the shorter side; the length of the other is therefore $x+0{\cdot}4$ in., and the length of the diagonal must therefore be $\sqrt{x^2+(x+0{\cdot}4)^2}$ in., using Pythagoras' theorem.

Hence $$\sqrt{x^2+(x+0{\cdot}4)^2}=1{\cdot}7\,;$$

$$\therefore\ x^2+(x+0{\cdot}4)^2=(1{\cdot}7)^2,$$

i.e. $$x^2+x^2+0{\cdot}8x+0{\cdot}16=2{\cdot}89$$

or $$2x^2+0{\cdot}8x-2{\cdot}73=0. \quad \text{......................(1)}$$

---

* The reason for the double sign ± will be appreciated if it is remembered that both $(+z)^2$ and $(-z)^2$ are equal to z^2.

The roots of (1) are $\dfrac{-0{\cdot}8 \pm \sqrt{(0{\cdot}8)^2 - 4 \times 2 \times (-2{\cdot}73)}}{2 \times 2}$,

i.e. $\dfrac{-0{\cdot}8 \pm \sqrt{22{\cdot}48}}{4}$.

$$\begin{array}{r|r} & 4\ 741 \\ \hline & 22\ 4800 \\ 87 & 6\ 48 \\ 944 & 3900 \\ 948 & 124 \end{array}$$

$\sqrt{22{\cdot}48}$ lies between 4 and 5.

$\sqrt{22{\cdot}48} = 4{\cdot}741$ approximately ;

$\therefore$ the roots of (1) are

$$\frac{-0{\cdot}8 + 4{\cdot}741}{4} \quad \text{and} \quad \frac{-0{\cdot}8 - 4{\cdot}741}{4},$$

i.e. $0{\cdot}985$ and $-1{\cdot}385$.

$\therefore$ $x = 0{\cdot}99$ to two significant figures, the negative value $-1{\cdot}385$ being inadmissible, as x represents a side of a rectangle.

Hence the sides are 0·99 in. and 1·39 in. long to the nearest hundredth of an inch.

This problem is also suitable for a graphical solution. (See p. 64.)

Sum and Product of the Roots of a Quadratic. Let α, β be the two roots of the equation

$$ax^2 + bx + c = 0, \quad \text{.........................(1)}$$

$$\text{so that } \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{.......................(2)}$$

$$\text{and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad \text{......................(3)}$$

Adding (2) and (3), $\alpha + \beta = -\dfrac{b}{a}$.

Multiplying (2) and (3),

$$\alpha\beta = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2};$$

$$\therefore \alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

That is, the sum of the roots equals $-\left(\dfrac{\text{the coefficient of } x}{\text{the coefficient of } x^2}\right)$

and the product of the roots equals $\left(\dfrac{\text{the constant term}}{\text{the coefficient of } x^2}\right)$.

The following method of obtaining these results is valuable, as it can be applied to any equation of higher degree. (See Ex. 3, p. 14.)

The equation whose roots are α and β must be

$$(x-\alpha)(x-\beta)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

since the left-hand side vanishes either when $x=\alpha$ or when $x=\beta$.

Now (4) is $$x^2-x(\alpha+\beta)+\alpha\beta=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and (1) may be written $$x^2+\frac{b}{a}x+\frac{c}{a}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

And (5) and (6) are different forms of exactly the same equation.

So, comparing, we see that $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

The first formula is a useful check on the accuracy of numerical solutions and should always be used.

Character of the Roots of a Quadratic. Examination of (2) and (3) above shews that:

1. If $b^2>4ac$, the roots are **real**; in particular, if b^2-4ac is a perfect square, the roots are **rational.**
2. If $b^2=4ac$, the roots are **equal.**
3. If $b^2<4ac$, the roots are **unreal** or **imaginary.**

The definition of a rational number will be found on page 83.

Imaginary numbers are numbers like $\sqrt{-16}$, $\sqrt{-x}$ (where x is positive).

Ex. 1. *If α, β are the roots of the equation $3x^2-2x-1=0$, find the equation whose roots are $2\alpha+\beta$ and $\alpha+2\beta$.*

We have $$\alpha+\beta=\tfrac{2}{3} \quad\text{and}\quad \alpha\beta=-\tfrac{1}{3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let $x^2+px+q=0$ be the equation whose roots are

$$2\alpha+\beta \quad\text{and}\quad \alpha+2\beta.$$

Then $$3\alpha+3\beta=\text{the sum of the roots}=-p,$$

and $$2\alpha^2+5\alpha\beta+2\beta^2=\text{the product of the roots}=q.$$

From (1), $$3\alpha+3\beta=2\,;$$

$$\therefore\ p=-2.$$

Again, $$2\alpha^2+5\alpha\beta+2\beta^2=2(\alpha+\beta)^2+\alpha\beta$$

$$=2(\tfrac{2}{3})^2-\tfrac{1}{3},\ \text{using (1)},$$

$$=\tfrac{5}{9}\,;$$

$$\therefore\ q=\tfrac{5}{9},$$

and the required equation is

$$x^2-2x+\tfrac{5}{9}=0,$$

i.e. $$9x^2-18x+5=0.$$

Ex. 2. *If* α, β *are the roots of the equation* $ax^2+bx+c=0$, *find the equation whose roots are* $\dfrac{\alpha}{\beta}$ *and* $\dfrac{\beta}{\alpha}$.

We have $$\alpha+\beta=-\frac{b}{a}\quad\text{and}\quad\alpha\beta=\frac{c}{a}.\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let $x^2+px+q=0$ be the equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

Then $$-p=\text{the sum of the roots}$$

$$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$$

$$=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{b^2/a^2-2c/a}{c/a},\ \text{using (1)},$$

$$=\frac{b^2-2ac}{ac}.$$

And $$q=\text{the product of the roots}$$

$$=\frac{\alpha\beta}{\beta\alpha}$$

$$=1.$$

The required equation is therefore

$$x^2 - \frac{b^2-2ac}{ac}\,.\,x + 1 = 0,$$

$$\text{i.e.} \qquad acx^2 - (b^2-2ac)x + ac = 0.$$

Ex. 3. *If α, β, γ, are the roots of the equation*

$$x^3 + px^2 + qx + r = 0,$$

shew that $\alpha+\beta+\gamma = -p$, $\alpha\beta+\beta\gamma+\gamma\alpha = q$, and $\alpha\beta\gamma = -r$.

The equation whose roots are α, β, γ must be

$$(x-\alpha)(x-\beta)(x-\gamma) = 0,$$

$$\text{i.e.} \qquad x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma = 0.$$

Comparing this with the given equation, we have the result stated.

Simultaneous Quadratics. If two simultaneous equations are given, one of which is of the first degree and the other of the second, they can always be solved as follows:

Ex. 1. *Solve the equations*

$$2x - 3y = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x^2 - 2xy - 3y^2 = -3. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1), $$x = \frac{1+3y}{2}.$$

Substituting in (2),

$$\left(\frac{1+3y}{2}\right)^2 - 2y\left(\frac{1+3y}{2}\right) - 3y^2 = -3;$$

$$\therefore \frac{1+6y+9y^2}{4} - (y+3y^2) - 3y^2 + 3 = 0;$$

$$\therefore 1 + 6y + 9y^2 - 4y - 12y^2 - 12y^2 + 12 = 0;$$

$$\therefore -15y^2 + 2y + 13 = 0;$$

$$\text{whence}^* \qquad (y-1)(15y+13) = 0;$$

$$\therefore y = 1 \quad \text{or} \quad -\tfrac{13}{15}.$$

---

* This method of solving by factors should be used whenever possible instead of using the formula.

And, using (1): if $y=1$, $x=2$;

if $y=-\frac{13}{15}$, $2x=1-\frac{13}{5}$;

$\therefore\ x=-\frac{4}{5}$;

$$\therefore \text{ the roots are } \begin{Bmatrix} x=2 \\ y=1 \end{Bmatrix}, \quad \begin{Bmatrix} x=-\frac{4}{5} \\ y=-\frac{13}{15} \end{Bmatrix}.$$

Verification. Since the values of x were determined from (1), we only verify that the roots satisfy (2).

Taking the first solution, the left-hand side of (2) becomes

$$2^2-2\,.\,2\,.\,1-3\,.\,1^2,$$

which is -3.

Taking the second solution, the left-hand side of (2)

$$=\left(-\frac{4}{5}\right)^2-2\left(-\frac{4}{5}\right)\left(-\frac{13}{15}\right)-3\left(-\frac{13}{15}\right)^2$$

$$=\frac{16}{25}-\frac{8\,.\,13}{5\,.\,15}-\frac{3\,.\,13^2}{15\,.\,15}$$

$$=\frac{16}{25}-\frac{13(24+39)}{15\,.\,15}$$

$$=\frac{16}{25}-\frac{13\,.\,7}{5\,.\,5}$$

$$=\frac{16-91}{25}$$

$$=-3.$$

When both equations are of the second degree, there is no general method of solution. Some typical solutions are appended.

Ex. 2. *Solve the equations*

$$3y^2-5x^2=7, \quad \ldots\ldots\ldots\ldots(1)$$

$$3xy-4x^2=2. \quad \ldots\ldots\ldots\ldots(2)$$

Eliminate the constants on the right of the equations by multiplying (1) by 2 and (2) by 7 and subtracting.

Thus $2(3y^2-5x^2)-7(3xy-4x^2)=0$;

$\therefore\ 6y^2-21xy+18x^2=0$;

$\therefore\ 2y^2-7xy+6x^2=0$;

$\therefore\ (2y-3x)(y-2x)=0.$ ....(3)

Now equation (3) has been obtained from (1) and (2), and the solutions of (1) and (2) are the solutions of (3) and *either* of the first two equations.

From (3), either

$$2y-3x=0$$

or

$$y-2x=0.$$

The required roots are therefore the solutions of

$$\text{I.}\quad \left.\begin{aligned}2y-3x&=0,\\3xy-4x^2&=2,\end{aligned}\right\}\quad \text{and}\quad \text{II.}\quad \left.\begin{aligned}y-2x&=0,\\3xy-4x^2&=2.\end{aligned}\right\}$$

Using the method of Ex. 1., equations I. give

$$3x\left(\frac{3x}{2}\right)-4x^2=2;$$

$$\therefore\ x^2=4;$$

$$\left.\begin{aligned}\therefore\quad x&=\pm 2\\ \text{and}\quad y&=\pm 3.\end{aligned}\right\}$$

Equations II. give

$$3x(2x)-4x^2=2;$$

$$\therefore\ x^2=1;$$

$$\left.\begin{aligned}\therefore\quad x&=\pm 1\\ \text{and}\quad y&=\pm 2.\end{aligned}\right\}$$

The complete solution of (1) and (2) is therefore

x	2	−2	1	−1
y	3	−3	2	−2

These should be verified by substitution in (1) and (2).

Ex. 3. *Solve the equations*

$$x^2-y^2=3x-2y \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x^2-y^2=2x+y. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

[L.U.]

Subtracting (2) from (1), $x-3y=0.$ ....(3)

Equations (1) and (2) are therefore equivalent to equation (3) and *one* of the original equations, *e.g.* (1).

Using (3) and substituting for x in equation (1),

$$9y^2 - y^2 = 9y - 2y\ ;$$

$$\therefore\ 8y^2 - 7y = 0\ ;$$

$$\therefore\ y(8y - 7) = 0\ ;$$

whence $y = 0$ or $y = \frac{7}{8}$,

and, using (3), $x = 0$ or $x = \frac{21}{8}$.

These values should be verified.

Ex. 4. *Solve the equations*

$$x^2 - xy + x = 1, \quad \text{.........................(1)}$$

$$xy - y^2 + y = 1. \quad \text{.........................(2)}$$

Subtract (2) from (1), and we have

$$x^2 - 2xy + y^2 + x - y = 0\ ;$$

$$\therefore\ (x - y)(x - y + 1) = 0\ ;$$

$$\therefore\ x - y = 0$$

or $x - y + 1 = 0$.

As in Ex. 2, we have

$$\text{I.}\quad \left.\begin{aligned} x - y &= 0, \\ xy - y^2 + y &= 1, \end{aligned}\right\} \quad \text{and} \quad \text{II.}\quad \left.\begin{aligned} x - y + 1 &= 0, \\ xy - y^2 + y &= 1\cdot \end{aligned}\right\}$$

Using the method of Ex. 1, equations I. give

$$y^2 - y^2 + y = 1\ ;$$

$$\therefore\ y = 1$$

and $x = 1$.

Equations II. give $y(y - 1) - y^2 + y = 1\ ;$

$$\therefore\ y^2 - y - y^2 + y = 1.$$

But the left-hand side of this equation is identically 0, so that equations II. give no solution.

The importance of verification by substituting in the *original* equations cannot be emphasised too strongly; for the method of solution sometimes gives rise to answers which do *not* satisfy the original equations. The explanation of such extraneous solutions can only be given after carefully analysing each step in the work.

EXERCISES II.

1. Solve the equations (i) $\frac{x+3}{x-3}-\frac{x-5}{x+5}=\frac{16}{x}$;

(ii) $5x^2+6x=27$. [I.C.E.]

2. Solve the equation $\frac{3x+2}{1-x}=\frac{1+x}{x}$,

giving the roots to two decimals. [I.M.E.]

3. Solve the equations

$$\text{(i)}\quad \frac{3}{x-3}-\frac{4}{x-4}+\frac{5}{x-1}=0\,;$$

$$\text{(ii)}\quad x^2+xy=4,$$
$$y^2+xy=5.$$

[I.C.E.]

4. Find, in simplest form, the relation between a and b for which the values of x given by the equations

$$ax+3a+4x+13=0,$$
$$bx+3x+4b+13=0$$

will be equal.

If this relation is satisfied, and if also $5a=2b$, determine the numerical values of a, b, x. [There are two sets of solutions, both of which should be given.] [I.M.E.]

5. If $ax^2+bx+c=0$, test the following:

Relation between coefficients.	Nature of the roots.
a, b, c have same sign	Both negative
a differs in sign from c	One negative, one positive
a, c differ in sign from b	Both positive

Examples 6–17 *are taken from papers set at the Inter B.Sc.* [*L.U*].

6. Find the condition that the equation $x^2+px+q=0$ may have (i) real roots, (ii) equal roots.

Find the equation whose roots are respectively equal to the sum and the difference, taken positively, of the roots of the equation $x^2-2ax+a^2-b^2=0$.

7. Find the condition that the roots of the equation

$$ax^2+bx+c=0$$

may be (i) real, (ii) rational.

Find the equation whose roots are respectively equal to the sum and the product of the roots of the equation $17x^2-3x+14=0$.

8. Express the sum and the difference of the roots of the equation $ax^2+bx+c=0$ in terms of the coefficients.

Form the quadratic equation whose roots are the squares of those of the given equation; and verify that the expression for the difference of its roots is the product of the two expressions obtained above.

9. Find the sum and product of the roots of the equation $ax^2+bx+c=0$ rationally in terms of a, b and c.

If $a(p+q)^2+2bpq+c=0$ and $a(p+r)^2+2bpr+c=0$, express $q+r$ and qr in terms of p, a, b and c.

10. Solve the equations $\left.\begin{aligned}3x^2+4xy-y^2&=1,\\ 2x+3y&=1.\end{aligned}\right\}$

11. Solve the equations (i) $y(2x-3)=9-2x$, $2x-3y=3$;

(ii) $x^2-xy+y^2=3$, $x^2+3xy=7$.

12. Solve the equations (i) $\dfrac{1}{x}+\dfrac{1}{y}=6$; $6x-y=1$;

(ii) $x^2-2xy+8y^2=8$, $3xy-2y^2=4$.

13. Solve the equations $\left.\begin{aligned}2x-3y&=7,\\ x^2+xy+2y^2&=28.\end{aligned}\right\}$

14. α and β being the roots of $x^2+px+q=0$, express

$$\frac{1}{(p\alpha+q)^2}+\frac{1}{(p\beta+q)^2}$$

in terms of p and q.

Simplify the expression when (i) the roots are equal, (ii) one is twice the other.

15. Solve the equations $2x^2-3xy-y^2=4$, $3x+y=1$.

16. If α and β are the roots of the quadratic equation

$$x^2+px+q=0,$$

prove that $p=-(\alpha+\beta)$ and $q=\alpha\beta$.

Prove that the roots of $(x+q)^2=p^2x$ are α^2 and β^2 and that the roots of $q(x+1)^2=p^2x$ are α/β and β/α.

17. Prove that in the equation

$$3x^2-2x(\alpha+2\beta)+\beta(2\alpha+\beta)=0,$$

(i) one root is independent of α and (ii) four times the product of the roots divided by β, and the result diminished by the sum of the roots is independent of β.

Find the relation between α and β so that one root may be twice the other.

18. Solve the equations

$$\frac{xy}{pq}-\frac{x}{p}=1-\frac{y}{q},$$
$$x+y=p+q.$$

19. Solve the equations (i) $\begin{cases} x^3+y^3=243, \\ x+y=9. \end{cases}$

(ii) $\begin{cases} x^3-y^3=91, \\ x^2y-xy^2=30. \end{cases}$

(iii) $\begin{cases} (x-3)(y-2)=6, \\ (2x+1)(y-1)=-12. \end{cases}$

(iv) $\begin{cases} x^2-5y^2=-1, \\ 3xy+7y^2=1. \end{cases}$

20. If α and β are the roots of the equation $x^2-lx+m=0$, find the equations whose roots are

(i) 2α and 2β; (ii) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$;

(iii) $p\alpha$ and $p\beta$; (iv) $\alpha+p$ and $\beta+p$;

(v) α^2 and β^2; (vi) $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

21. Solve the equations

(i) $y^4-5y^2+4=0$; (ii) $x(x+2)-\dfrac{35}{x(x+2)}=34$;

(iii) $9z^4-13z^2+4=0$.

22. If α, β are the roots of the equation $ax^2+bx+c=0$, find the equations whose roots are (i) $\alpha-1, \beta-1$; (ii) $\dfrac{1+\alpha}{1-\alpha}, \dfrac{1+\beta}{1-\beta}$.

23. Solve the equations

$$3x + y = 1, \qquad x^2 + xy - y^2 + 16x + 1 = 0. \qquad \text{[L.U.]}$$

24. Solve the equation $ax^2 + 2bx + c = 0$.

If the roots of this equation are α and β, construct the equation whose roots are $\alpha^2 + \dfrac{1}{\beta}$ and $\beta^2 + \dfrac{1}{\alpha}$. [B.U.]

25. Shew that the equation

$$ax^2 + bx + c = 0$$

has always real roots if a and c have opposite signs. What do you know about the signs of the roots in such a case?

Find the greatest value of $8x - 7x^2$, and the least value of $8x^2 - 7x$, illustrating your answer by careful graphs. [S.W.]

26. Establish the formula for the roots of the equation

$$ax^2 + bx + c = 0.$$

Shew that the roots must be real if a, c have opposite signs.

Shew that the roots of

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x-p} = 0$$

are real for all values of p. [S.W.]

27. Shew how to find the sum, the sum of the squares, and the sum of the cubes of the roots of a quadratic equation in terms of the coefficients in the equation, without actually solving the equation.

If x_1, x_2 are the roots of the equation $6x^2 - 6x + 1 = 0$, prove that

$$\tfrac{1}{2}(a + bx_1 + cx_1^2 + dx_1^3) + \tfrac{1}{2}(a + bx_2 + cx_2^2 + dx_2^3)$$

is equal to $a + b/2 + c/3 + d/4$ for all values of a, b, c, d. [Man.]

28. Divide $x^4 - y^4 - x^2y^2 + 2xy^3 - 4xy + 4y^2 - 4$ by $x^2 - xy + y^2 - 2$.

If both factors of the former expression are equal to xy, find its numerical value. [B.U.]

29. What is the condition that the equations

$$ax^2 + 2hx + b = 0 \quad \text{and} \quad a'x^2 + 2h'x + b' = 0$$

should have a common root?

Determine for what values of k the equations

$$(2k - 5)x^2 - 4x - 15 = 0,$$
$$(3k - 8)x^2 - 5x - 21 = 0$$

have a common root, and solve the equations in each case. [S.W.]

30. A man walks at the rate of x miles an hour, and rides at the rate of y miles an hour; how long does he take to perform a journey of n miles if he walks half the time and rides half the time?

Prove that if he had walked half the way and ridden half the way he would have taken altogether a longer time. [S.W.]

31. What is meant by the statement "A varies inversely as B"? Give a simple example.

The cost of running a steamer per day varies, for engine power, as the square of the speed. The other expenses are at the rate of ten pounds per day. When tho boat makes a certain voyage in 10 days the profit with a given cargo is £70; but when the voyage takes 16 days the profit is only £64. Find the time when the profit is £74. [S.W.]

CHAPTER II.

THE USE OF SQUARED PAPER: COORDINATES: THE STRAIGHT LINE.

In this chapter some elementary examples and exercises in the use of squared paper are given for revision. It is important to choose plain scales which can be read accurately and rapidly: the whole utility of the graphical method of representing two or more correlated quantities depends on the suitable selection and correct and clear figuring of the scales employed. Details will be found in the authors' *Practical Mathematics*. The illustrations are reduced from graphs drawn on decimally divided paper, but all the decimal subdivisions, *except the fifth*, are omitted in the reduced figures.

Ex. 1. *A certain insurance office gives assurance of £100 at death for the following yearly premiums:*

Premium	£2. 3*s*. 1*d*.	£2. 6*s*. 6*d*.	£2. 11*s*. 9*d*.	£2. 18*s*. 2*d*.
Age of person insured at time of taking policy	21	25	30	35

Premium	£3. 6*s*. 3*d*.	£3. 16*s*. 4*d*.	£4. 10*s*. 7*d*.	£5. 13*s*. 8*d*.
Age of person insured at time of taking policy	40	45	50	55

Illustrate graphically.

On the horizontal axis, two main divisions represent 5 years. The premiums when decimalised are:

£2·154, £2·325, £2·588, £2·908, £3·313, £3·817, £4·529, £5·683.

On the vertical axis, sums of money ranging from £2 to £5·7 have to be represented. Each main division is therefore taken to represent £0·2; each sub-division on the original figure representing £0·02 and the third decimal place being estimated.

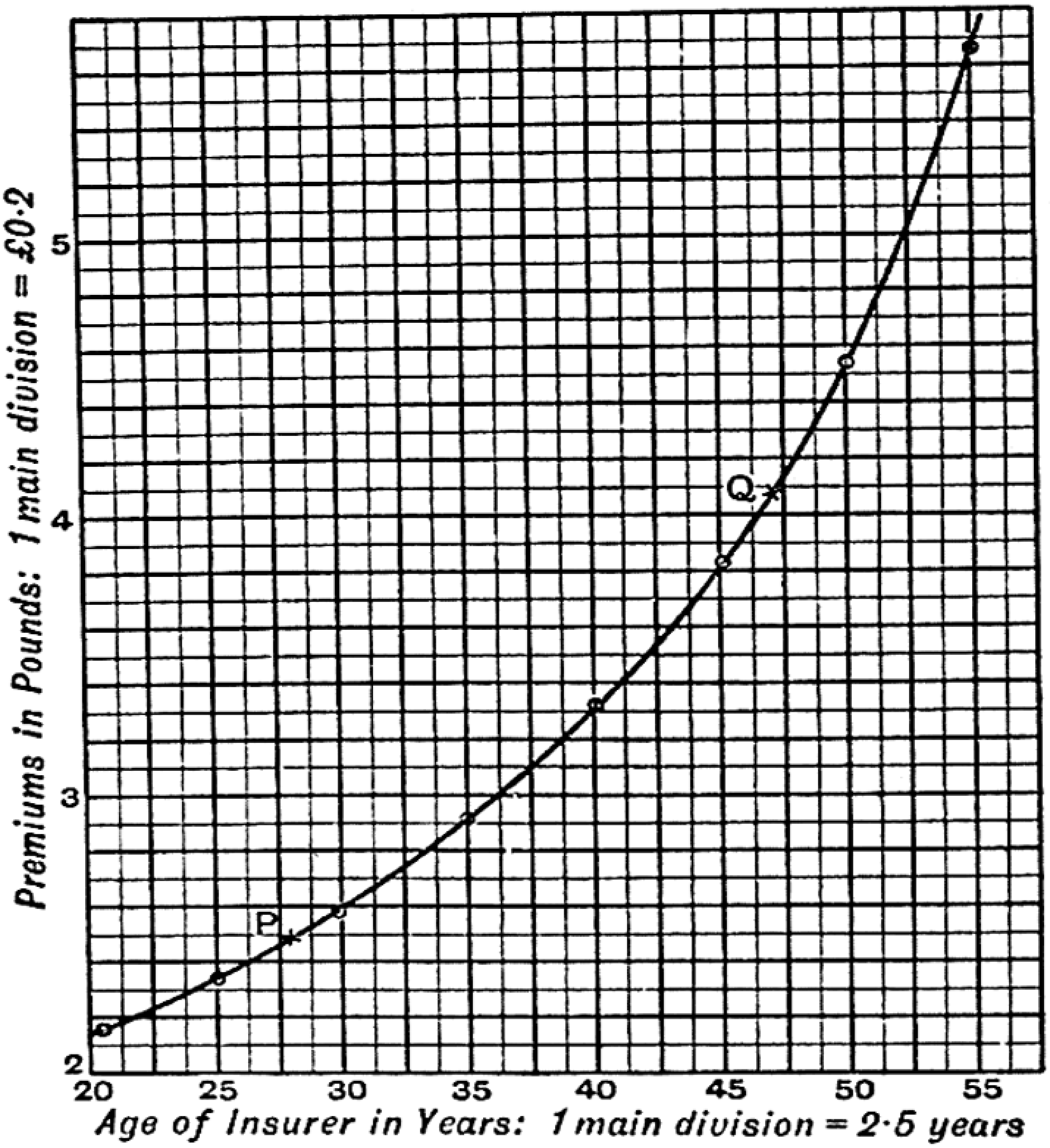

FIG. 1.—Variation of life insurance premium with age.

The diagram (Fig. 1) represents the plotted points, and it will be noticed that they lie on a regular curve, which has been drawn. The curve is useful for **interpolation**, *i.e.* it can be used for making a fair guess at the premium which would be charged at the same office at ages other than those given in the table.

Thus, at 28, the premium would not differ much from that shewn by the point P on the diagram, viz. £2·485 or £2. 9*s*. 8*d*.; the premium at 47 is shewn by the point Q, and is nearly £4. 1*s*. 7*d*.

The premium increases at a rapidly increasing rate. Thus, an increase of 5 years from 25 increases the premium by 5*s*. 3*d*., an increase of 5 years from 40 increases the premium by 10*s*. 1*d*., while an increase of 5 years from 50 increases the premium by 23*s*. 1*d*. The mere numerical values of these increases could be obtained from the table by calculation, but an inspection of the graph reveals this increase and its variations at once, and it is clearly connected in some way with the **steepness** of the graph.

Ex. 2. *Trace the changes in the value of the expression*

$$(x-1)(x-2)(x-3).$$

Value of x	$x-1$	$x-2$	$x-3$	Approx. value of $(x-1)(x-2)(x-3)$
− 1	− 2	− 3	− 4	− 24·00
− 0·5	− 1·5	− 2·5	− 3·5	− 13·13
0	− 1	− 2	− 3	− 6·00
0·2	− 0·8	− 1·8	− 2·8	− 4·03
0·5	− 0·5	− 1·5	− 2·5	− 1·88
0·7	− 0·3	− 1·3	− 2·3	− 0·90
1·0	0	− 1·0	− 2·0	0
1·2	0·2	− 0·8	− 1·8	0·29
1·5	0·5	− 0·5	− 1·5	0·38
1·7	0·7	− 0·3	− 1·3	0·27
2·0	1·0	0	− 1	0
2·3	1·3	0·3	− 0·7	− 0·27
2·5	1·5	0·5	− 0·5	− 0·38
2·8	1·8	0·8	− 0·2	− 0·29
3·0	2·0	1·0	0	0
3·2	2·2	1·2	0·2	0·53
3·5	2·5	1·5	0·5	1·88
3·7	2·7	1·7	0·7	3·21
4·0	3·0	2·0	1·0	6·00
4·5	3·5	2·5	1·5	13·13
5·0	4·0	3·0	2·0	24·00

Fig. 2 shews the points obtained, values of the expression being plotted vertically against values of x plotted horizontally.

Note carefully that

if $x < 1$ the values of the expression are $-^{\text{ve}}$,
if $x > 1$ and < 2 ,, ,, ,, ,, $+^{\text{ve}}$,
if $x > 2$ and < 3 ,, ,, ,, ,, $-^{\text{ve}}$,
while if $x > 3$,, ,, ,, ,, $+^{\text{ve}}$.

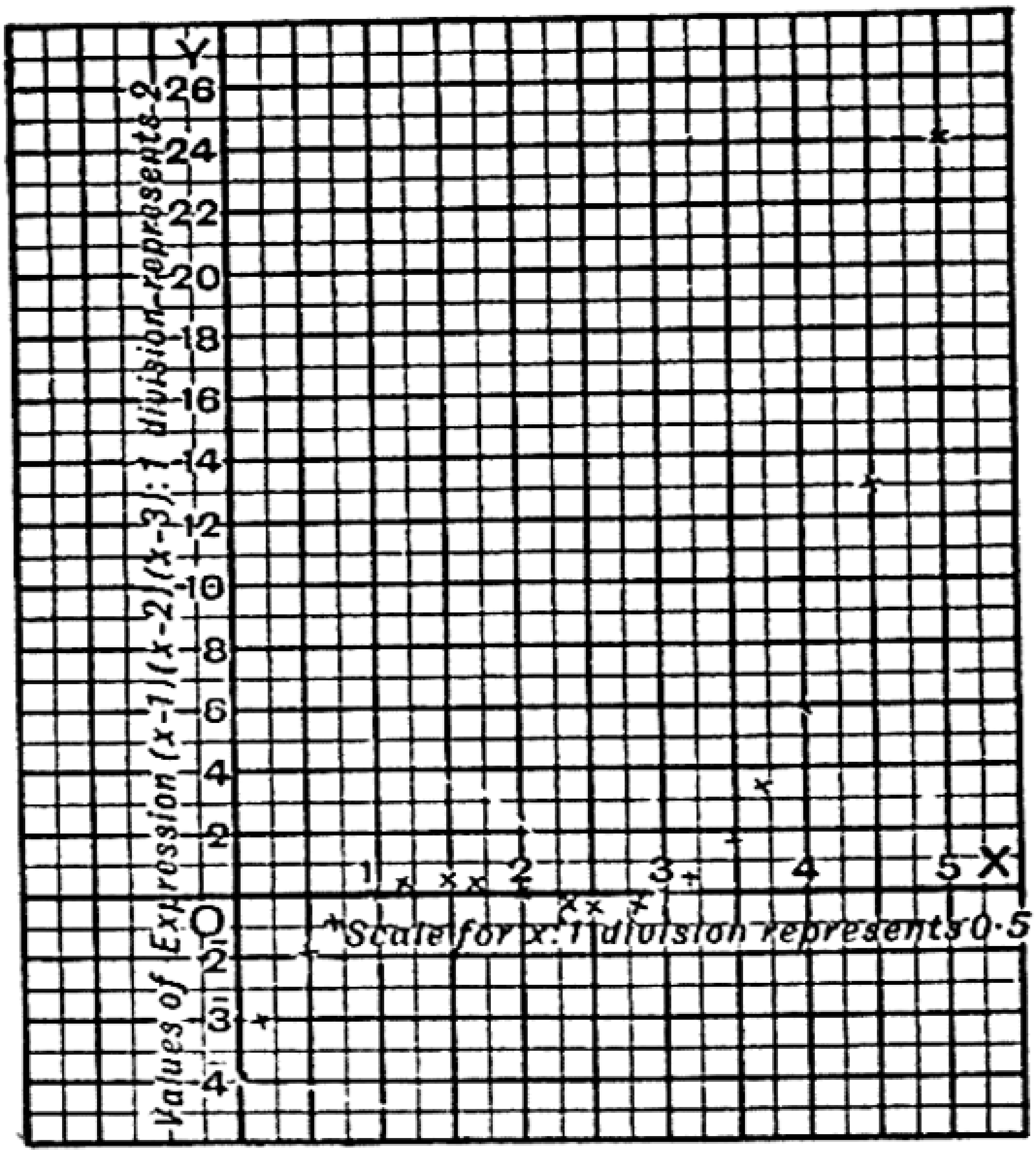

FIG. 2.

Further, there is a regularity about the plotted points; that is to say, they appear to lie on a continuous curve. To test if this is the case, other values of x are taken, and the portion of the figure lying between the values 0·5 and 3·5 for x is plotted on a larger scale as shewn in Fig. 3. In this diagram, points shewn in Fig. 2 are distinguished thus, ⊙; the new points being shewn by crosses thus, ×; and in every case the intermediate points lie on a regular continuous curve passing through the original points.

Values of x	0·5	0·7	0·9	1·0	1·1
Values of $(x-1)(x-2)(x-3)$, to three significant figures	−1·875	−0·897	−0·231	0	+0·171
Values of x	1·2	1·4	1·5	1·6	
Values of $(x-1)(x-2)(x-3)$, to three significant figures	+0·288	+0·384	+0·375	+0·336	

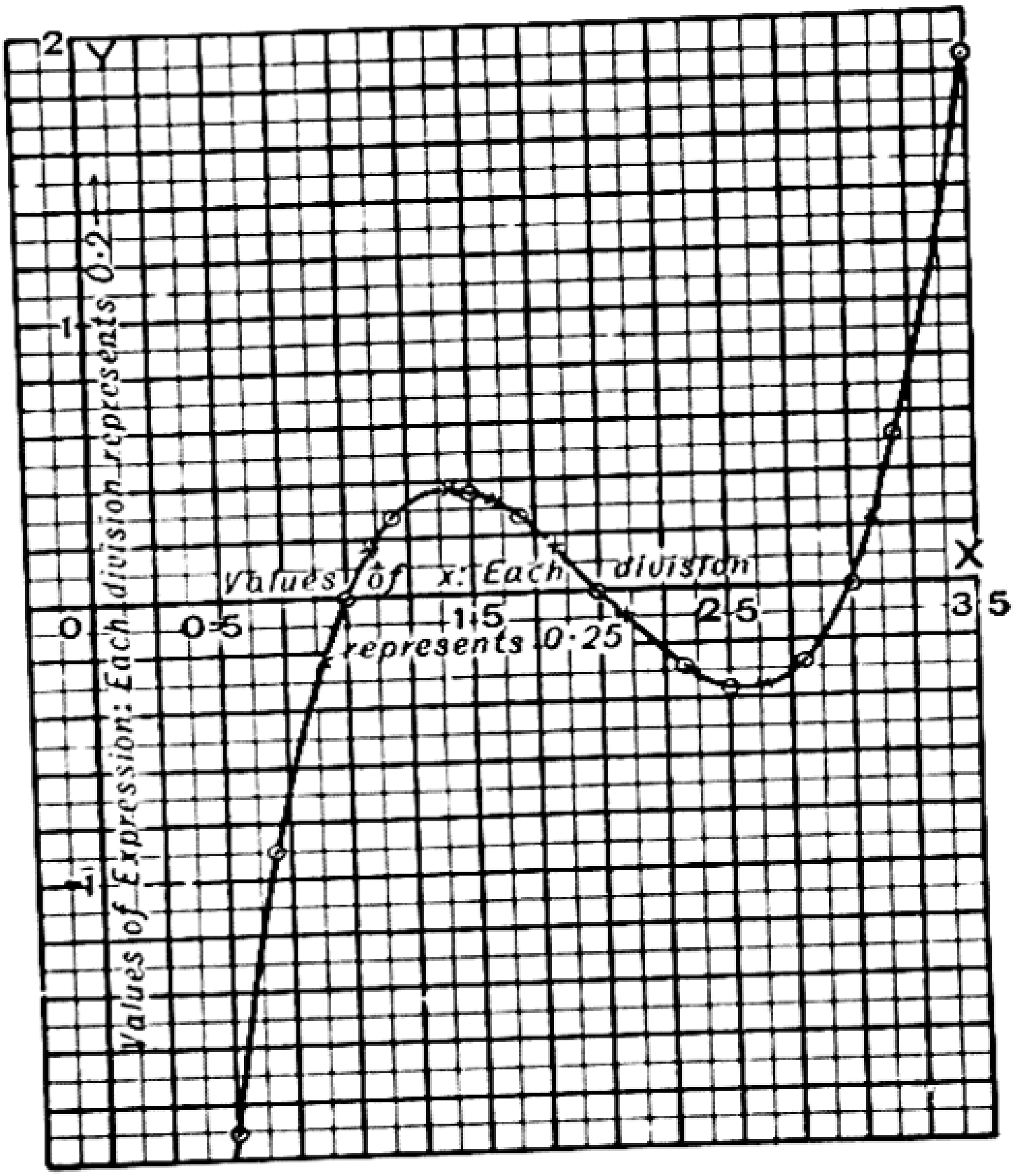

Fig. 3.—Graph of $(x-1)(x-2)(x-3)$.

Values of x	1·8	2·0	2·1	2·2	2·3	2·5
Values of expression, to three figures	+0·192	0	−0·099	−0·192	−0·273	−0·375
Values of x	2·6	2·8	3·0	3·1	3·2	3·5
Values of expression, to three figures	−0·384	−0·288	0	+0·231	+0·528	+1·875

As x increases, it is clear (1) that the expression increases to about 0·385 (when $x \doteqdot 1{\cdot}42$); it then decreases to $-0{\cdot}385$ (when $x \doteqdot 2{\cdot}6$); after which it increases without limit; (2) that the expression is equal to 0 when $x=1$, when $x=2$ and also when $x=3$.

(The symbol $\doteqdot$ is read "is approximately equal to.")

Ex. 3. *Trace the changes in the value of* $1+\sqrt{1+x^2}$ *as* x *alters.*

Notice that since $(-a)^2$ has the same value as $(+a)^2$, we need only consider positive values of x.

Values of x	x^2	$1+x^2$	$\sqrt{1+x^2}$	$1+\sqrt{1+x^2}$
0	0	1	1	2
0·2	0·04	1·04	1·02	2·02
0·5	0·25	1·25	1·12	2·12
1·0	1·0	2·0	1·41	2·41
1·5	2·25	3·25	1·80	2·80
2·0	4·0	5·0	2·23	3·23
2·2	4·84	5·84	2·42	3·42
3·0	9·0	10·0	3·16	4·16
3·5	12·25	13·25	3·64	4·64
4·0	16·0	17·0	4·12	5·12
4·6	21·16	22·16	4·71	5·71
5·0	25·0	26·0	5·10	6·10
6·0	36·0	37·0	6·08	7·08
6·8	46·24	47·24	6·87	7·87
7·0	49·0	50·0	7·07	8·07
8·0	64·0	65·0	8·06	9·06
9·0	81·0	82·0	9·06	10·06
10·0	100·0	101·0	10·05	11·05
11·0	121·0	122·0	11·05	12·05
12·0	144·0	145·0	12·04	13·04

The points determined by the first and last columns are shewn in Fig. 4. They evidently lie on a perfectly continuous and regular curve; that is to say, if we found the value of $1+\sqrt{1+x^2}$ for, say, $x=5{\cdot}473$, we should get a number k lying between 6 and 7, such that if the point $x=5{\cdot}473$ and $y=k$ were plotted it would lie very nearly on the straight line which passes through the points AB in the figure.

The least possible value of the expression $1+\sqrt{1+x^2}$ is 2·0, and it takes this value when x is 0. For all other values of x, whether $+^{ve}$ or $-^{ve}$, $1+\sqrt{1+x^2}>2$. This number 2 is the minimum value of the expression $1+\sqrt{1+x^2}$.

There is no maximum value, as the larger we take x the larger $1+\sqrt{1+x^2}$ becomes, and this is true whether x be $+^{ve}$ or $-^{ve}$.

For values of $x>2$ and <10 the points lie very approximately on a straight line.

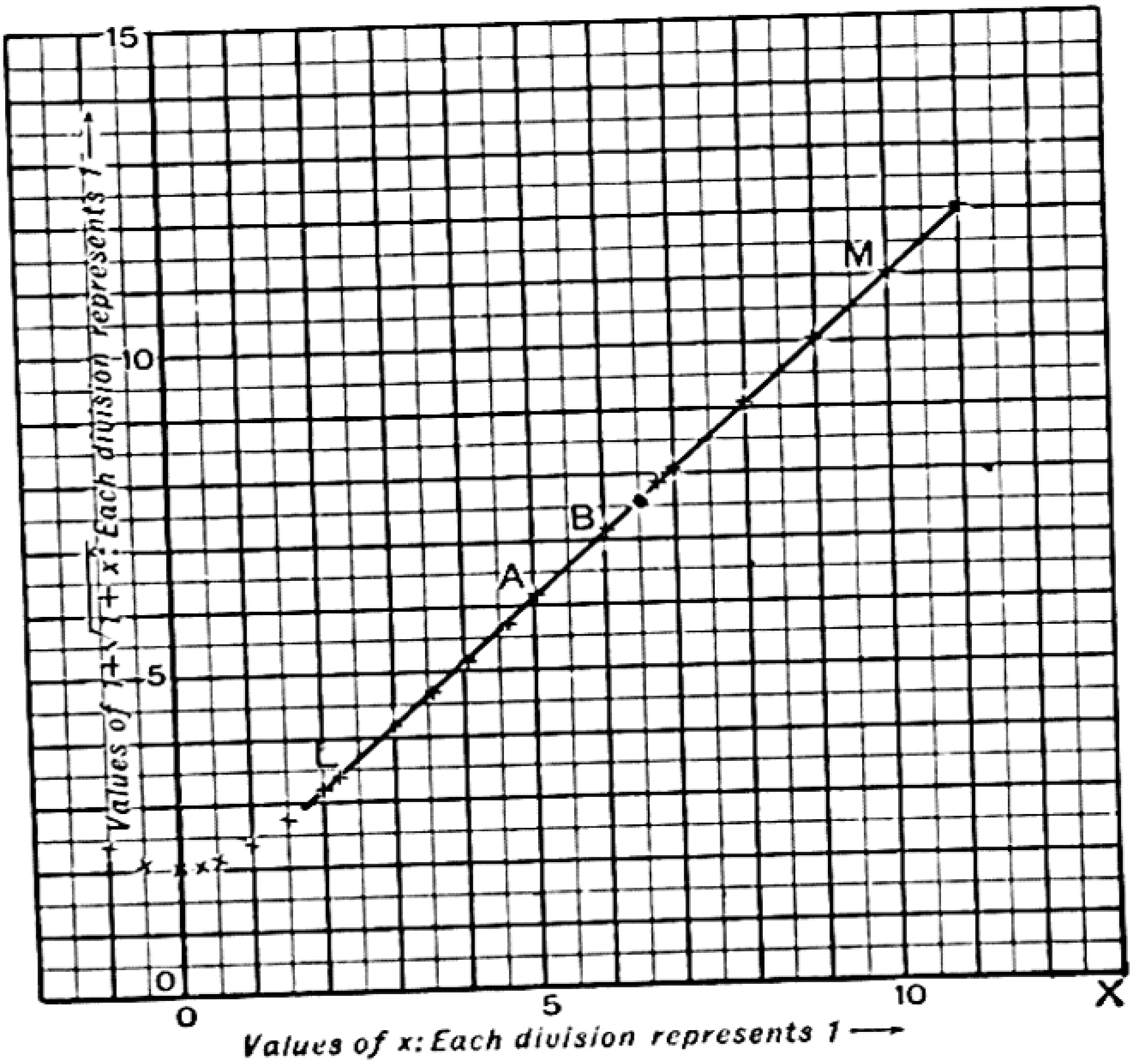

FIG. 4.—Graph of $1+\sqrt{1+x^2}$.

Ex. 4. *The total cost C of a ship per hour (including interest and depreciation on capital, wages, coal, etc.) is in pounds* $C=4+S^3/1000$, *where S is the speed in knots or nautical miles per hour. The time in hours spent in a passage of, say,* 3000 *miles is* $3000/S$, *so that the total cost of the passage is this time multiplied by C. Express this algebraically in terms of S.*

Find what this amounts to for various speeds. For what speed is it a minimum? [B.E.]

Total cost of passage in pounds = time of passage × cost per hour

$$= \frac{3000}{S} \times C$$

$$= \frac{3000}{S}\left(4 + \frac{S^3}{1000}\right).$$

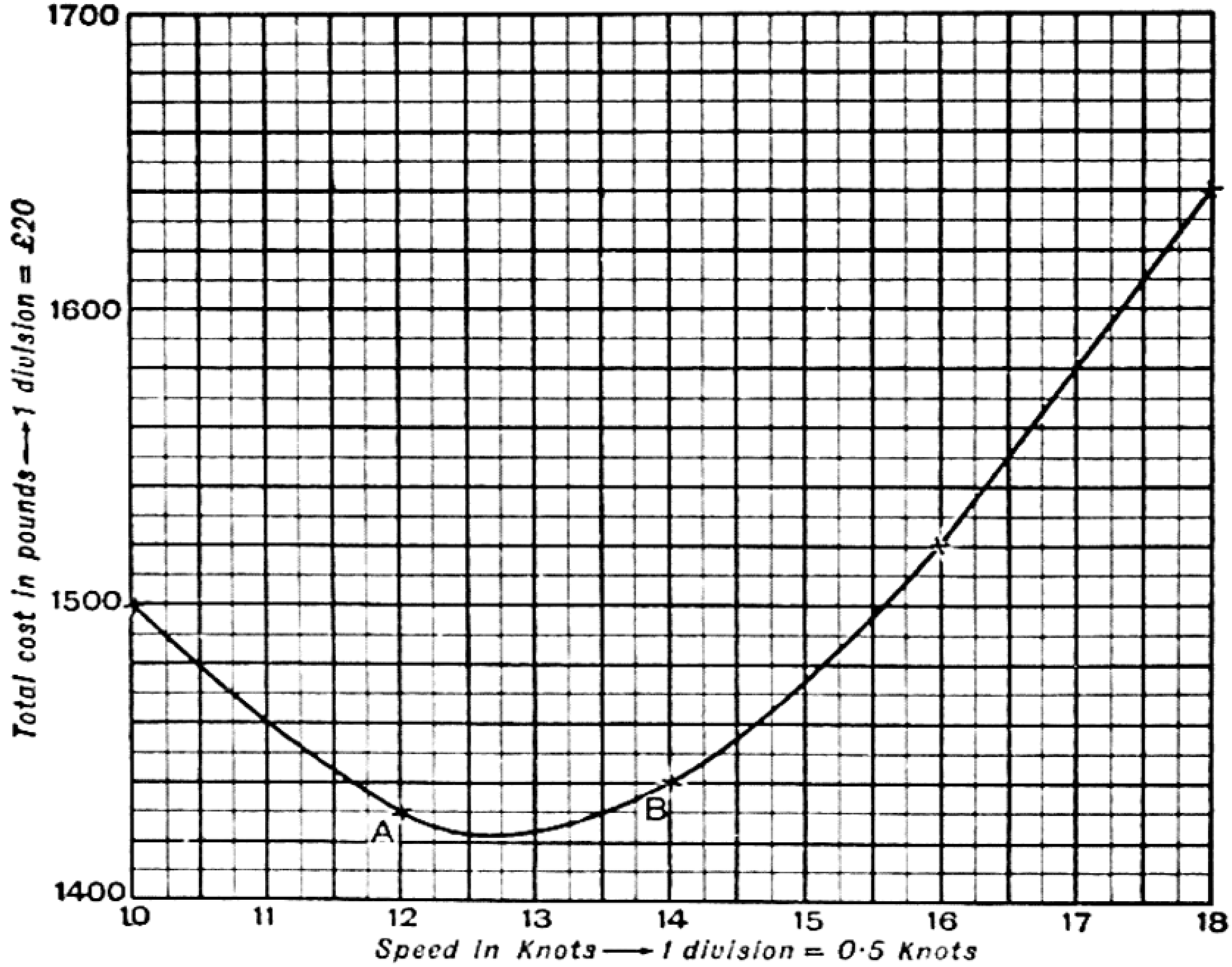

FIG. 5.—Variation of cost of journey with speed.

Calculating to three significant figures in the first instance, we have for values of S between 10 and 18.

S	S^3	$\frac{3000}{S}$	$4+\frac{S^3}{1000}$	Total cost in £
10	1000	300	5	1500
12	1728	250	5·73	1430
14	2744	214	6·74	1440
16	4096	188	8·10	1520
18	5832	167	9·83	1640

S is plotted against total cost to the scales indicated in Fig. 5. From this curve the speed giving the minimum cost is seen to lie between 12·5 knots and 12·8 knots. A portion of the curve lying between the points A and B is shown magnified in Fig. 6, the requisite calculations being made to four significant figures.

S	S^3	$\frac{3000}{S}$	$4+\frac{S^3}{1000}$	Total cost in £
12	1728	250	5·728	1432
12·5	1953	240	5·953	1429
13	2197	230·8	6·197	1430
12·6[*]	2000	238·1	6·000	1429

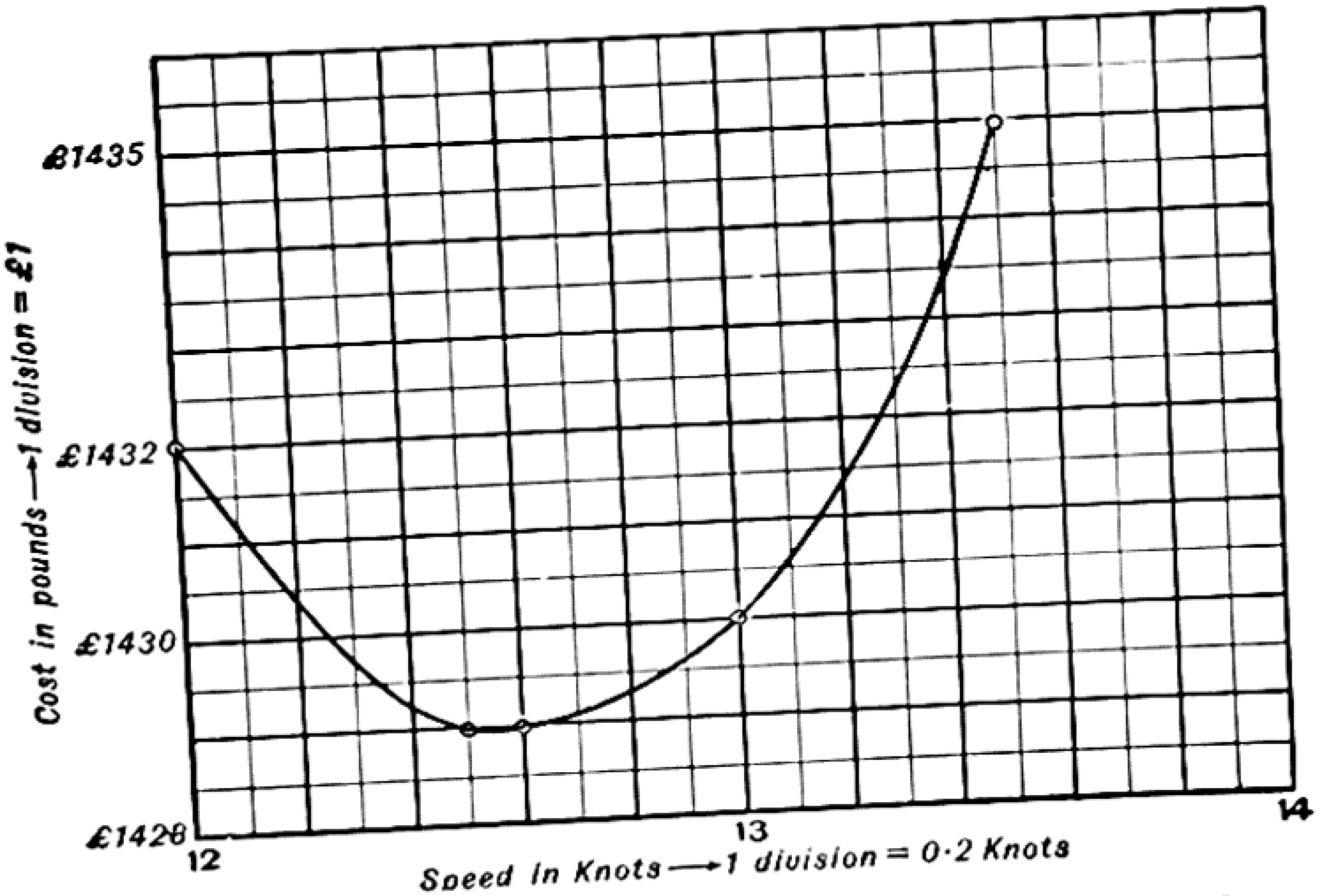

FIG. 6.—Enlargement of part of curve between A and B in Fig. 5.

From Fig. 6, the speed giving the minimum total cost lies between 12·5 and 12·6 knots, and nearer the latter. Note that at the best speed a slight variation in the speed does not make much difference in the total cost of the journey.

[*] As the total cost for 13 knots was greater than that for 12·5 knots, the cost for 12·6 knots was calculated.

Coordinates. Any two distinct numbers can be represented on squared paper by means of a single point. These two numbers are the **coordinates** of the point, and are represented by the distances of the point from two straight lines at right angles called the **axes of coordinates,** or **axes of reference,** or, briefly, the **axes.** The point of intersection of the axes is the **origin.** The axes are distinguished as the X (or horizontal) axis and the Y (or vertical) axis. The distance of a point from the vertical axis is the **abscissa** of the point, its x coordinate, or, briefly, its x; the distance of a point from the horizontal axis is its **ordinate,** the y coordinate of the point, or, briefly, its y.

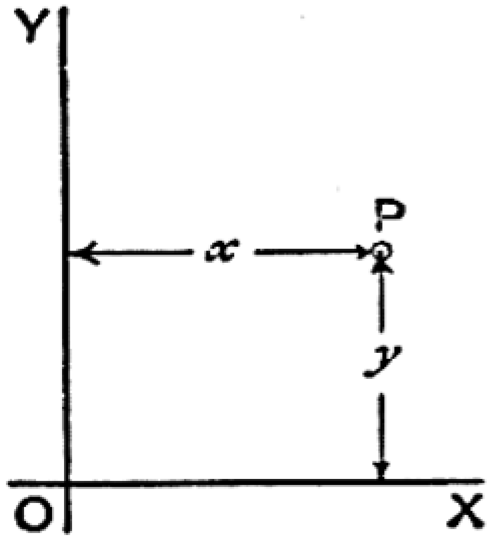

FIG. 7.

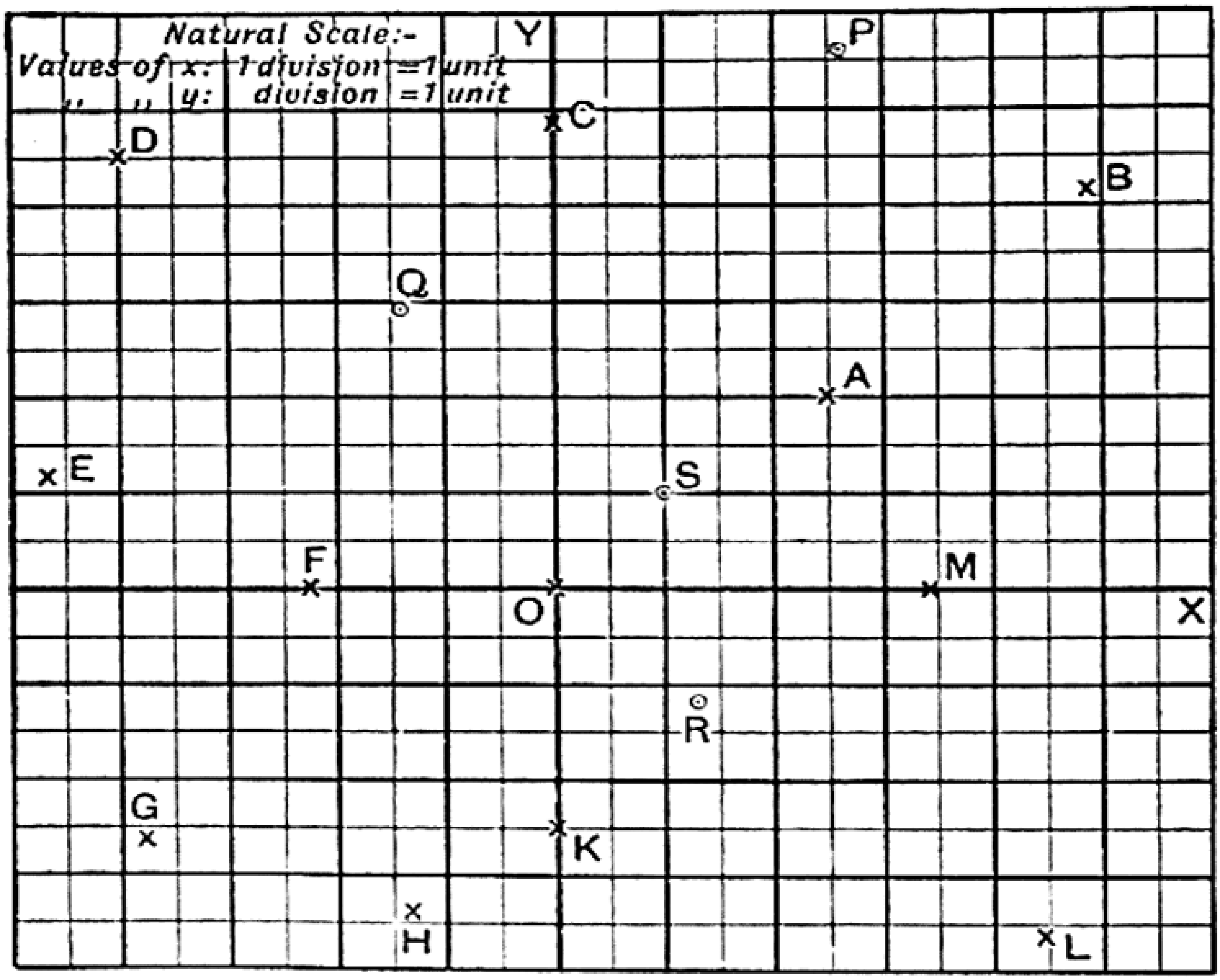

FIG. 8.

These **coordinates** are sometimes called **rectangular coordinates.** For polar coordinates, see p. 156.

In Fig. 7 (x, y) are the rectangular coordinates of the point P. This means that the distance of P from OY is x, and from OX is y. Consistently with previous work, if the x of a point is negative, the point is to the *left* of the vertical axis; if the y of a point is negative, the point is *below* the horizontal axis.

Thus, in Fig. 8, P is the point 2·6, 5·6;
Q is the point $-1{\cdot}4$, 2·9;
R is the point 1·3, $-1{\cdot}2$;
F is the point $-2{\cdot}3$, 0.

Ex. 1. *What is known about the position of the point or points for which $x = 2$, and for which y is not known? What does $y = 1$ mean?*

Whatever y may be, x is to be 2. That is to say, the point is always to be $+2$ units from the vertical axis; hence, it may be anywhere on the line AB, which is parallel to OY and 2 units to its right (Fig. 9).

The x of each point on AB is 2, and all points where $x = 2$ must lie on AB.

Hence $x = 2$ is represented by AB, and conversely the line AB may be called the line $x = 2$.

Similarly $y = 1$ means the line CD, which is parallel to OX and 1 unit above it. Conversely, the line CD is represented by the equation $y = 1$.

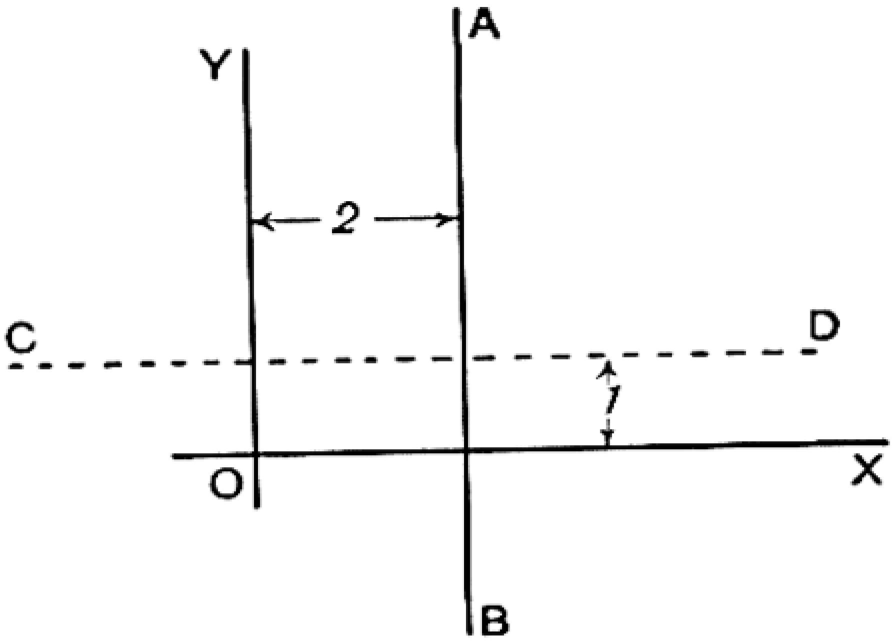

FIG. 9.

Ex. 2. *What equation, if any, represents the X-axis?*

Now, if a point lies on OX, its distance from OX is zero. That is to say, the y-coordinate of every point on OX is 0, no matter what its x-coordinate may be.

Hence $y = 0$ can only refer to points lying on OX,
and $\therefore\ y = 0$ represents the axis OX.

EXERCISES III.

1. Give the coordinates of the points A, B, C, etc. (Fig. 8), indicated by crosses.

2. Represent on squared paper the points: 1, 2; 2·5, 3·7; −0·2, 2·2; 1·7, −2·4; −1·5, −2·0; 0, −2·2; 3, 0; 0·5, −0·7.

3. What is represented by $x=0$?

4. What is represented by $x=y$?

5. What is represented by $x=2y$?

6. What is represented by $x+y=0$?

7. What line represents $y=2·7$?

8. What line represents $3y=2x$?

9. If, in Fig. 8, axes are drawn through the point A parallel to OX and OY, what would be the coordinates of the points B, C. D and A with reference to the new axes, the scale of the figure being used?

10. The axes are usually at right angles, but it is sometimes convenient to use oblique axes, *i.e.* axes which are not at right angles. In this case the coordinates of a point are measured parallel to the axes. Draw axes at (i) 60°, (ii) 135°, and plot the points 1, 2; 2, −3; 3, −1; −4, −2.

Typical Graphs. All regular continuous simple graphs must belong to one or other of the standard types illustrated in Fig. 10.

The following general inferences may be made if we imagine x to increase steadily, and trace the corresponding changes in y.

Let the initial value of x considered be x_1, and let the corresponding value of y be y_1. Let the second value of x be x_1+t and let the second value of y be y_1+s. The third value of x will be x_1+2t, since x is assumed to increase steadily.

Let the third value of y be y_1+s+s'.

Thus, in Fig. 11, $ON=x_1$, $NP=y_1$.

$$ON_1=x_1+t, \text{ where } NN_1=t.$$

And $N_1P_1=y_1+s$, where $Q_1P_1=s$.

Also $ON_2 = x_1 + 2t$, since $N_1N_2 = NN_1 = t$.
And $N_2P_2 = y_1 + s + s'$.

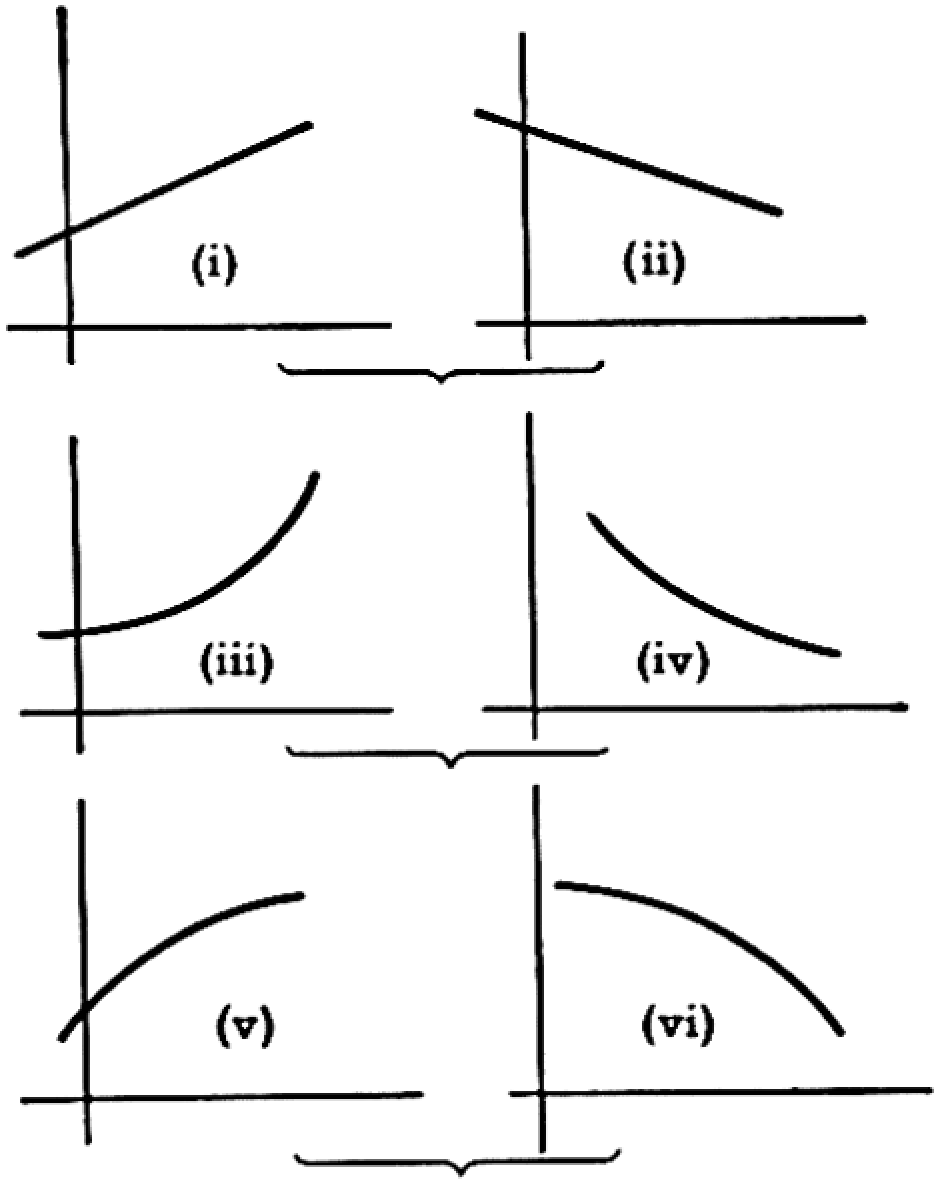

FIG. 10.—Typical Graphs.

CASE I. The linear graph sloping upwards to the right (Fig. 11). Through P, P_1, P_2, draw lines parallel to the axes as in the figure. Then

$$PQ_1 = Q_1Q_2 = t,$$
$$P_1Q_1 = s = R_2Q_2,$$
$$P_2R_2 = s'.$$

Now, inspection will shew that in all cases $s = s'$.

[For considering $\triangle^s PQ_1P_1$, $P_1R_2P_2$;

$$PQ_1 = P_1R_2 = t,$$
$$\angle P_1PQ_1 = \angle P_2P_1R_2$$

and $\angle PQ_1P_1 = \angle P_1R_2P_2 = 90°$.

Hence the triangles are identically equal and $P_1Q_1 = P_2R_2$, *i.e.* $s = s'$.]

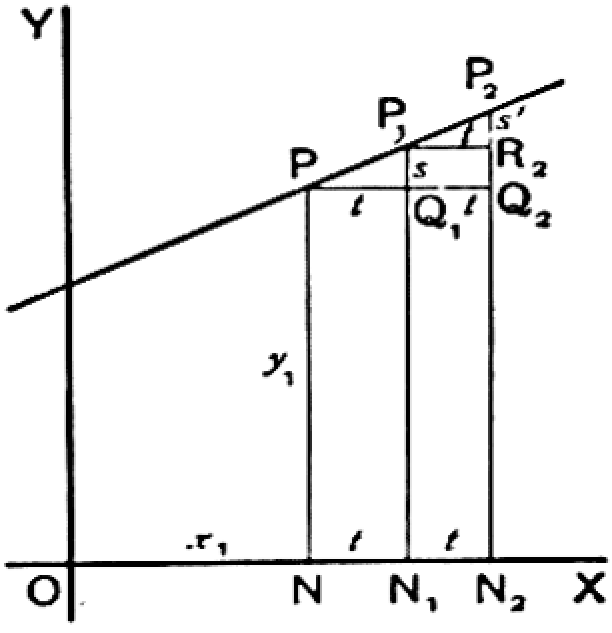

FIG. 11.

Hence, when x is x_1+2t, y will be y_1+2s, so that as x increases steadily, y also increases steadily.

Thus, if y is increased by 0·5 when x is increased by 1,
then y will be ,, 1 ,, ,, 2,
and y ,, ,, 5 ,, ,, 10;

or generalising,

If y is increased by k when x is increased by h,
then y will be ,, nk ,, ,, nh.

If any experimental results, or other numbers, are represented graphically by a straight line, it may be inferred that if successive increments in x be equal, then the corresponding successive changes in y will also be equal.

This is expressed mathematically by saying that **y increases with x at a steady or uniform rate.**

From the figure, it will be seen that the line has **uniform slope** or that its **gradient is constant.** It is convenient to **measure the slope by the rise per unit horizontal distance.**[*]

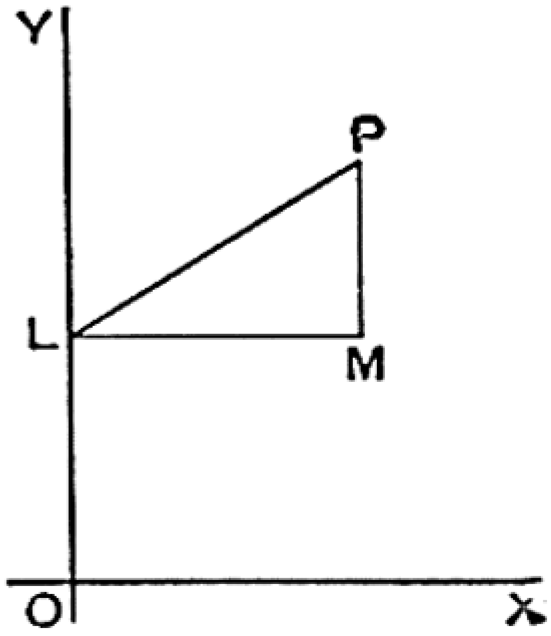

FIG. 12.

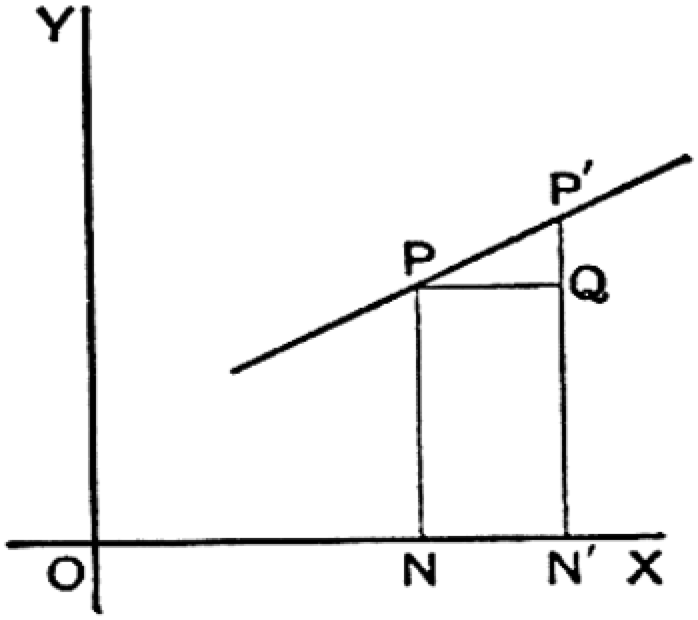

FIG. 13.

Thus, in Fig. 12, the line LP rises by the amount PM in the horizontal distance LM; the slope is measured by the fraction PM/LM.

Again, in Fig. 13, suppose the coordinates of P to be x, y, and those of P' to be $x+\Delta x$, $y+\Delta y$ [read 'x plus delta-x, y plus

---

[*] In practice, gradients are measured by the number of feet *along the slope* required to give a difference of level of 1 foot. Thus, posts are erected along the side of a railway for the information of the engine driver, as shewn in the figure

delta-y'], where Δx* denotes an increase in x (*i.e.* NN') and Δy denotes the corresponding increase in y (*i.e.* QP'). Then, since the slope is the rise in unit horizontal distance, and y increases by the amount Δy while x increases by Δx, it follows that the slope is measured by the fraction $\Delta y/\Delta x$.

Ex. Fig. 14 *shews a displacement-time graph for a point moving with uniform speed. Shew that* $\Delta s/\Delta t$, *calculated at any point of the graph, measures the rate at which the point is moving.*

Ot is the axis of time, OS the axis of displacement; evidently displacements are not measured from the starting-point, since when $t=0$, $s=OA$.

From the graph, in the time represented by NN', *i.e.* by PQ, the displacement increases from PN to RN'.

Using the notation already explained $NN'=\Delta t$, $RQ=\Delta s$.

Therefore in Δt units of time, the point moves Δs units of length.

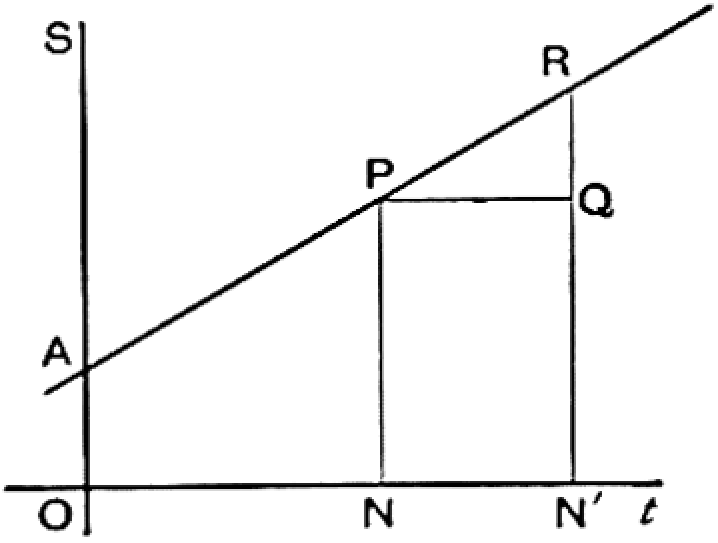

FIG. 14.

$\therefore$ in the unit of time the point moves $\dfrac{\Delta s}{\Delta t}$ units of length, *i.e.* the speed is $\dfrac{\Delta s}{\Delta t}$ units of length per unit of time, or the rate at which the point moves is measured by $\dfrac{\Delta s}{\Delta t}$.

Geometrically, $\dfrac{\Delta s}{\Delta t}$ measures the slope of the graph.

Hence, the slope of the displacement-time graph is a measure of the rate, or speed.

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* *N.B.* Δx does *not* mean "Δ multiplied by x"; and Δy does *not* mean "Δ multiplied by y."

CASE II. **The linear graph sloping downwards to the right.** Using the same notation as in Case I., it is clear from Fig. 15 that the same discussion applies, so that if y is y_1 when x is x_1, y will be $y_1 - s$ when x is $x_1 + t$, and will be $y_1 - 2s$ when x is $x_1 + 2t$. Therefore, *while x increases steadily, y decreases steadily.*

The line in this case *slopes downward*, and may consistently be said to have *negative upward slope*, or, briefly, *negative slope*.

In this case, if y increases by a negative amount Δy (*i.e.* y decreases) while x increases by a positive amount Δx, $\Delta y/\Delta x$ measures the slope and also the **rate of increase of y with respect to x.**

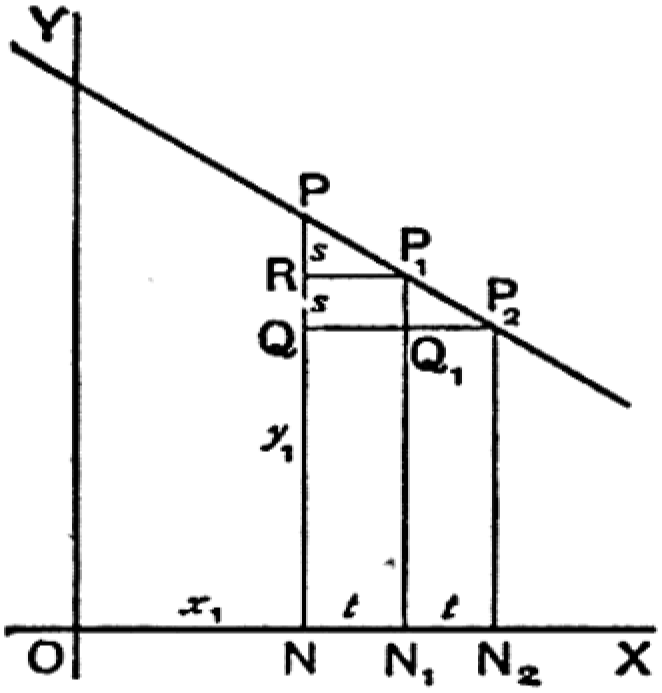

FIG. 15.

$\frac{\Delta y}{\Delta x}$ is negative, so that the rate of increase of y with respect to x and the slope are both negative.

CASE III. **The graph concave upwards and sloping upwards to the right** (Fig. 16). Geometrically, it is clear that the curve gets steeper as x increases.

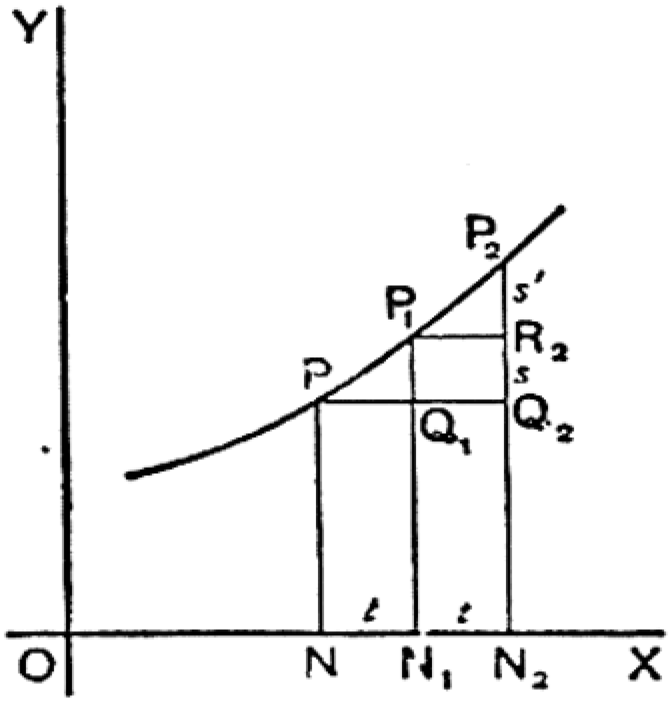

FIG. 16.

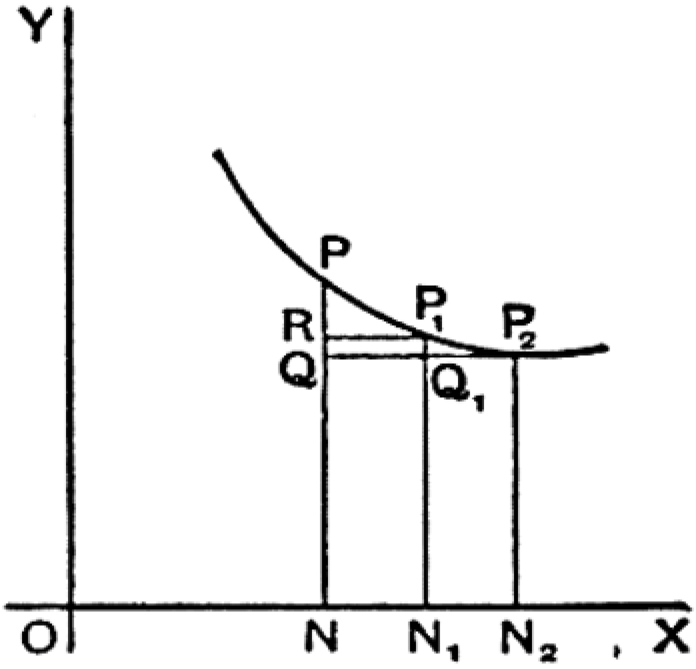

FIG. 17.

Using the same notation as before,

$$ON = x_1, \quad ON_1 = x_1 + t, \quad ON_2 = x_1 + 2t,$$
$$NP = y_1, \quad N_1P_1 = y_1 + s, \quad N_2P_2 = y_1 + s + s',$$

where $Q_2R_2 = s$ and $R_2P_2 = s'$, and $s' > s$, so that y increases more rapidly than x.

$$\text{At the point } P, \frac{\Delta y}{\Delta x} \text{ is measured by } \frac{P_1Q_1}{PQ_1}.$$

$$\text{At the point } P_1, \frac{\Delta y}{\Delta x} \text{ is measured by } \frac{P_2R_2}{P_1R_2}.$$

Comparing these fractions, since $P_2R_2 > P_1Q_1$ and $PQ_1 = P_1R_2$, therefore $\frac{\Delta y}{\Delta x}$ increases as x increases. Hence **y increases with x, but at an increasing rate.**

CASE IV. **The graph concave upwards and sloping downwards to the right** (Fig. 17). On comparing with Case II., it will be clear that $\frac{\Delta y}{\Delta x}$ is negative, and diminishes numerically as x increases. Hence **y diminishes as x increases, but at a diminishing rate.**

CASE V. **The graph convex upwards and sloping upwards to the right** (Fig. 18). Comparing with Case I., $\frac{\Delta y}{\Delta x}$ is positive, but diminishes as x increases. Further, **y increases as x increases, but at a diminishing rate.**

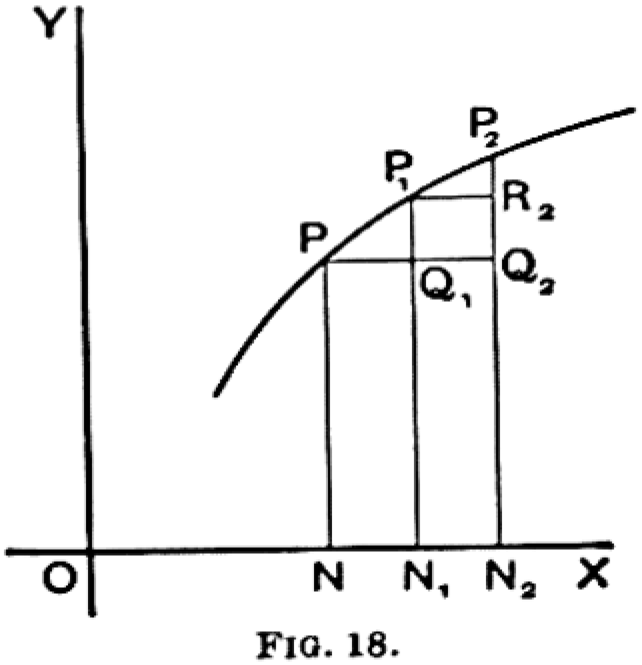

FIG. 18.

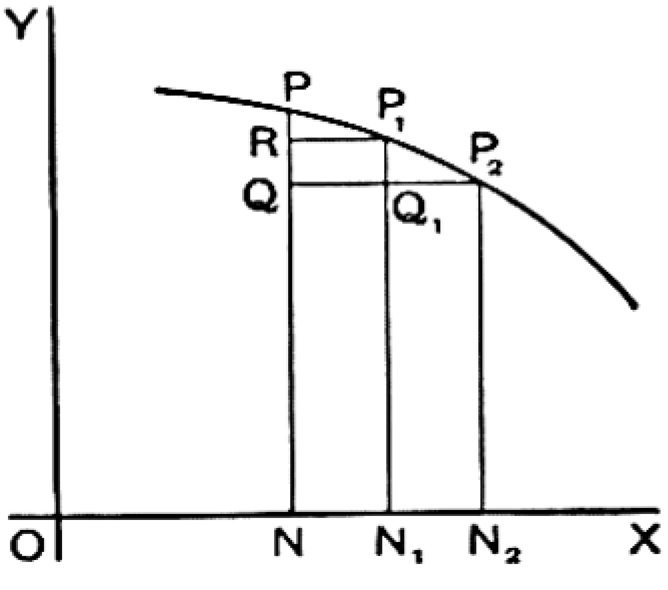

FIG. 19.

CASE VI. **The graph convex upwards and sloping downwards to the right** (Fig. 19). Comparing with Cases II. and IV., $\frac{\Delta y}{\Delta x}$ is negative, and increases numerically as x increases. Further, **y diminishes as x increases, and at an increasing rate.**

Detailed Consideration of the Linear Graph.

The linear, or straight line graph, discussed in Cases I. and II., pp. 35-38, is so important that it is worth while to consider it in some detail.

Ex. 1. *Represent the variation in the value of the expression* $a + bx$ *when* $a = 1{\cdot}5$, $b = 2$ *for different values of* x, *by plotting on squared paper.*

Let y denote the value of the expression. In order to find the value of y corresponding to any particular value of x, we have to multiply that value of x by 2 and add 1·5. There is no restriction whatever on the particular values of x which may be chosen; but it is best to arrange them in order of either decreasing or increasing magnitude. The work should be set out in tabular form, as follows.

Values of x	−5·0	−4·5	−2·0	−1·5	−0·5	0	0·5	1·0	1·5	2·0	etc.
Values of bx	−10	−9	−4	−3	−1	0	1	2	3	4	etc.
Value of a	1·5	1·5	1·5	1·5	1·5	1·5	1·5	1·5	1·5	1·5	etc.
Values of y	−8·5	−7·5	2·5	−1·5	+0·5	1·5	2·5	3·5	4·5	5·5	etc.

The bottom row is obtained by adding the numbers in the second and third rows.

The two sets of numbers which are to be plotted are the values of x and the values of y given in the above table. Representing them on squared paper, all the points will be found to lie on a straight line as in Fig. 20. **Further, if any value of x whatever be selected, and the corresponding value of y calculated, the point representing these values of x and y will also be found to lie on the same straight line.**

E.g. if
$$x = 1{\cdot}7, \quad y = 1{\cdot}5 + 2 \times 1{\cdot}7$$
$$= 4{\cdot}9,$$
and the point 1·7, 4·9 lies on the line, as shewn at A.

Again, if
$$x = -3{\cdot}4, \quad y = 1{\cdot}5 - 6{\cdot}8$$
$$= -5{\cdot}3,$$
and the point −3·4, −5·3 lies on the line, as shewn at P.

Further, we can use the graph to determine without calculation the value of y corresponding to any given value of x, or to find the value of x which causes y to have a certain value.

Thus, to find y when $x = 2{\cdot}2$, cut off $ON = 2{\cdot}2$ and erect the ordinate at N. Then NQ, which is $5{\cdot}9$, is the required value of y.

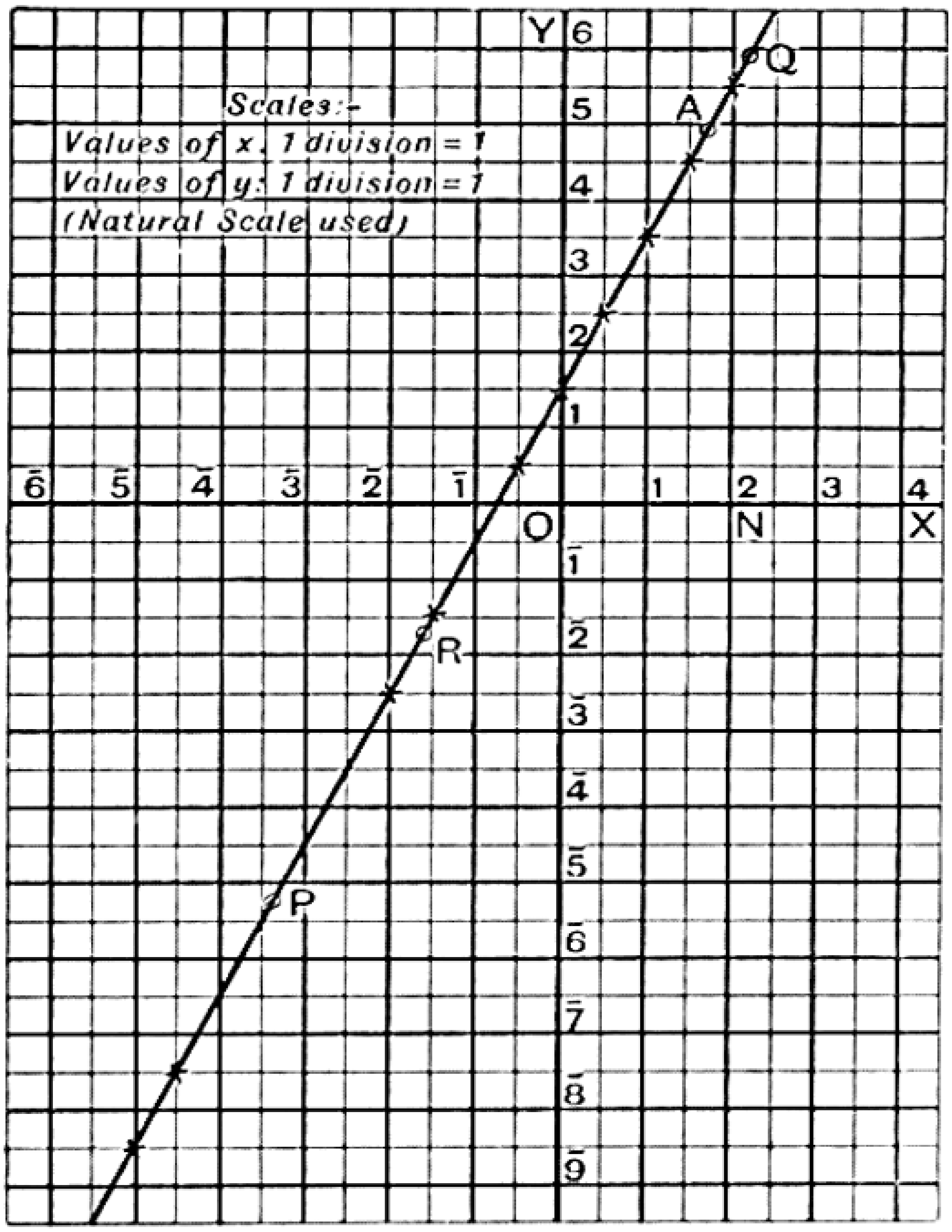

FIG. 20.—Graph of $y = 1{\cdot}5 + 2x$.

The value of x which makes $y = -1{\cdot}7$ is $-1{\cdot}6$, as shewn by the point R.

The value of x which makes $y = 0$ is $-0{\cdot}75$, as the graph cuts the axis of x (where $y = 0$) at the point midway between $-0{\cdot}7$ and $-0{\cdot}8$.

It should be noted that $x = -0{\cdot}75$ is the solution of the equation $1{\cdot}5 + 2x = 0$.

Further, $x = -1{\cdot}6$ is the solution of $1{\cdot}5 + 2x = -1{\cdot}7$.

Ex. 2. *Represent graphically the expression* $a+bx$, *where* $a=0·37$ *and* $b=0·53$.

Let y denote the value of the expression. Then we have:

x	-4	$-2·5$	-2	-1	0	1	2	3	$4·5$	6	etc.
bx	$-2·12$	$-1·325$	$-1·06$	$-0·53$	0	$0·53$	$1·06$	$1·59$	$2·385$	$3·18$	etc.
y	$-1·75$	$-0·955$	$-0·69$	$-0·16$	$0·37$	$0·9$	$1·43$	$1·96$	$2·755$	$3·55$	etc.

the bottom row being obtained by adding the value of a (0·37) to each term of the second row.

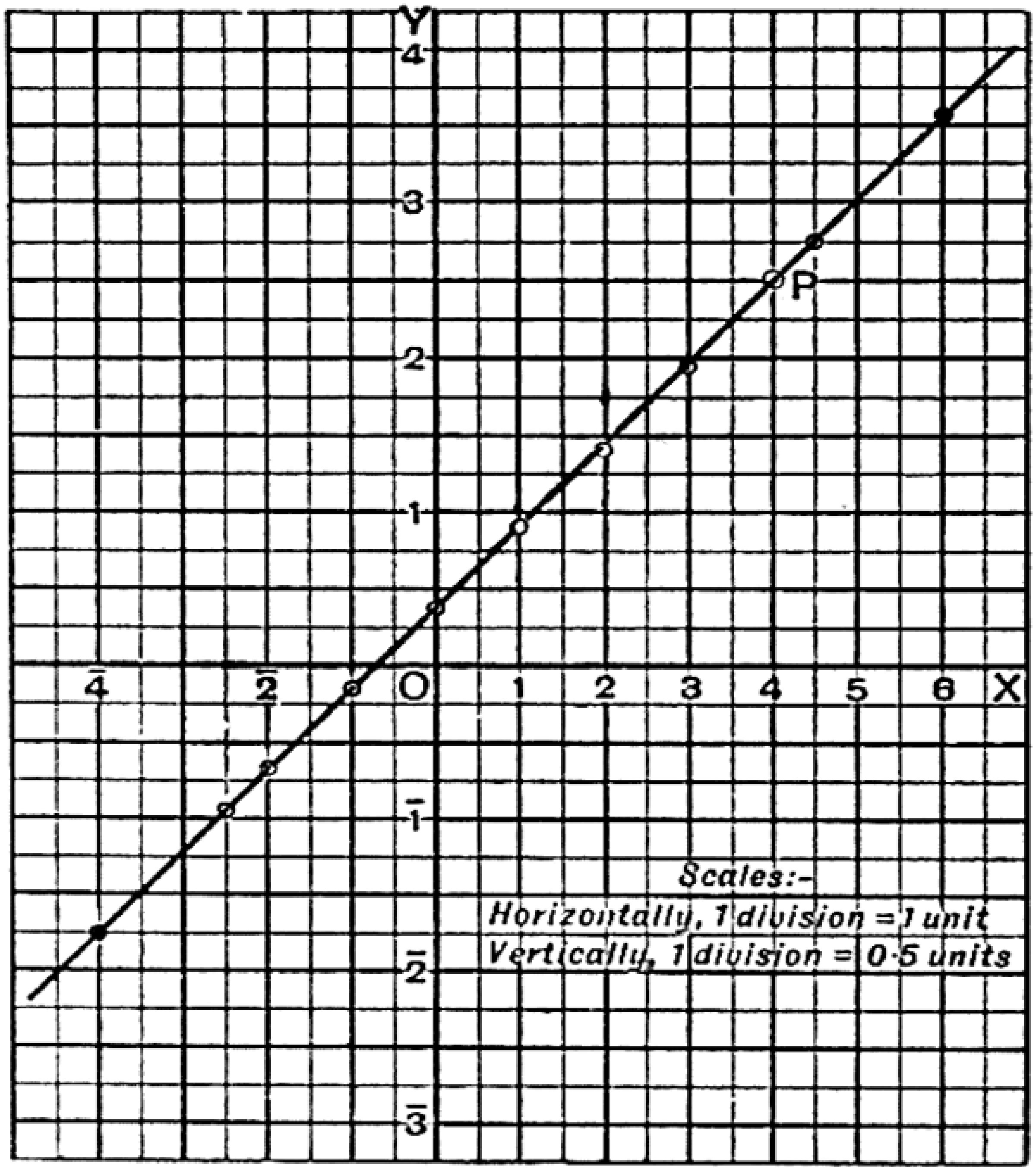

FIG. 21.—Graph of $y=0·37+0·53x$.

Fig. 21 shews the graph obtained by plotting these numbers. It will be noticed that in this figure the scale of y is double the scale for x, the actual graph on a natural scale having only half the slope shewn.

It will be seen that the graph cuts the axis of x almost exactly where x is $-0{\cdot}7$.

That is to say, $x = -0{\cdot}7$ is an approximate solution of the equation $0{\cdot}37 + 0{\cdot}53x = 0$.

Note further that if we want to solve the equation

$$2{\cdot}5 = 0{\cdot}37 + 0{\cdot}53x$$

we look for the value of x which makes $y = 2{\cdot}5$. This is at P, where very approximately $x = 4$, and this is an approximate solution of the given equation.

In the same way, the solution of the equation

$$0{\cdot}37 + 0{\cdot}53x = 3$$

is approximately $x = 5$.

The Equation of a Straight Line. It will be found that the **graph connecting y, the value of any expression of the form $a + bx$, with the value of x, is always a straight line.** Hence an expression of this form is called a **linear expression.** Alternatively, the graph may be said to represent the equation

$$y = a + bx.$$

Conversely, any straight line whatever may be represented algebraically by an equation of this form.

With this assumption, the graph of such an expression may be plotted rapidly by evaluating y for **any two** values of x, plotting the points whose coordinates are thus determined, and joining these points by a straight line. This will save time and labour in succeeding examples.

Of course it is unnecessary to use the letters a and b. Any linear equation connecting x and y can be put into the form

$$y = \textbf{a constant} + \textbf{(another constant)} \,.\, x,$$

and it is important to discover the geometrical meaning of the two constants.

The Significance of the Constants a and b. A line satisfies **two** conditions when considered geometrically: it can be made to pass through two points, or to pass through one point and have a given slope. In the expression $a + bx$ there are **two** undetermined constants, the coefficients a and b. Since $y = a + bx$ always represents a straight line, it may be made to represent **any** straight line by suitably choosing the values of a and b.

Ex. 1. *Determine the geometrical significance of the constants a and b in the equation* $y = a + bx$.

I. Take a series of equations in which the b's are always the same and the a's are different, and draw the graphs.

Fig. 22 shews the graphs of the following straight lines :—

(i) $y = 0\cdot2 + 0\cdot35x$. (ii) $y = 2\cdot6 + 0\cdot35x$.
(iii) $y = 1\cdot7 + 0\cdot35x$. (iv) $y = 0\cdot5 + 0\cdot35x$.
(v) $y = 0 + 0\cdot35x$. (vi) $y = -0\cdot5 + 0\cdot35x$.
(vii) $y = -1\cdot2 + 0\cdot35x$.

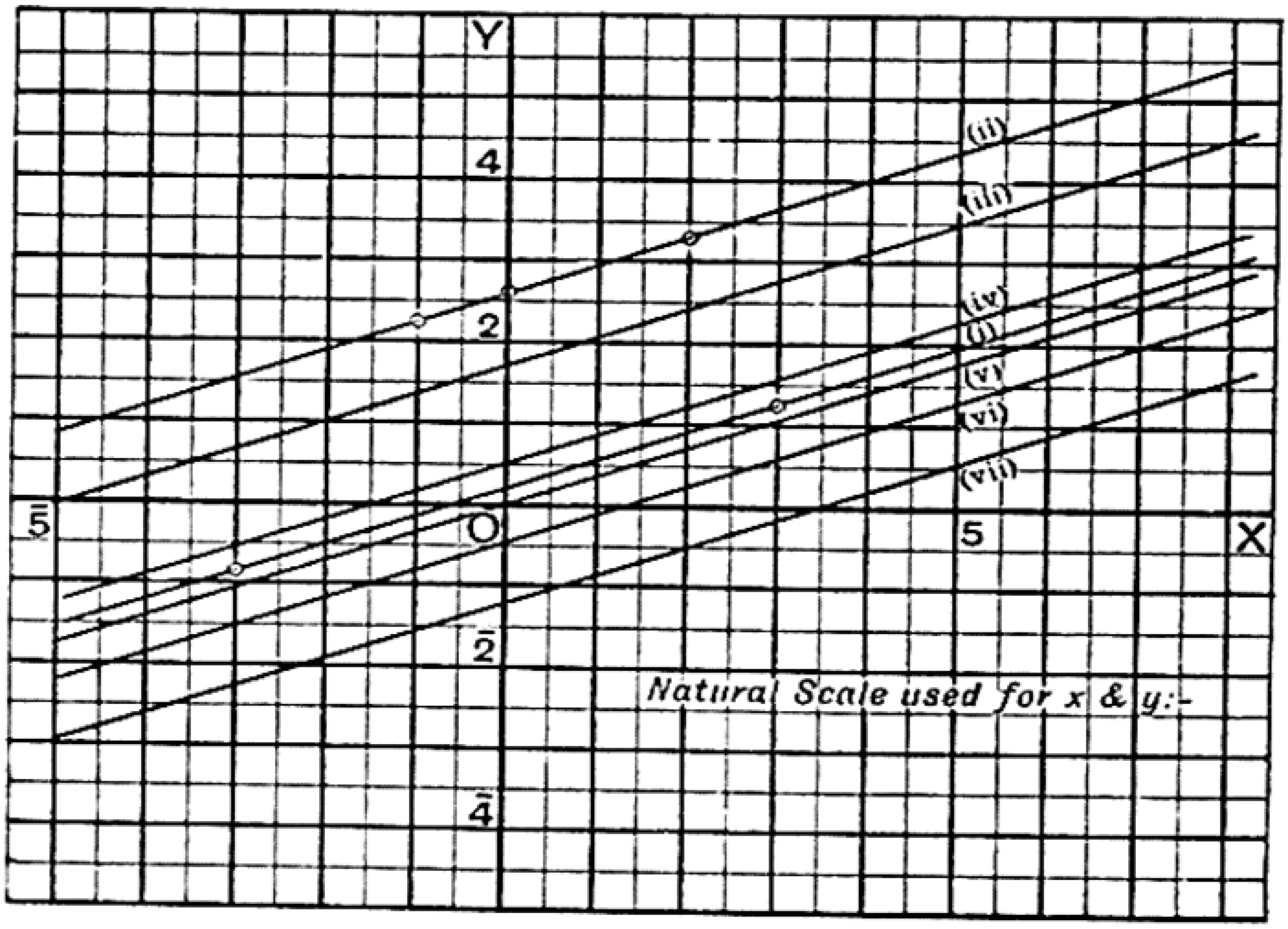

FIG. 22.

For line (i), let $x = -3$, then $y = 0\cdot2 - 1\cdot05 = -0\cdot85$;
,, $x = 3$, ,, $y = 0\cdot2 + 1\cdot05 = 1\cdot25$.

The points $-3, -0\cdot85$; $3, 1\cdot25$, fix the first line.

For line (ii), let $x = -1$, then $y = 2\cdot6 - 0\cdot35 = 2\cdot25$;
,, $x = 2$, ,, $y = 2\cdot6 + 0\cdot7 = 3\cdot3$.

The points $-1, 2\cdot25$; $2, 3\cdot3$, fix the second line.

Points on the other lines are found in a similar way and plotted.

The lines have **the same slope and different positions.**

If $a = 0$, the line passes through the origin; if a is positive, the line cuts OY above the origin; if a is negative, the line cuts OY below the origin.

II. Take a series of equations in which the a's are always the same and the b's different, and draw the graphs.

Fig. 23 shews the graphs of the following straight lines :—

(i) $y=1+3{\cdot}0x$. (ii) $y=1+2{\cdot}4x$.
(iii) $y=1+1{\cdot}2x$. (iv) $y=1+0{\cdot}6x$.
(v) $y=1$. (vi) $y=1-0{\cdot}5x$.
(vii) $y=1-x$. (viii) $y=1-2x$.

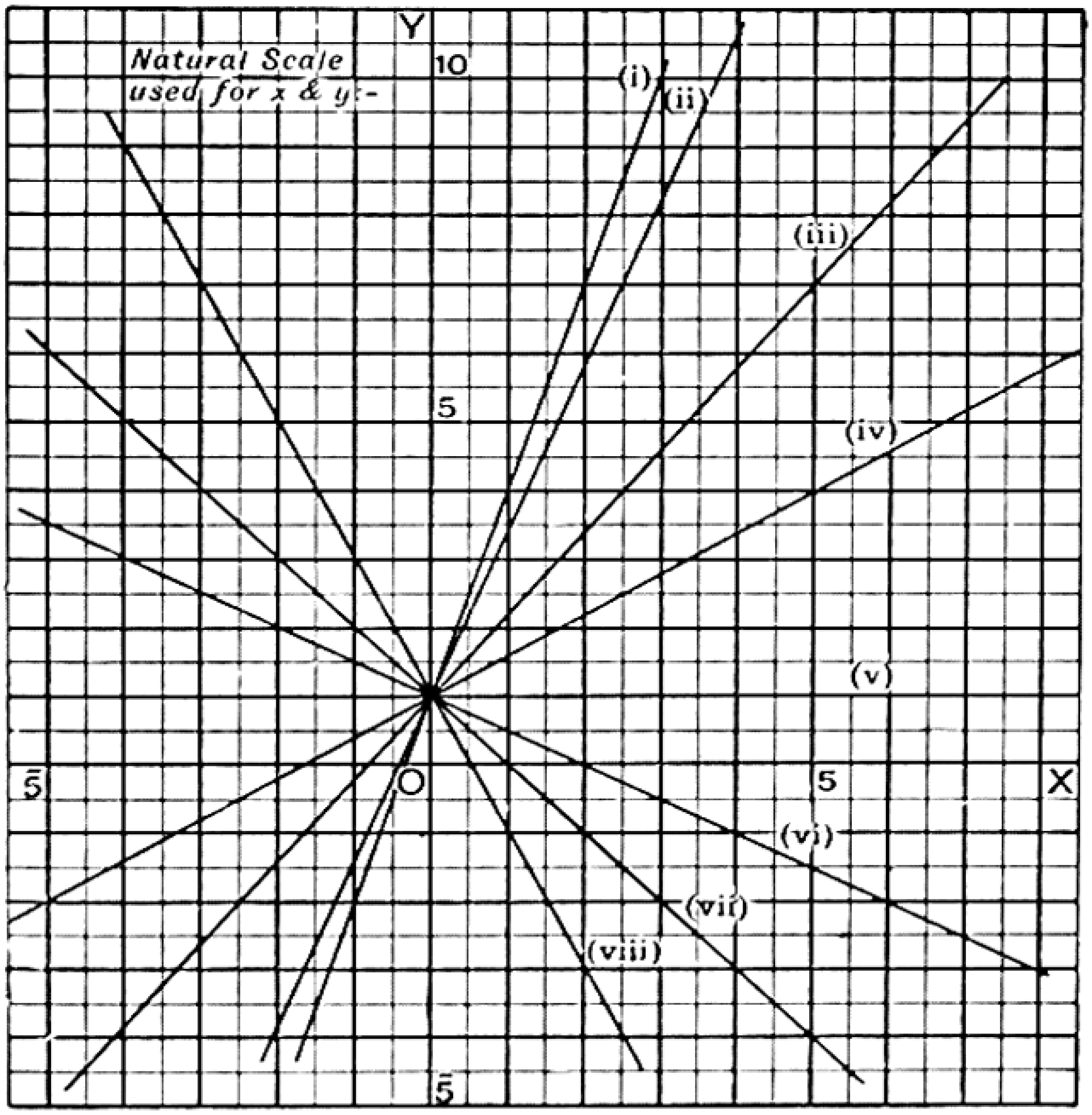

FIG. 23.

The lines have been found in the same way as before. Note **that the resulting lines have one point common and different slopes.** If b is 0, the line is parallel to the axis of x ; if b is positive, the line has positive slope ; if b is negative, the line has negative slope.

Hence, it may be inferred that a is connected with the **position** of the line $y=a+bx$, while b is connected with its **inclination or slope.**

In the equation $y=a+bx$, if x takes the value 0, $a+bx$ becomes $a+b\,.\,0$, *i.e.* a. That is to say, $y=a$ if $x=0$. But x is only 0 along the vertical axis. **Hence a is the value of y where the graph cuts the axis OY.** Thus, in Fig. 24, a is the intercept OR on the vertical axis. Obviously if $a=0$ the graph passes through the origin, as the intercept is zero.

Further, the intercept is positive or negative, *i.e.* the graph cuts OY above or below OX, as a is positive or negative.

Again, since $y=a+bx$, we have, subtracting a from each side,

$$y-a=bx.$$

Dividing by x, we have

$$b=(y-a)/x.$$

In Fig. 24, $ON=x$ and $PN=y$, where P is any point on the graph. By what has gone before, $OR=a$.

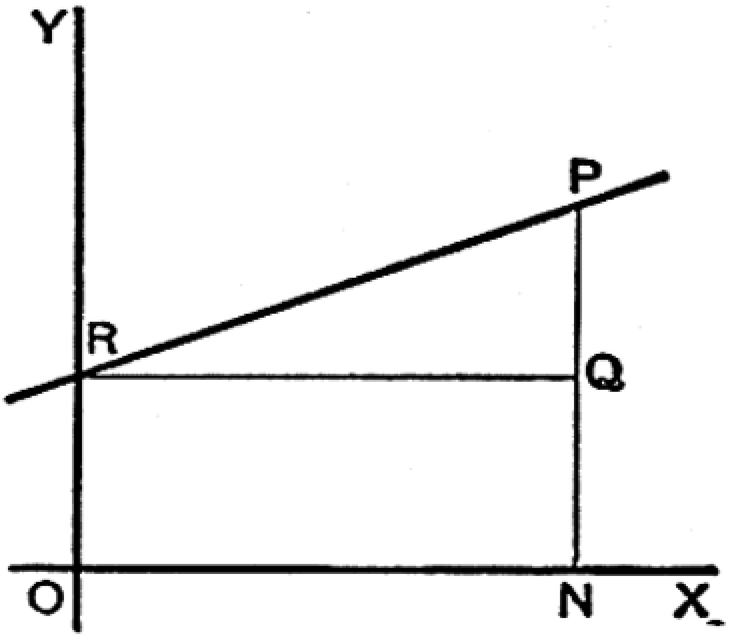

FIG. 24.

Therefore $\dfrac{y-a}{x}$ is $\dfrac{PN-OR}{ON}$ or $\dfrac{PN-QN}{RQ}$, that is $\dfrac{PQ}{RQ}$.

Hence b is PQ/RQ.

And on p. 36 it was shewn that the fraction PQ/RQ measured the rise in unit horizontal distance or slope of the line RP.

Hence b in the equation $y=a+bx$ **denotes the slope of the graph** representing the equation.

Thus, the equation of any linear graph is $y=a+bx$, where a is the intercept on OY and b is the slope.

The most convenient way of finding b is to read off the coordinates of any two points A, B and calculate b by evaluating $\Delta y/\Delta x$.

Attention must be paid to the scales of x and y in reading the value of a and in evaluating $\Delta y/\Delta x$.

Ex. 2. *Write down the equations of the graphs* (i) *and* (ii) *shewn in* Fig. 25.

In line (i) the intercept on the axis of Y is OP. $\therefore\ a=6{\cdot}0$. The coordinates of P are 0, 6·0; the coordinates of Q are 4·8, −4·0.

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{-4{\cdot}0-6{\cdot}0}{4{\cdot}8}=\frac{-10}{4{\cdot}8}\doteqdot -2{\cdot}08\,;\quad \therefore\ b\doteqdot -2{\cdot}08.$$

The required equation is $y=6{\cdot}0-2{\cdot}08x$.

As a check, find y where x is 3. We have

$$y = 6{\cdot}0 - 6{\cdot}24 = -0{\cdot}24,$$

and at the point R, x is 3 and y is $-0{\cdot}24$.

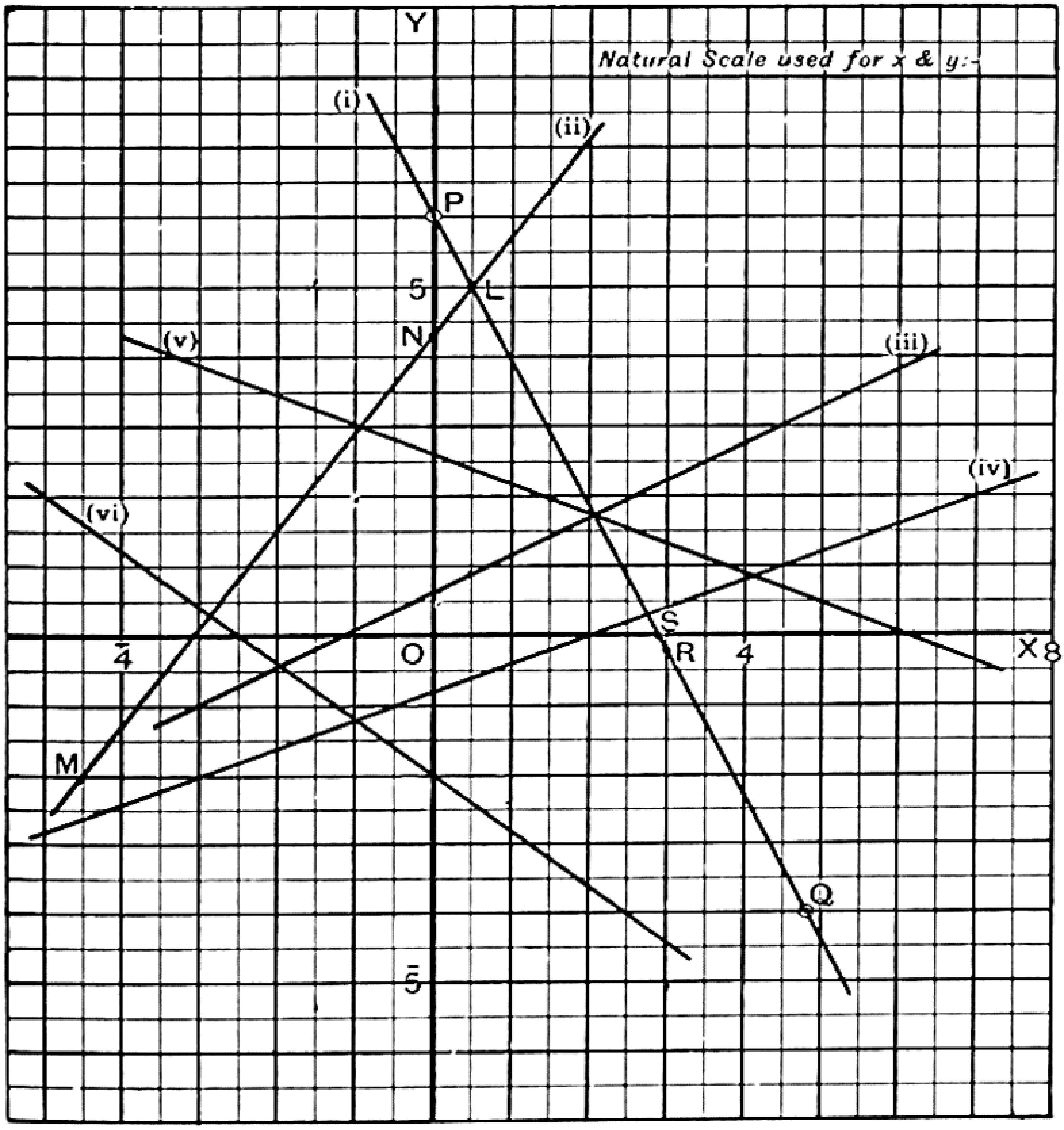

FIG. 25.

In line (ii) the intercept on the vertical axis is $4{\cdot}3$; $\therefore\ a = 4{\cdot}3$. The coordinates of M are $-4{\cdot}5,\ -2$; the coordinates of N are $0,\ 4{\cdot}3$.

$$\therefore\ b = \frac{\Delta y}{\Delta x} = \frac{4{\cdot}3 - (-2)}{0 - (-4{\cdot}5)} = \frac{6{\cdot}3}{4{\cdot}5} = 1{\cdot}40.$$

$\therefore$ the equation of line (ii) is $y = 4{\cdot}3 + 1{\cdot}40x$.

Ex. 3. *Solve the equation* $6 - 2{\cdot}08x = 0$.

The graph of $y = 6 - 2{\cdot}08x$ is line (i) (Fig. 25).

Line (i) cuts the axis of x at S, and at S, $x = 2{\cdot}9$ very approximately. Also at S on this line $6 - 2{\cdot}08x = 0$, since $y = 0$.

Hence $x = 2{\cdot}9$ is the approximate solution of the equation

$$6 - 2{\cdot}08x = 0.$$

Ex. 4. *Solve the equations* $y = 6 - 2{\cdot}08x$, $y = 4{\cdot}3 + 1{\cdot}40x$.

Lines (i) and (ii) (Fig. 25) represent these equations.

The point L is common to both lines. That is to say, at L the same values of x and y satisfy the two equations. But at L, $x = 0{\cdot}5$, $y = 5{\cdot}0$. Therefore $x = 0{\cdot}5$, $y = 5{\cdot}0$ are approximate solutions of the simultaneous equations.

EXERCISES IV.

1. Plot graphs of the following expressions or **functions** of x:

(i) $1{\cdot}5 + 2{\cdot}3x$. (ii) $-1{\cdot}35 + 2{\cdot}7x$.
(iii) $0{\cdot}86 - 0{\cdot}55x$. (iv) $3{\cdot}1 - 2{\cdot}5x$.
(v) $2{\cdot}7 + 0{\cdot}53x$. (vi) $-2{\cdot}1 - 0{\cdot}23x$.

2. Write down the equation of each of the lines shewn in Fig. 25.

3. Draw the graphs of $2x + 3$ and $16 - (2x - 3)$. What is the x-coordinate of the point common to these graphs?

4. What value of x makes the expressions $1{\cdot}3x - 2$ and $4{\cdot}3 - 2{\cdot}1x$ have the same value?

5. Solve the following equations graphically:

(i) $\frac{5}{4}x = \frac{3}{2}$. (ii) $\dfrac{x-1}{4} + \dfrac{x+3}{5} = 8$.
(iii) $5(x - 11) - 3(x + 4) = 1$. (iv) $4{\cdot}37 + 1{\cdot}59x = 0$.

6. Solve the following equations graphically:

(i) $\left.\begin{aligned} y &= 4 - 2x, \\ y &= 0{\cdot}5 - 1{\cdot}5x. \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 5x + 3y &= 21, \\ 3x - 7y &= -5. \end{aligned}\right\}$
(iii) $\left.\begin{aligned} y &= 4{\cdot}3 - 2{\cdot}1x, \\ y &= -0{\cdot}5 - x. \end{aligned}\right\}$ (iv) $\left.\begin{aligned} 3y &= 2{\cdot}8x + 5{\cdot}6, \\ 5y &= 19 - 2{\cdot}7x. \end{aligned}\right\}$

7. Solve
$$\left.\begin{aligned}\frac{x}{3}+\frac{y}{4}+2=0,\\ y=3x+10,\end{aligned}\right\}$$
and verify your result by drawing the graphs of the equations on squared paper.

8. Find the equations of the diagonals of the square whose corners are the points (0, 0), (a, 0), (a, a), (0, a).

9. Plot the points (1, 3) and (– 5, 7) ; draw the straight line which bisects their join at right angles, and find its equation.

10. Draw the graphs of the lines $3x-4y=12$ and $x+y=0$; draw the two bisectors of the angles between them, and find their equations.

11. Plot the points A, (3, 4) ; B, (6, 10) ; and C, (10, 2). Let P, Q, R, be the feet of the perpendiculars from A, B, C, respectively on OX. By using the formula for the area of a trapezium,

Area = ½(sum of parallel sides) × (distance between them),

calculate the area of each of the trapeziums $APQB$, $BQRC$, $APRC$, and deduce the area of the triangle ABC.

12. Using the method of Ex. 11, find the areas of the triangles whose vertices are the points :

(i) (5, 6) ; (7, 12) ; (13, 3).
(ii) (– 3, 4) ; (0, 9) ; (7, 1).
(iii) (– 1, – 2) ; (2, 6) ; (8, 0).

13. Find a formula for the area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

14. Calculate the area of a quadrilateral whose vertices are the points (0, 3), (3, 1), (10, 2) and (6, 10).

15. Shew that $\frac{1}{2}\{(x_1-x_3)(y_2-y_4)+(x_2-x_4)(y_3-y_1)\}$ represents the area of the quadrilateral whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4).

The Determination of Laws from Observation. A valuable application of the work just done is the determination of the numerical relation existing between two sets of numbers which have been determined experimentally, or in other ways, and are represented graphically by a straight line. In the first case, there would undoubtedly be experimental errors, so that the plotted points would not lie exactly on a straight line. All that is necessary in order that a determination of the relation may be possible, is that a straight line can be drawn so as to lie evenly among the points; that is to say, the points should not trend away from a straight line.

For example, the trend of the points in Fig. 26 is undoubtedly linear, although no single point lies on the straight line drawn. If the observed values of x are correct, the **probable error of observation** in y may be determined by seeing what change in y is required to bring the point on to the line.

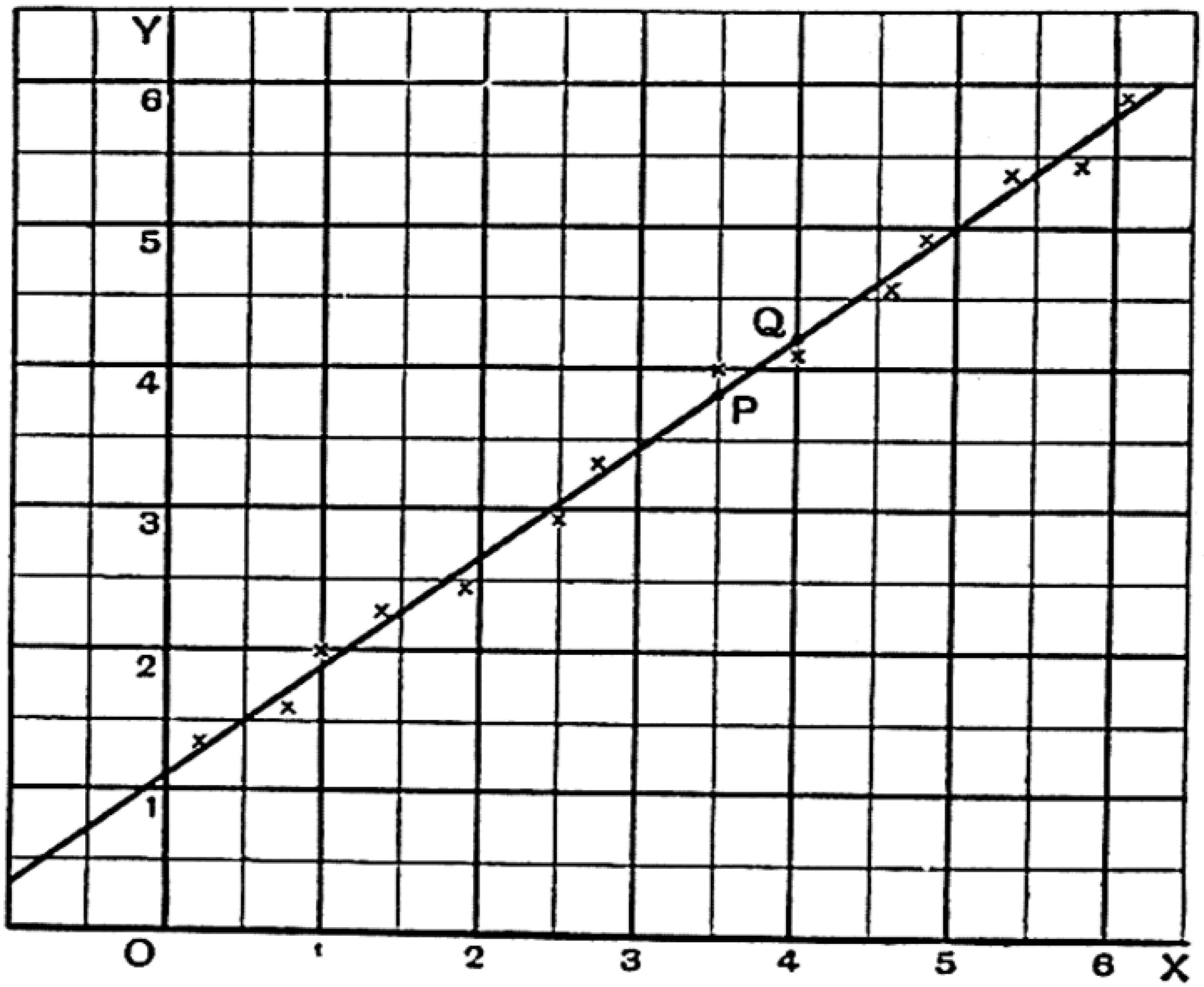

FIG. 26.

Thus, when $x=3{\cdot}5$, the observed value of y is $4{\cdot}0$, and the probable error is $+0{\cdot}2$; *i.e.* the value of y is too great by $0{\cdot}2$, as shewn at the point P. Again, when x is $4{\cdot}0$, the observed value of y was $4{\cdot}1$, and the probable error is $-0{\cdot}1$; *i.e.* the value of y is too small by $0{\cdot}1$, as shewn at the point Q.

If the graph represents experimental results, it will shew when and where large errors occur, and suggest reforms in the method of experimenting. But if the plotted points persist in deviating from a straight line, no matter how carefully observations are made, the assumption of a linear law is not justified.

For example, in Fig. 27, the *law* is probably best represented by the broken curve, although the first five points lie exactly on the straight line. Had the experiment ceased after the first seven observations, a linear law would probably have been inferred. Again, in Fig. 28, the law is probably *not* linear, although there are as many points above as below the line which has been drawn. (See Case V., p. 39.)

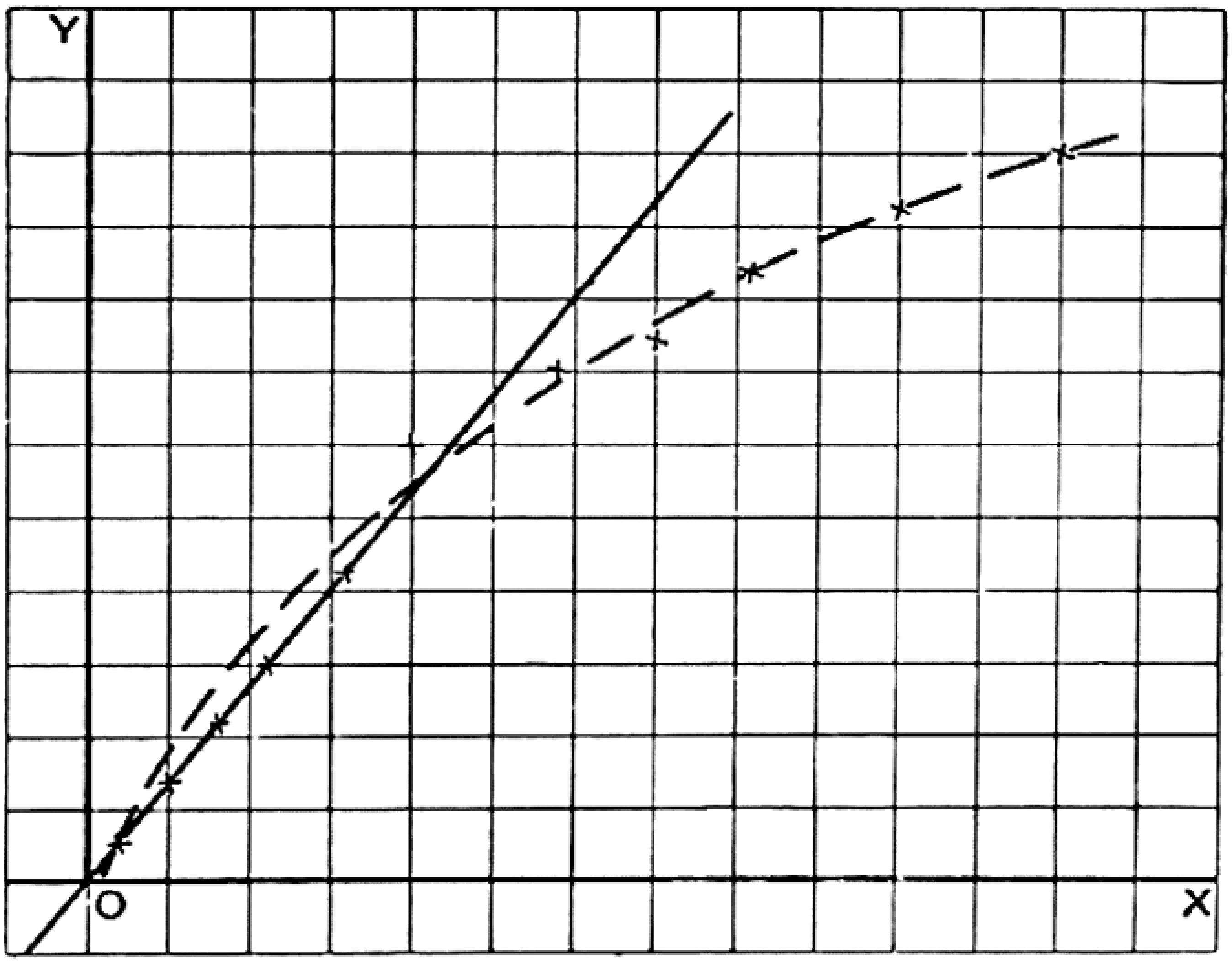

FIG. 27.

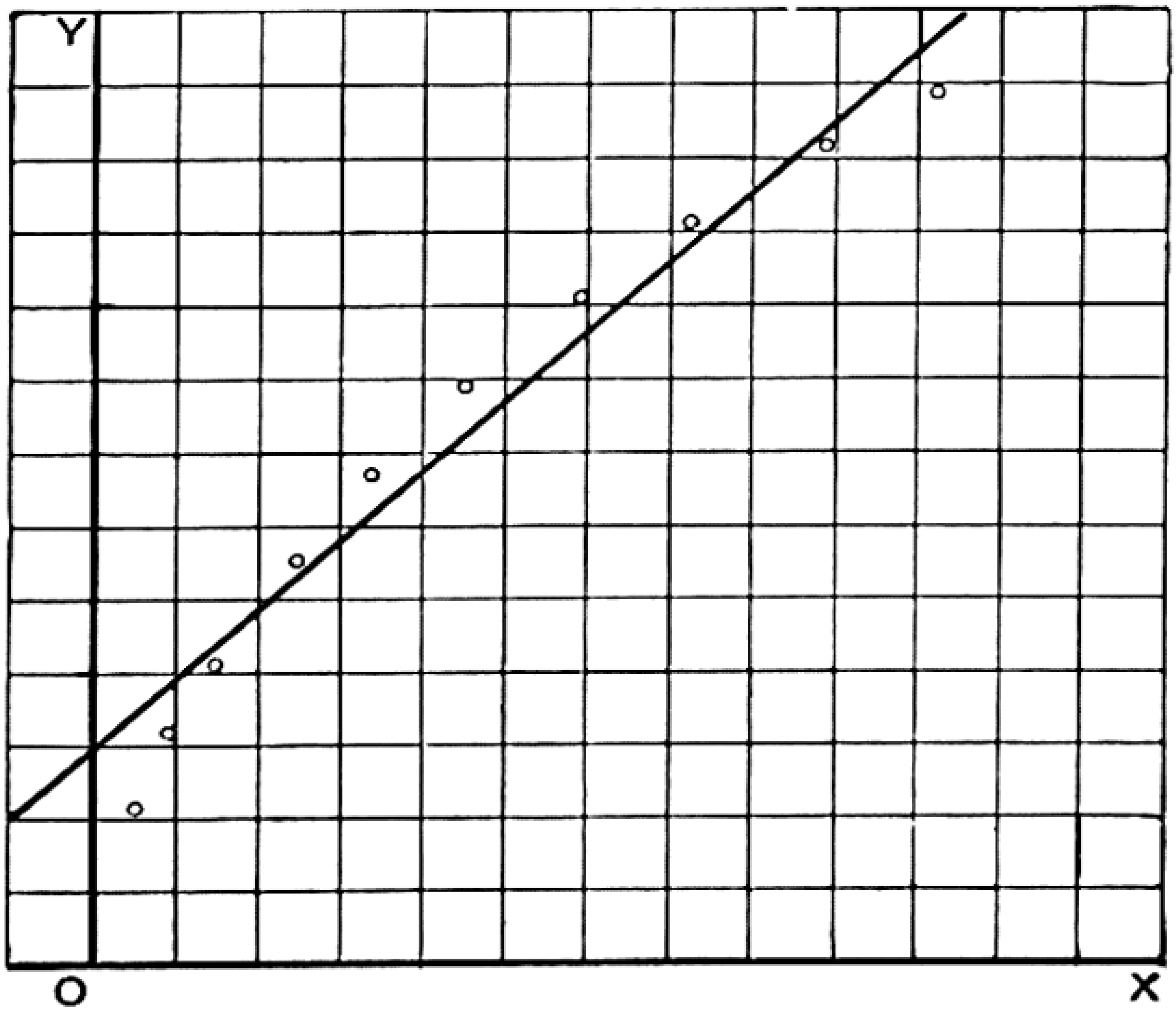

FIG. 28.

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The best way of determining if the observed values are connected by a linear relation, is to stretch a black thread over the plotted points and move it about until it passes through as many as possible; then see that the plotted points do not trend away from the thread. If the points are arranged about the thread irregularly with no great deviation from it in any instance, the law is probably linear. Two points through which the thread passes are now marked, and joined by a straight line. The equation of this line gives very approximately the 'law' or numerical relation connecting the given numbers.

Ex. 1. *In testing a large single cylinder gas engine, Prof. Perry obtained the following numbers for the Indicated and the Brake Horse-Power. What law seems to connect the* B.H.P. *with the* I.H.P.?

Brake Horse-Power, B - -	16	57	95	99	117
Indicated Horse-Power, I -	35	73	114	120	139

Note that we are to connect B with I.

It is convenient to plot values of I along the horizontal axis and values of B along the vertical axis; so that I will be used where we have hitherto used x and B where we have hitherto used y.

Choosing the scales indicated in Fig. 29, and plotting the points, the black thread gives the line shewn.

We cannot read off the intercept 'a' on the vertical axis without using a much larger sheet of squared paper, as this intercept is negative. The alternative method for determining a in such cases should be carefully noted.

First determine 'b.' Choosing the points P, Q, we see that ΔB (*i.e.* PR) is 107, while ΔI (*i.e.* QR) is 110;

$$\therefore \quad \frac{\Delta B}{\Delta I}, \ i.e.\ b = \frac{107}{110} \doteqdot 0{\cdot}97.$$

Now use the equation $B = a + bI$ to determine a as follows: $b = 0{\cdot}97$, and at the point P, B is 120, while I is 140.

Therefore $$120 = a + 0{\cdot}97 \times 140;$$

$$\therefore \quad a = 120 - 135{\cdot}8 = -15{\cdot}8.$$

Therefore the required law is $B = -15{\cdot}8 + 0{\cdot}97I$.

Its accuracy may be tested by substituting pairs of values for B and I from the given table.

From the table, when I is 73, B is 57.

From the equation, when I is 73, B is $-15{\cdot}8 + 0{\cdot}97 \times 73$,

$$i.e. \qquad B \doteqdot 54{\cdot}9.$$

Therefore, if the values of I are assumed to be correct, the percentage error in the observed value of B is $+3{\cdot}8\,\%$.

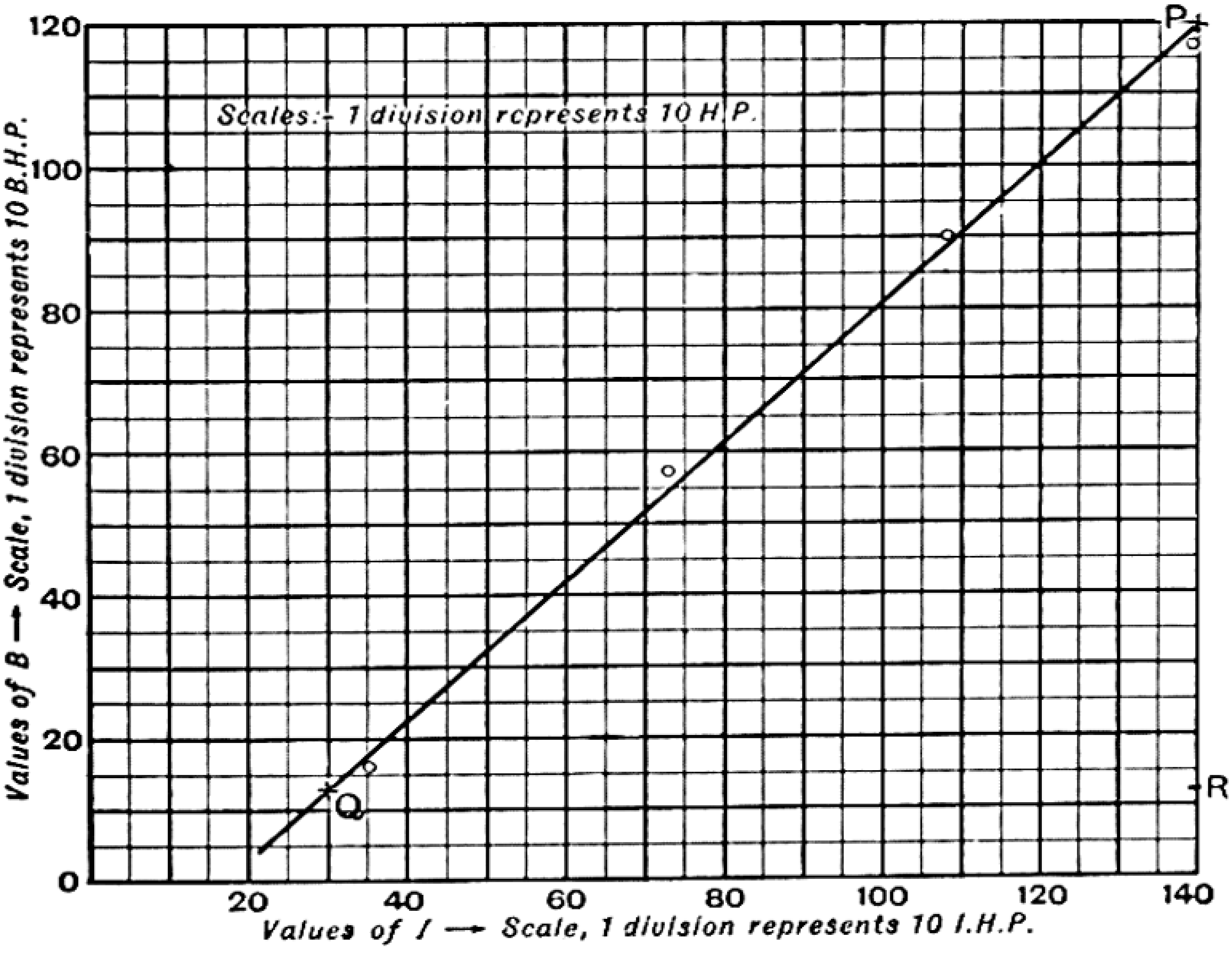


FIG. 29.—Graph connecting B.H.P. and I.H.P.

Again, from the table, when I is 35, B is 16.

From the equation, when I is 35, B is $-15{\cdot}8 + 0{\cdot}97 \times 35$,

$$i.e. \qquad B \doteqdot 18{\cdot}2,$$

and the percentage error in the observed value of B is

$$-1{\cdot}2\,\%.$$

The graph shews whether the errors are positive or negative.

Ex. 2. *The following tests were made on a steam-electric generator; W is the weight of steam in pounds used per hour; K is the output in kilowatts:*

K	3942	3105	1907	910
W	80400	68100	50200	35100

Find if there is a simple approximate law connecting W with K. State the meaning of W/K in words; call it w. Express w in terms of K. [B.E.]

Plotting values of K horizontally (since we have to connect W with K) for a range 800 to 4000, values of W vertically for a range 30000 to 80000, and using the scales shewn in the figure, we have points as shewn (Fig. 30). The straight line was obtained with a black thread as previously explained, and it will be seen that the required law is of the form

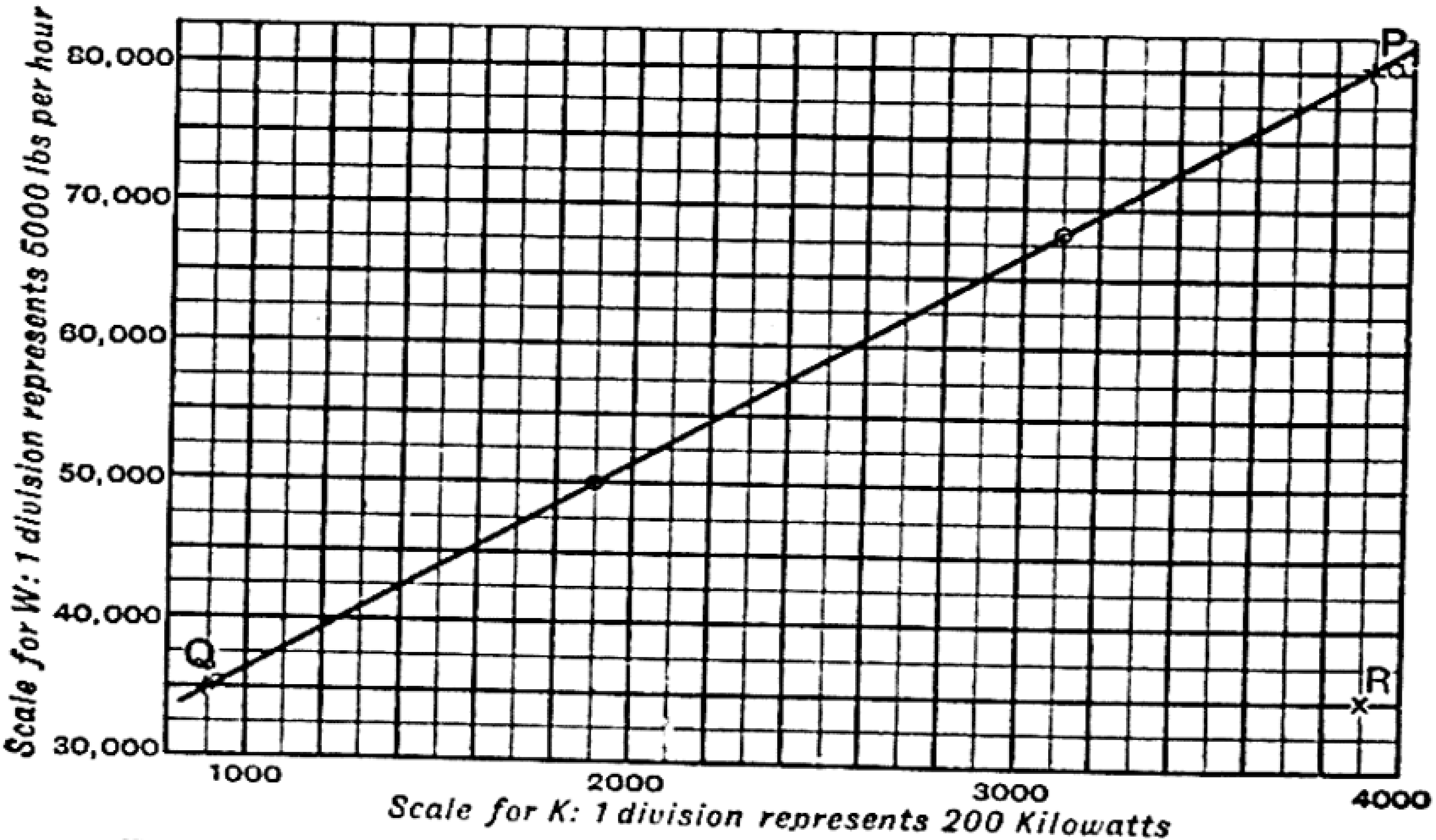


FIG. 30.—Graph connecting steam used per hour with electric output.

$$W = a + bK.$$

a cannot be read directly from the graph, since the real axes are not shewn. But the real axes are parallel to the lines shewing the chosen scales in the figure, and consequently b can be found just as if these lines were the real axes.

For b the points P, Q were selected.

$$\Delta y = PR \doteqdot 45000,$$

$$\Delta x = QR \doteqdot 3000;$$

$$b = \frac{\Delta y}{\Delta x} \doteqdot \frac{45000}{3000} \text{ cr } 15.$$

Now, at the point P, $W = 80000$, $K = 3900$; using the equation $W = a + bK$, and substituting 15 for b, we have

$$80000 \doteqdot a + 15 \times 3900;$$

$$\therefore\ a \doteqdot 80000 - 58500 \text{ or } 21500.$$

The approximate law is consequently

$$W = 21500 + 15K.$$

Testing this by evaluating W, where K is 1907,

$$W \doteqdot 21500 + 28600$$

$$\doteqdot 50100,$$

which agrees fairly well with the experimental number, the error being approximately 0·2 %.

$\frac{W}{K}$ is of course $\frac{\text{steam used in lb. per hour}}{\text{output in kilowatts}}$, and is therefore the steam used *per kilowatt* in lb. per hour, or pounds of steam per kilowatt-hour.

We have now to obtain an equation connecting w and K, where $w = \frac{W}{K}$.

Dividing the equation $W = 21500 + 15K$ by K,

$$\frac{W}{K} = \frac{21500}{K} + 15.$$

Substituting w for $\frac{W}{K}$, $w = \frac{21500}{K} + 15$,

or, in words, the number of pounds of steam used per kilowatt-hour is obtained approximately by adding to 15 the quotient of 21500 divided by the output in kilowatts.

A graph connecting w and K should be plotted and compared with that shewn in Fig. 30.

Ex. 3. *The following values of x and y were observed, and it was important to see if any simple algebraic law connected them :*

x	0·5	1·7	3·0	4·7	5·7	7·1	8·7	9·9	10·6	11·8
y	148	186	265	326	388	436	529	562	611	652

Find the law and give the probable error in y when x is 5·7.

The scales shewn in Fig. 31 were chosen and a black thread used to obtain the best line.

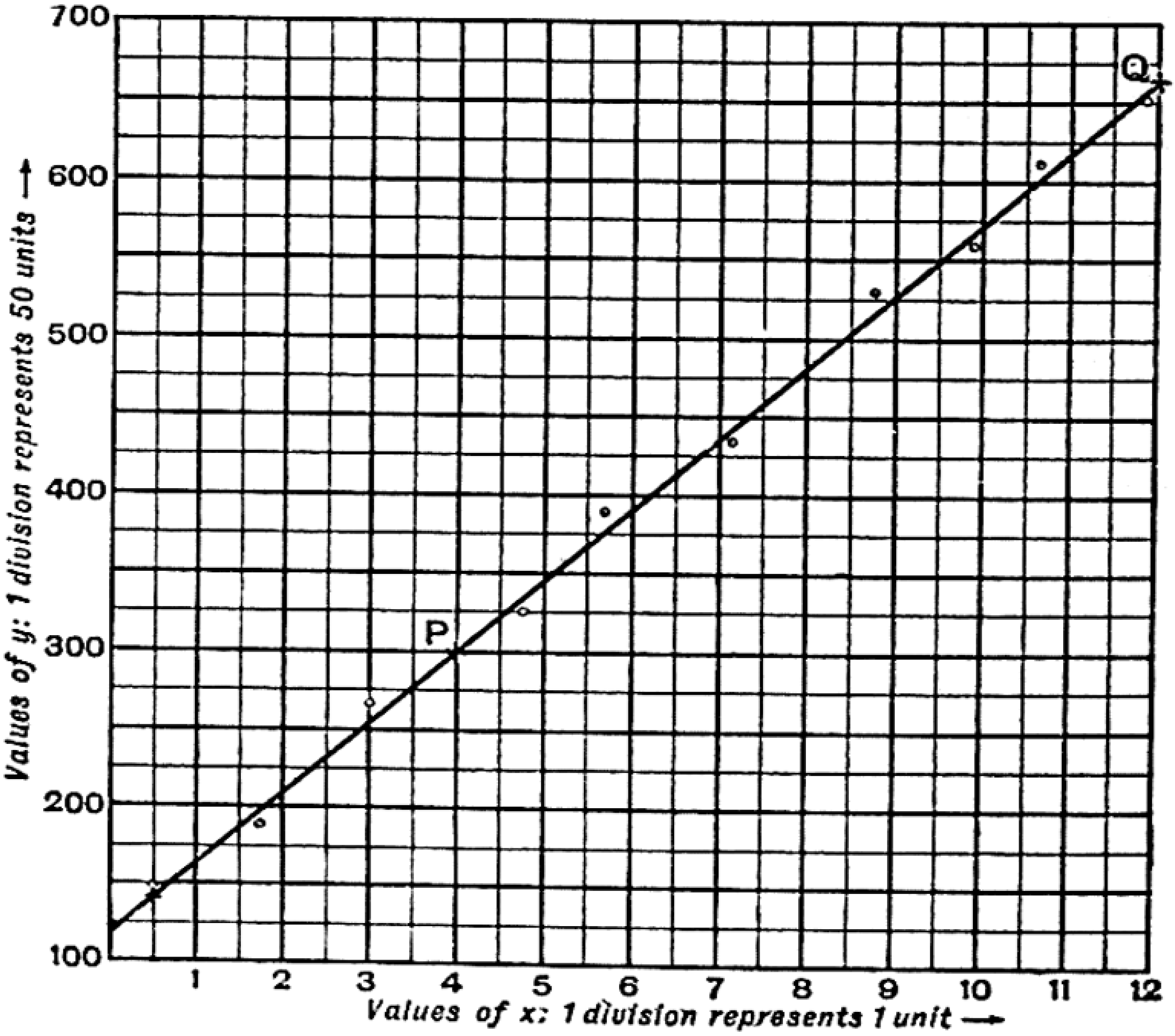


FIG. 31.

At P, $x=4$, $y=300$; at Q, $x=12$, $y=665$;

$\therefore$ $\Delta x=8$, $\Delta y=365$ and b, *i.e.* $\Delta y/\Delta x \doteqdot 45{\cdot}6$.

The line cuts $x=0$ where $y \doteqdot 117$, $\therefore$ $a \doteqdot 117$.

The required law is $y=117+45{\cdot}6x$.

Where x is 5·7 the value of y is 378 from the graph, and the probable error in the observed value is 10.

Ex. 4. *A part of the graph of* $1+\sqrt{1+x^2}$ *shewn in* Fig. 4, p. 29, *is approximately linear. What law represents it?*

From the graph, it is clear that for values of x between 2 and 10 the equation of the line LM may be used instead of

$$y=1+\sqrt{1+x^2}.$$

For LM, $\Delta y=11-3{\cdot}2=7{\cdot}8$ and $\Delta x=8$;

$$\therefore\ b,\ i.e.\ \Delta y/\Delta x \doteqdot 0{\cdot}98.$$

The line cuts the axis of Y where $y=1{\cdot}23$; $\therefore\ a=1{\cdot}23$.

Hence, for the given range of values of x, we have

$$y=1{\cdot}23+0{\cdot}98x.$$

For many purposes this linear expression for y would be more convenient to use than the expression $1+\sqrt{1+x^2}$.

Ex. 5. *The freezing and boiling points of water are marked 0°, 100° respectively on the Centigrade thermometer, and 32°, 212° respectively on the Fahrenheit thermometer. What equation may be used to convert Fahrenheit readings to Centigrade?*

In Fig. 32 Centigrade temperatures are measured horizontally; Fahrenheit temperatures vertically.

The points A, representing 32° F. = 0° C., and B, representing 212° F. = 100° C., were first plotted.

The straight line AB represents the connection between Fahrenheit and Centigrade temperatures.

Assuming that

$$F=a+bC,\quad a=32,$$

$$b=\frac{\Delta F}{\Delta C}=\frac{212-32}{100-0}=1{\cdot}8;$$

$$\therefore\ F=32+1{\cdot}8C.$$

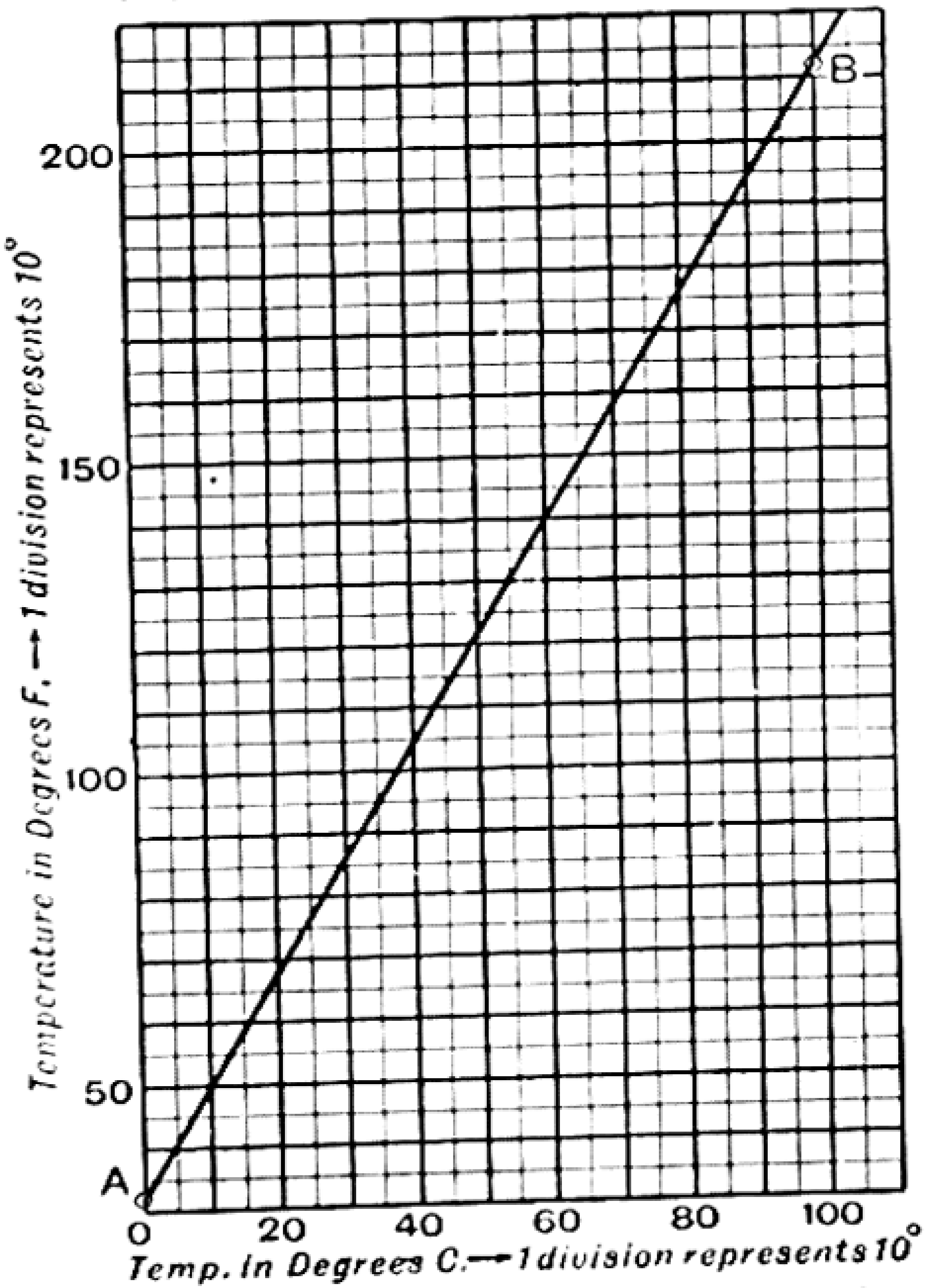


FIG. 32.—Conversion of thermometric scales.

This is frequently expressed in the form $\frac{F-32}{9}=\frac{C}{5}$.

EXERCISES V.

1. A certain machine has been made in several sizes, the largest selling at £135 and the smallest at £24. It has been found that the smaller sizes sell best, so a revised scale of prices is made, according to which the largest size of machine would sell at £100 and the smallest size at £30. Assume a linear connection between the two sets of prices and deduce an equation giving N, the new price of any machine in pounds, in terms of O, its original price. What is the new price of machines costing originally £30, £55 and £100?

2. The cost of making a certain machine consists of an initial expense plus a fixed sum for each machine. To make 25 machines costs £885, to make 40 machines costs £1335. Find the initial expense and the cost per machine.

3. If W = weight of feed water in lb. per hr. and P = power in kilowatts; if $P=50$ when $W=3800$ and if $P=100$ when $W=5100$, assume a linear relation between W and P, and find it. What is W when $P=70$? [B.E.]

4. An examiner has given marks to papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law, converting the highest number of marks into 250 and the lowest into 100; shew how he may do this, and state the converted marks for papers already marked 60, 100, 150. Use squared paper, or mere algebra, as you please. [B.E.]

5. A factory produces per week L yards of material at a total cost of C pounds.

L	7200	5500	4240	2000
C	190	163	143	107

Plot L and C. Find the cost *per yard* when $L=6000$.
If C and L are connected by an equation

$$C=aL+b,$$

find the most probable values of a and b. [M.C.U.]

6. In an experiment with a screw jack the following results were obtained:

Load W lifted in lb. -	100	120	140	160	180	200
Power P applied in lb. -	12·6	13·8	15·7	17·6	19·6	21·5

Plot these observations, and find the values of a and b which satisfy the equation

$$W = a + bP.$$

[M.C.U.]

7. The following numbers are thought to follow a law like

$$y = a + bx.$$

Test if this is so, and find the most probable values of a and b.

x	2·30	3·10	4·00	4·92	5·91	7·20
y	25·0	39·1	50·3	67·2	85·6	105·0

[B.E.]

8. The following observed values of x and y are thought possibly to fulfil the law

$$y = a + bx.$$

Try if this is so. Find the most probable values of a and b. What is the probable error of each value of y?

x	10	20	30	40	50	60	70	80
y	180	303	492	603	807	919	1131	1200

[B.E.]

9. A steam engine when its load is P (horse-power) uses W lb. of steam per hour. The following measurements were made:

P	124·6	62	19·8
W	2750	1490	640

Is there any apparent law connecting W and P? If so, what is it? [B.E.]

10. When a weaving factory is producing Y yd. of material per week, C is the total cost in pounds of the factory per week, including wages, interest on capital, rates, etc. What is the probable law connecting C and Y?

Y	8100	5900	4050	1980
C	198	167	134	98

What is the cost per yard when $Y = 5000$? [B.E.]

11. The following corresponding values of two numbers were measured in a laboratory, and there are errors of observation. Plot on squared paper. If there is probably a law like $y=a+bx$, find the best values of a and b.

x	100	200	300	400	500	600	700	800
y	17·3	25·0	34·3	42·8	51·2	59·9	68·0	77·3

[B.E.]

12. Values of x and y are given. Find the simple linear law connecting them and the probable error in y when $x=8{\cdot}7$.

x	1·7	3·0	4·7	5·7	7·1	8·7	9·9	10·6
y	182	261	323	383	433	525	560	606

13. The following numbers were obtained in an experiment:

K	4·30	5·38	6·63	8·64	9·26	12·05	14·84	18·25
L	1	1·7	2·5	3·8	4·2	6·0	7·8	10·0

Find the probable law connecting K with L, and the probable values of K when $L=3$, 5 and 8.

14. Plot $\qquad 3y=4{\cdot}8x+0{\cdot}9.$

Plot $\qquad y=2{\cdot}24-0{\cdot}7x.$

Find the point where they cross. What angle does each of them make with the axis of x? At what angle do they meet? [B.E.]

15. At an electricity works, where new plant has been judiciously added, if W is the annual works cost in millions of pence, and T is the annual total cost, and U the number of millions of electrical units sold, the following results have been found:

U	W	T
0·3	0·47	0·78
1·2	1·03	1·64
2·3	1·70	2·73
3·4	2·32	3·77

Find approximately the laws connecting T and W with U. Also find the probable values of W and T when U becomes 5, if there is the same judicious management. [B.E.]

16. Within what limits may we take $2{\cdot}4+\sqrt{x^2+5{\cdot}76}$ as nearly equivalent to $2{\cdot}95+0{\cdot}98x$?

17. The time taken by a star to move from South to West depends upon an angle called its declination. Corresponding values of these times and angles are given below. Plot the angles vertically at 10° to 1 inch against the times horizontally at 40 minutes to 1 inch; and from your graph find (1) the time taken by Aldebaran, whose declination is 16°, to move from S. to W., and (2) the declination of a star which performs this movement in 2 hours 48 minutes.

Declination in degrees - -	0	10	19	28	34	40	46	48	50
Time in minutes - - -	360	328	296	260	232	192	140	112	76

[I.C.E.]

18. Corresponding values of two related numbers, m and k, are given below. Calculate, to one decimal place, the values of $(mk)^2$ for each pair of corresponding values, and plot the values of $(mk)^2$ vertically against those of m horizontally, taking 1 inch to represent unity for m, and $\frac{1}{2}$ inch to represent unity for $(mk)^2$.

m	0·5	1	1·5	2	3	4	5	6	7
k	1·91	1·39	1·15	1	0·82	0·71	0·637	0·58	0·54

Shew from your graph that the relation between m and k is of the form $m^2k^2=am+b$ where a and b are constants; and find the values of a and b. [I.C.E.]

19. Plot the points given in the annexed table of values:

x	2	4·9	7	10·1	16	18	20
y	2·1	2·21	2·32	2·45	2·62	2·72	2·8

Draw the straight line lying most evenly among the points, and find its equation.

What would probably be the value of y when $x=21{\cdot}5$ and when $x=0$? [L.U.]

20. The following corresponding values of x and y were measured. There may be errors of observation. Test if there is a probable law

$$y = a + bx^2,$$

and, if this is the case, what are the probable values of a and b?

x	1·00	1·50	2·00	2·30	2·50	2·70	2·80
y	0·77	1·05	1·50	1·77	2·03	2·25	2·42

(B.E.)

21. It is thought that a certain set of observations of corresponding values of x and y are connected by a relation of the form $y = a/(1 + bx)$. If this is so, show that x and the reciprocal of y are connected by a linear relation, *i.e.* one of the form

$$1/y = mx + c.$$

Corresponding observations of x and y are given in the table:

x	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0
y	1·36	1·25	1·15	1·07	1·00	0·938	0·882	0·832	0·791	0·750

There are small errors in the observed values of y.

Shew that x and y are connected by a relation of the form $y = a/(1 + bx)$, and find the probable values of a and b.

CHAPTER III.

THE USE OF SQUARED PAPER (*continued*).

THE CIRCLE. GRAPHICAL SOLUTION OF EQUATIONS. GRAPHS OF FUNCTIONS. MAXIMUM AND MINIMUM VALUES OF QUADRATIC EXPRESSIONS.

WE have seen that any equation of the first degree in two variables represents a straight line. The circle is a simple geometrical figure, and it is well to be able to recognise its equation.

Distance between the two points (x_1, y_1) and (x_2, y_2). Let the coordinates of the point A be (x_1, y_1), of B (x_2, y_2) (Fig. 33).

Draw AC and BD perpendicular to OX and BN perpendicular to AC.

Clearly $BN = x_1 - x_2$ and $AN = y_1 - y_2$.

Using Pythagoras' Theorem, $AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

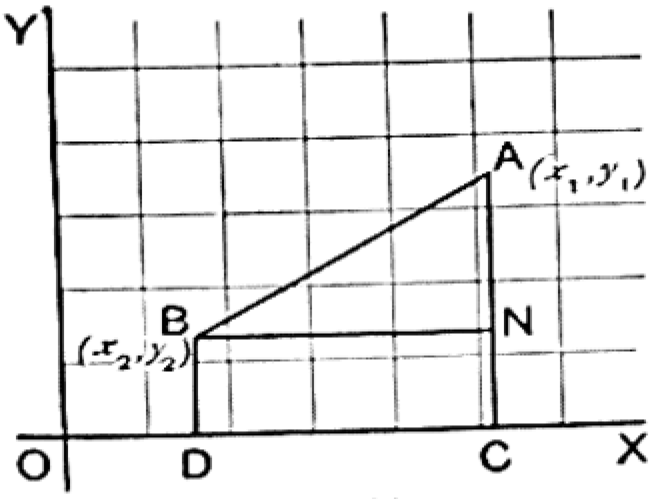


FIG. 33.

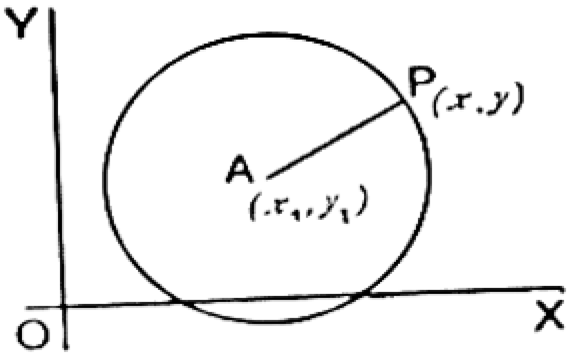


FIG. 34.

Equation of a Circle. A circle is completely determined by its centre and its radius.

Let (x_1, y_1) be the centre A of a circle whose radius is r. (Fig. 34.)

Let (x, y) be any point P on the circumference.

Then $AP=r$ and, using the above formula, we have

$$(x-x_1)^2+(y-y_1)^2=r^2.$$

This is the general equation of a circle: for the coordinates of *any* point on the circle satisfy it, while the coordinates of any point not on the circle do not satisfy it.

The equation may be written in the form

$$x^2+y^2-2xx_1-2yy_1+x_1{}^2+y_1{}^2-r^2=0.$$

Note that the numbers x_1, y_1 and r completely determine the circle.

Consequently, any equation of the form

$$x^2+y^2+2ax+2by+c=0,$$

where a, b and c are constants, must represent a circle. Hence, with rectangular axes the equation of any circle must have the coefficients of x^2 and y^2 equal and no term involving xy.

Ex. *Find the centre and radius of the circle*

$$2x^2+2y^2-4x+8y+3=0.$$

The given equation may be written

$$2(x^2-2x)+2(y^2+4y)+3=0,$$

i.e. $\quad 2[(x-1)^2-1]+2[(y+2)^2-4]+3=0,$

i.e. $\quad 2(x-1)^2+2(y+2)^2-7=0,$

i.e. $\quad (x-1)^2+(y+2)^2=3{\cdot}5.$

The centre of the circle is, therefore, the point $(1, -2)$ and the radius is $\sqrt{3{\cdot}5}$.

Graphical Solution of Equations.

Ex. 1. *Solve the equation* $2x^2+0{\cdot}8x-2{\cdot}73=0$ (see p. 10).

We start by drawing the graph of $y=2x^2+0{\cdot}8x-2{\cdot}73$.

Evidently, when $x=-2$, y is positive;

when $x=0$, y is negative;

and when $x=1$, y is positive.

Since y changes from a positive to a negative value as x passes from -2 to 0, clearly y should vanish for some value of x between -2 and 0.

Similarly, y should vanish for some value of x between 0 and 1.

Now, the values of x for which y vanishes are the roots of the given equation ; therefore the rough estimation just made gives the range of values of x for which y should be plotted.

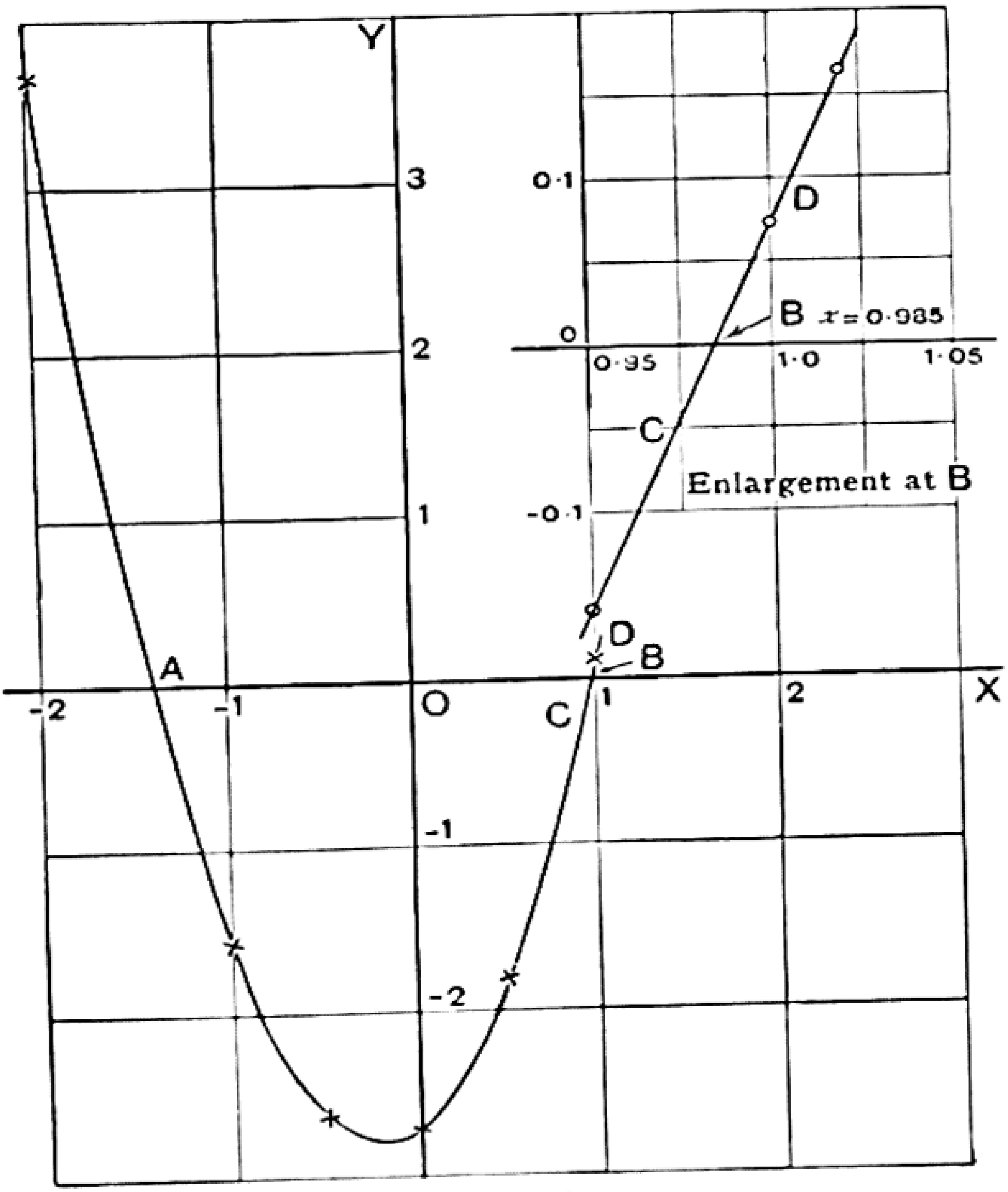


FIG. 35.

Arranging the work in tabular form :

x	− 2	− 1·5	− 1	− 0·5	0	0·5	1	1·5
$2x^2$	8	4·5	2	0·5	0	0·5	2	4·5
$0{\cdot}8x$	− 1·6	− 1·2	− 0·8	− 0·4	0	0·4	0·8	1·2
y	3·67	0·57	− 1·53	− 2·63	− 2·73	− 1·83	0·07	2·97

Fig. 35 shews that the roots are approximately − 1·4 and 1·0.

To obtain a closer approximation to the positive root, the portion of the curve CBD is enlarged as follows:

x	0·9	0·95	1·0
$2x^2$	1·62	1·805	2·0
$0{\cdot}8x$	0·72	0·76	0·8
y	$-0{\cdot}39$	$-0{\cdot}165$	0·07

Fig. 35 shews that a closer approximation to the positive root is 0·985.

A closer approximation to the negative root can be obtained in a similar way.

Ex. 2. *Solve the equation* $x^3-4x+2=0$ *by drawing the graphs of* $y=x^3$ *and of* $y=4x-2$ *for values of* x *between* -3 *and* $+2$.

Tabulating as in the last example for $y=x^3$:

x	-3	-2	-1	0	1	2
y	-27	-8	-1	0	1	8

In Fig. 36 these values are plotted with the graph of

$$y=4x-2,$$

the points of intersection of the two graphs being A, B and C.

Let OP, the abscissa of C, be denoted by p.

Since C lies on the graph of $y=x^3$, $CP=p^3$.

Since C lies on the graph of $y=4x-2$, $CP=4p-2$;

$$\therefore\ p^3=4p-2$$

or

$$p^3-4p+2=0.$$

That is, p satisfies the equation $x^3-4x+2=0$.

Similar consideration of the points A and B shews that the roots of the given equation are the abscissae of the points of intersection of the two graphs.

Fig. 36 shews that the roots are approximately $-2{\cdot}25$, $0{\cdot}6$ and $1{\cdot}64$.

To obtain a closer approximation to the root corresponding to the point A, take the following values :

x	$-2{\cdot}25$	$-2{\cdot}23$	$-2{\cdot}20$
y	$-11{\cdot}391$	$-11{\cdot}090$	$-10{\cdot}648$
$4x-2$	$-11{\cdot}0$	$-10{\cdot}92$	$-10{\cdot}8$

Fig. 36 shews that this root is approximately $-2{\cdot}216$.

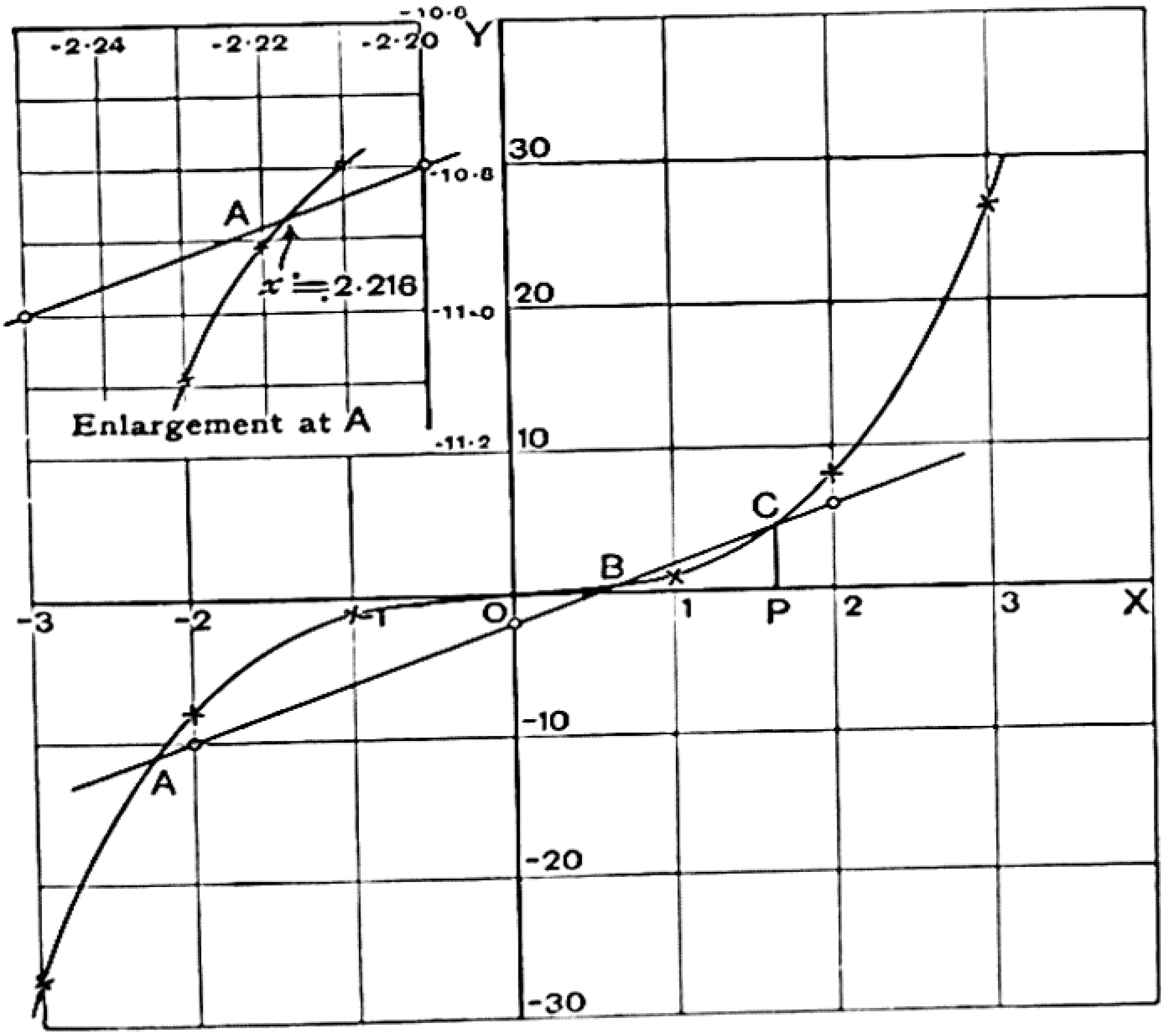


FIG. 36.

A similar method gives as the other roots $0{\cdot}539$ and $1{\cdot}675$ approximately.

Ex. 3. *Solve the equation* $x^2+9(1-x)^2=4$ *by drawing the graphs of* $x^2+y^2=4$ *and* $3x+y=3$.

Fig. 37 shews the circle $x^2+y^2=4$, whose centre is the origin and whose radius is 2; and the straight line $3x+y=3$. The points of intersection are A and B.

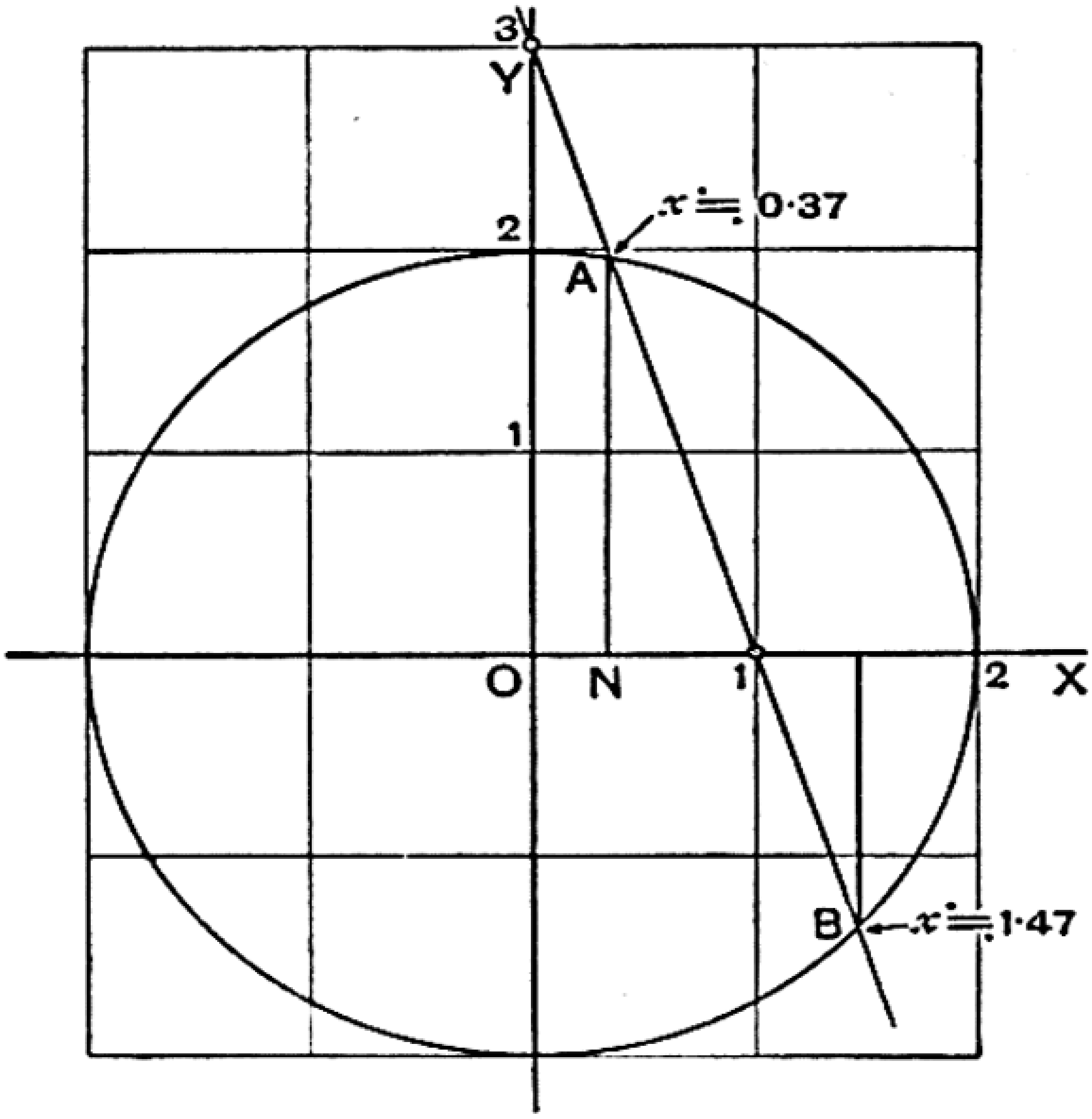


FIG. 37.

Let p be the abscissa of A. Since A lies on the circle,

$$p^2+AN^2=4. \quad (1)$$

Since A lies on the straight line,

$$3p+AN=3. \quad (2)$$

Substituting for AN from (2) in (1), $p^2+9(1-p)^2=4$.

Hence p satisfies the equation

$$x^2+9(1-x)^2=4.$$

Similarly the abscissa of B satisfies this equation.

The abscissae of A and B are approximately 0·37 and 1·47.

Proceeding as in the previous examples, closer approximations are 0·343 and 1·457.

Ex. 4. *Plot the graphs of* $x=\frac{6}{y}-y$ *and of* $y=3x-\frac{1}{x}$ *for values of* x *between* $-2{\cdot}5$ *and* $2{\cdot}5$, *and thus obtain the solutions of the equations*

$$\left.\begin{aligned} xy+y^2&=6,\\ 3x^2-xy&=1.\end{aligned}\right\}$$

To plot $x=\frac{6}{y}-y$ it is clearly better to choose values of y and calculate the corresponding values of x, as follows:

y	-4	-3	$-2\cdot5$	-2	$-1\cdot5$	$1\cdot5$	2	$2\cdot5$	3	4
$\frac{6}{y}$	$-1\cdot5$	-2	$-2\cdot4$	-3	-4	4	3	$2\cdot4$	2	$1\cdot5$
x	$2\cdot5$	1	$0\cdot1$	-1	$-2\cdot5$	$2\cdot5$	1	$-0\cdot1$	-1	$-2\cdot5$

To plot $3x - \frac{1}{x}$ we have

x	$-1\cdot2$	$-1\cdot0$	$-0\cdot5$	$-0\cdot2$	$0\cdot2$	$0\cdot5$	$1\cdot0$	$1\cdot2$
$\frac{1}{x}$	$-0\cdot833$	$-1\cdot0$	$-2\cdot0$	$-5\cdot0$	$5\cdot0$	$2\cdot0$	$1\cdot0$	$\cdot833$
y	$-2\cdot767$	$-2\cdot0$	$0\cdot5$	$4\cdot4$	$-4\cdot4$	$-0\cdot5$	$2\cdot0$	$2\cdot767$

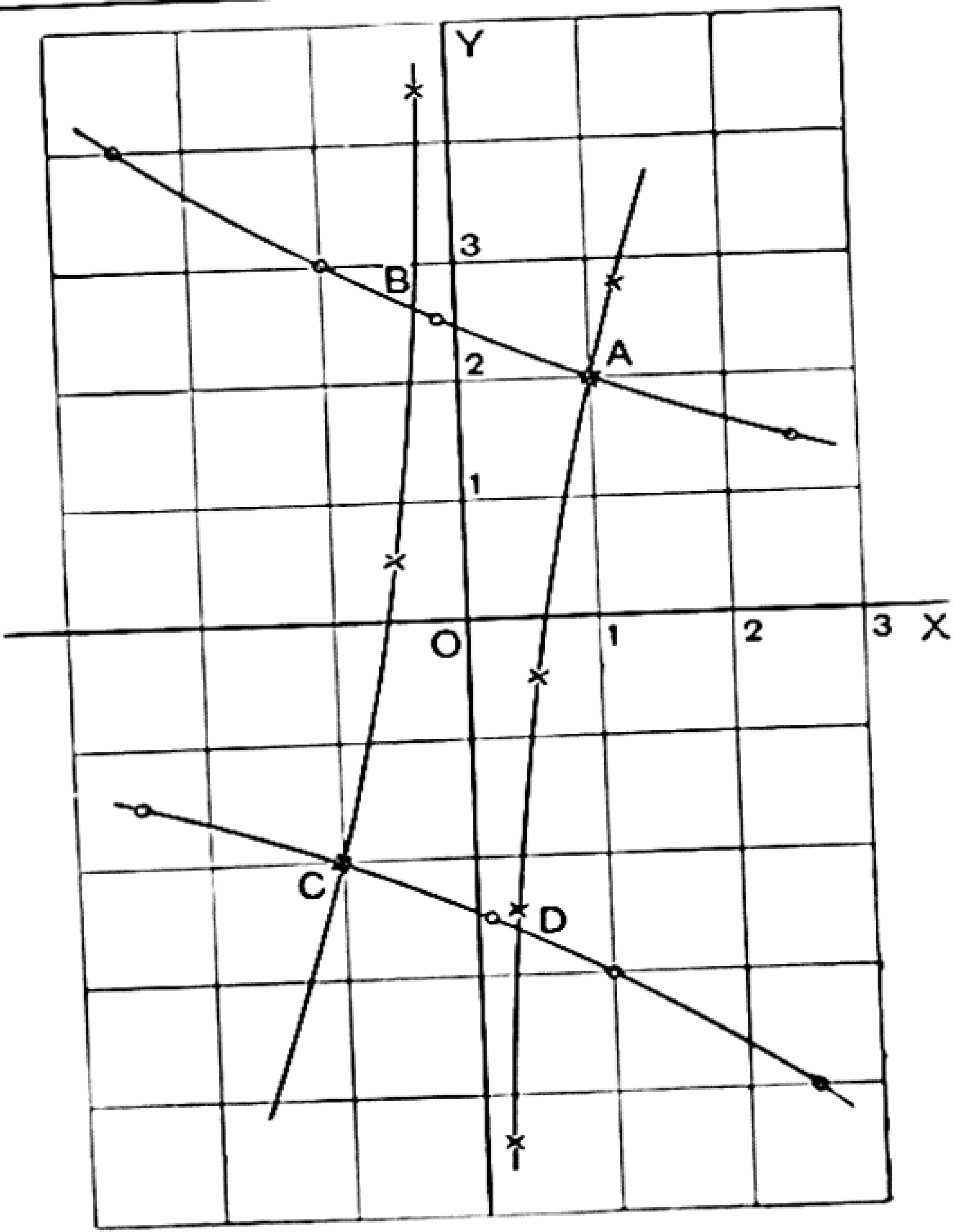


FIG. 38.

Fig. 38 shews that the graphs intersect at the points *A*, *B*, *C* and *D*.

At *A*,
$x=1, \; y=2.$

At *C*,
$x=-1, \; y=-2.$

These values satisfy the given equations, and are accurate solutions.

At *B*,
$x=-0\cdot3, \; y=2\cdot6.$

At *D*,
$x=0\cdot3, \; y=-2\cdot6.$

Closer approximations to the roots corresponding to the points *B* and *D* could be found by enlarging.

Ex. 5. *Draw on the same diagram, and with the same scale and axes, the graphs of* (1) $y=(x+1)^2$, (2) $x^2+y^2=16$.

Explain how you can find by means of these graphs the real roots of the equation $x^4+4x^3+7x^2+4x=15$, *and find these roots as accurately as you can.* [L.U.]

The graph of $x^2+y^2=16$ is a circle of radius 4, whose centre is at the origin.

This intersects the graph of $y=(1+x)^2$ in the points A and B (Fig. 39).

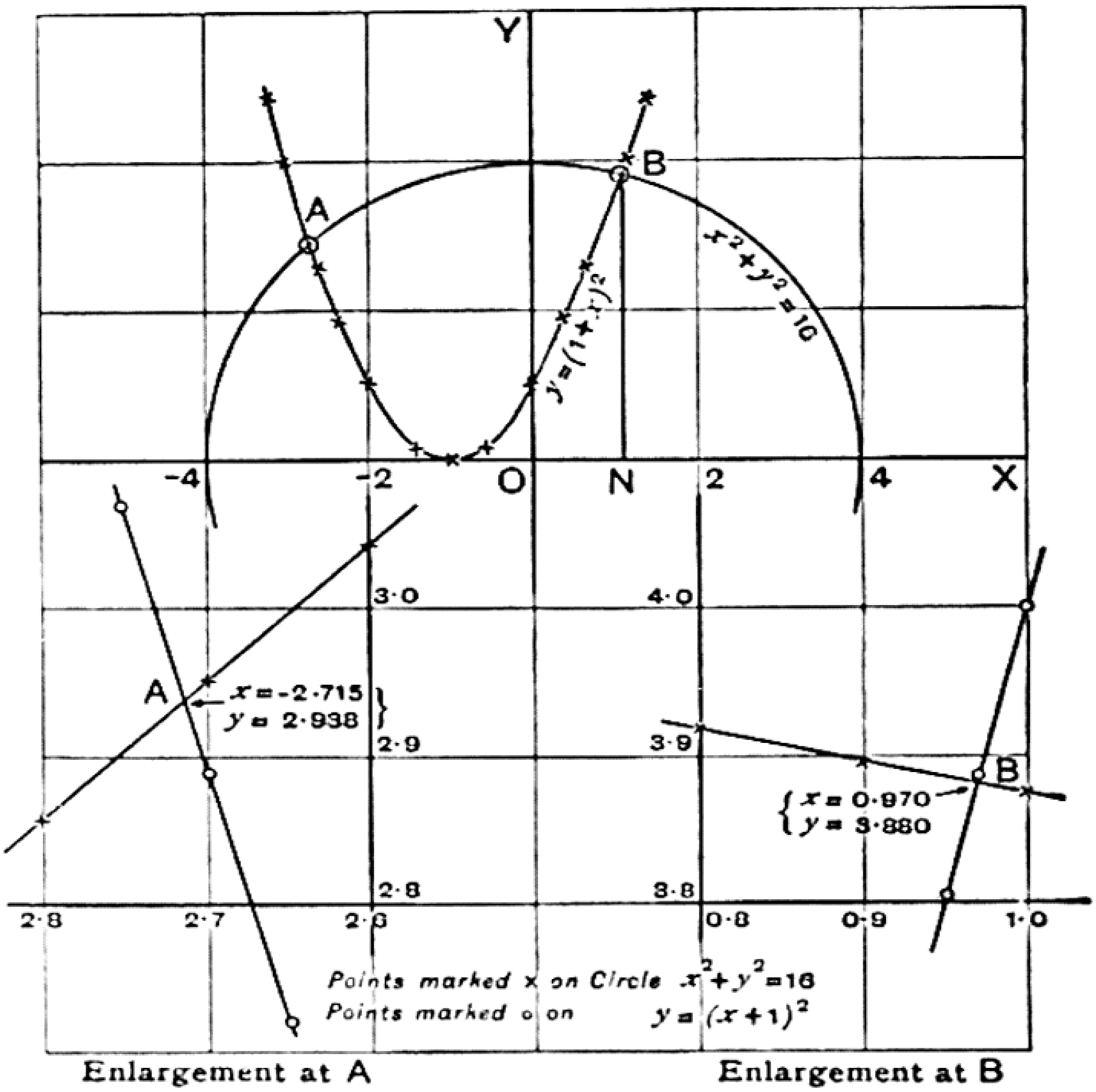


FIG. 39.

If $ON=p$, we have $BN=(p+1)^2$ and $p^2+BN^2=16$;

$$\therefore\ p^2+(p+1)^4=16, \qquad i.e.\ p^4+4p^3+7p^2+4p=15.$$

Thus p satisfies the given equation $x^4+4x^3+7x^2+4x=15$. So also does the abscissa of A. Therefore the roots are approximately $-2·7$ and $1·0$, and by enlarging at A and B closer approximations are $-2·715$ and $0·970$.

Ex. 6. *Shew that* $\dfrac{(x-1)(x+5)}{x-3}$ *cannot lie between* 2 *and* 18, *and draw a rough graph of the expression.*

Let
$$y=\frac{(x-1)(x+5)}{x-3}.$$

Rearranging, $\quad x^2+x(4-y)+3y-5=0. \qquad\qquad (1)$

Since x is real, the roots of equation (1) must be real.

Therefore $\quad (4-y)^2-4(3y-5)>0$ (see p. 12);

$$\therefore\ (y-2)(y-18)>0. \qquad\qquad (2)$$

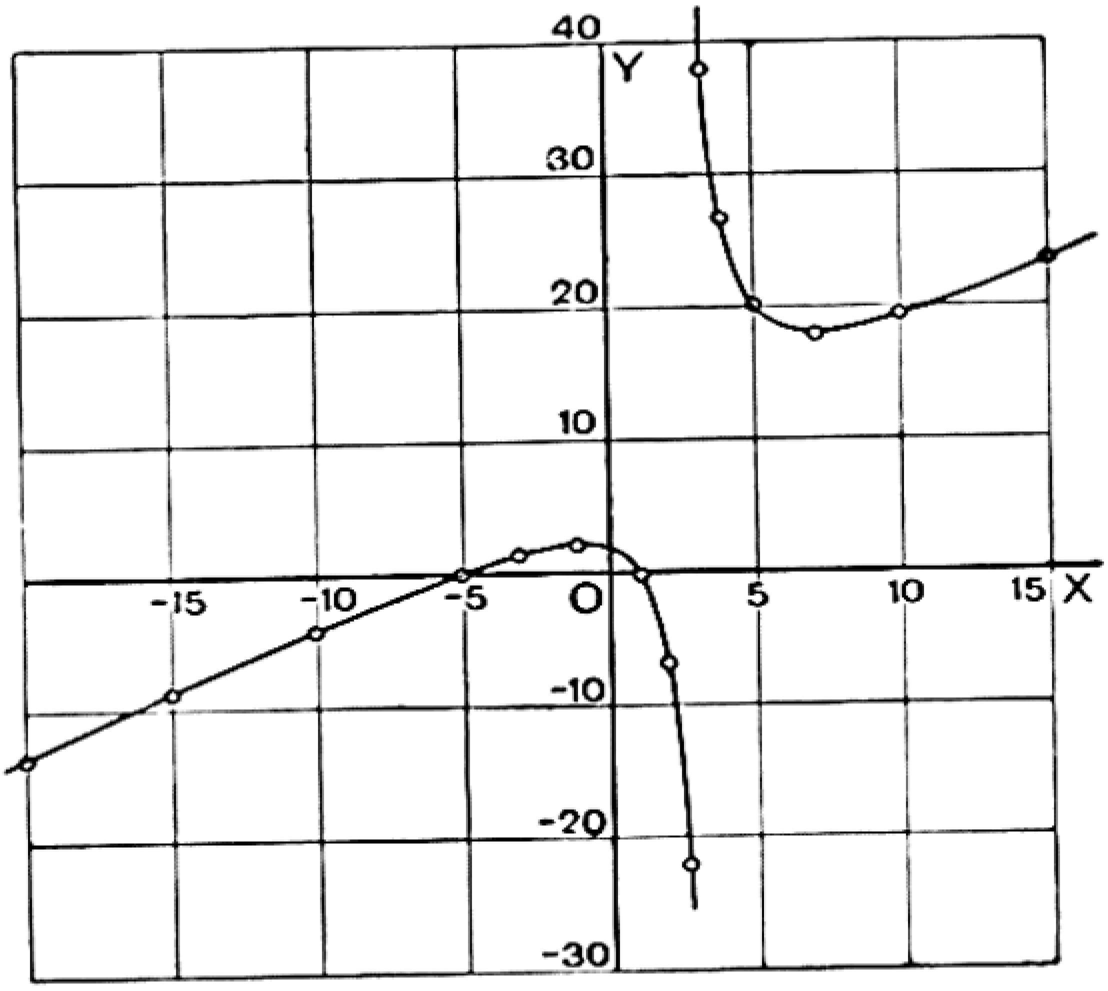


FIG. 40.

For this to be true, the factors on the left of (2) must be either both positive or both negative. If they are both positive,
$$y>18:$$
if they are both negative, $y<2$.

Hence y cannot lie between 2 and 18.

To draw a rough graph, consider the changes in y as x passes from large negative through zero towards large positive values, paying careful attention to the particular values of x at which y vanishes or becomes infinite, as follows:—

x	large and negative	<-5	-5	between -5 and 1	1	between 1 and 3	3	>3	large and positive
y	large and negative	negative	0	positive	0	negative	infinite	positive	large and positive

Further, when $y=2$, (1) becomes $\quad x^2+2x+1=0,$

giving $\quad x=-1;$

when $y=18$, (1) becomes $x^2-14x+49=0,$

giving $\quad x=7.$

We can now draw the graph illustrated in Fig. 40.

Ex. 7. *Shew that* $\dfrac{x^2+1}{x^2+x+1}$ *must lie between* 2 *and* $\frac{2}{3}$, *and draw a rough graph of the expression.*

Let
$$y=\frac{x^2+1}{x^2+x+1}.$$

Rearranging, we have
$$x^2(y-1)+xy+(y-1)=0 \quad\text{.....................(1)}$$

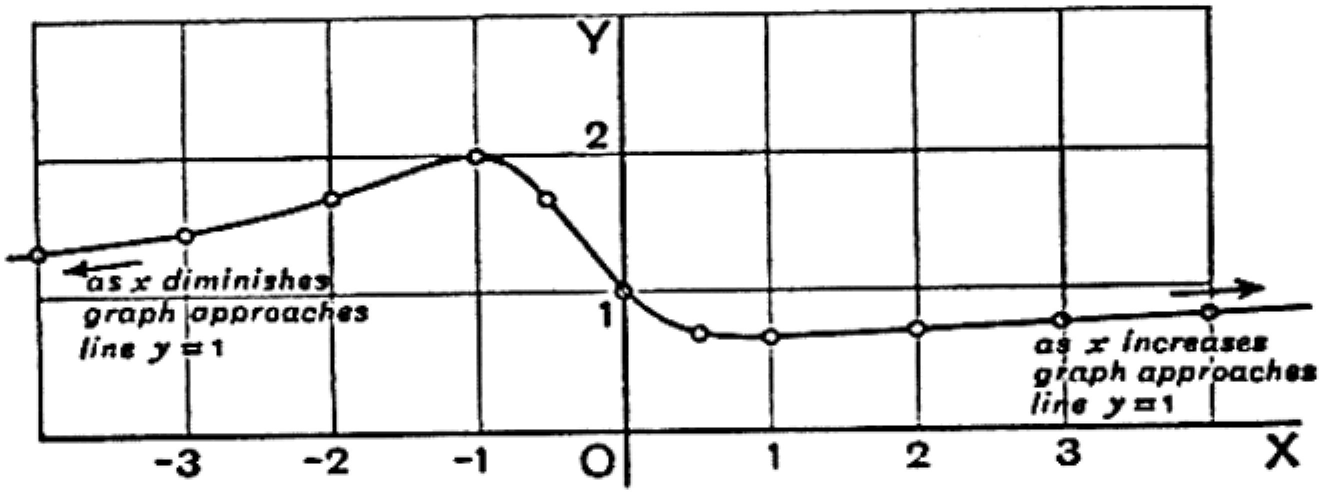


FIG. 41.

Since x is real, the roots of (1) must be real;
$$\therefore\ y^2-4(y-1)^2>0\,;$$
$$\therefore\ (y-2y+2)(y+2y-2)>0\,;$$
$$\therefore\ (2-y)(3y-2)>0. \quad\text{...............(2)}$$

Thus the factors on the left of (2) must be either both positive or both negative.

If both are negative $y>2$ and at the same time $y<\frac{2}{3}$, which is impossible.

If both are positive $y<2$ and at the same time $>\frac{2}{3}$; *i.e.* y lies between 2 and $\frac{2}{3}$ for all values of x.

Tabulating as in Ex. 6 :—

x	large and negative	large and positive
y	approximately 1	approximately 1

Further, when $y=2$, (1) becomes $x^2+2x+1=0$,
giving $x=-1$;
and when $y=\frac{2}{3}$, (1) becomes $-\frac{1}{3}x^2+\frac{2}{3}x-\frac{1}{3}=0$,
giving $x=+1$.

The graph is shewn in Fig. 41.

Ex. 8. *Shew that* $\frac{(x-1)(x-4)}{x-2}$ *can take any value positive or negative, and draw a rough graph.*

Let
$$y=\frac{(x-1)(x-4)}{x-2}.$$

Rearranging, we have
$$x^2-x(5+y)+(4+2y)=0.$$

As x is real, we have
$$(5+y)^2-4(4+2y)>0\,;$$
$$\therefore\ y^2+2y+9>0\,;$$
$$\therefore\ (y+1)^2+8>0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

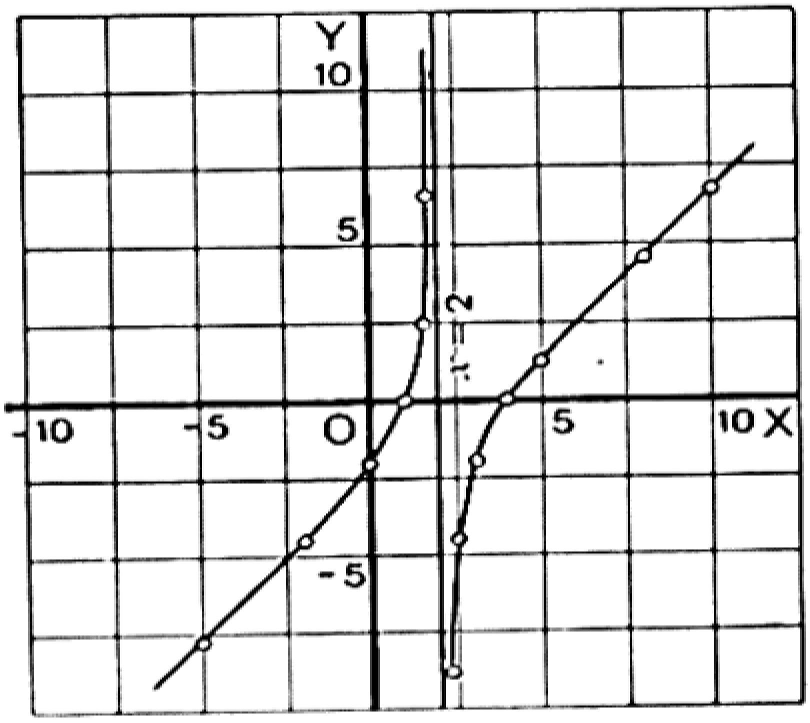


FIG. 42.

Clearly (1) is true for all values of y, since $(y+1)^2$ is positive whether y is negative or positive. Tabulating :—

x	large and negative	<1	1	between 1 and 2	2	between 2 and 4	4	>4	large and positive
y	large and negative	negative	0	positive	∞ *	negative	0	positive	large and positive

The graph is shewn in Fig. 42.

---

* This symbol may be read "infinite" or "infinitely large."

The Maximum or Minimum Value of a Quadratic Expression may be investigated as follows :—

Let $$y = ax^2 + bx + c.$$

Then $$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c,$$

i.e. $$y = \frac{4ac - b^2}{4a} + a\left(x + \frac{b}{2a}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

CASE I. a positive.

As x is real, $\left(x + \frac{b}{2a}\right)^2$ is positive or zero : it cannot be negative. Therefore, since a is positive, $a\left(x + \frac{b}{2a}\right)^2$ is positive or zero. Therefore, from (1), the minimum value of y is $\frac{4ac - b^2}{4a}$, and y takes this minimum value when $x = -\frac{b}{2a}$.

CASE II. a negative.

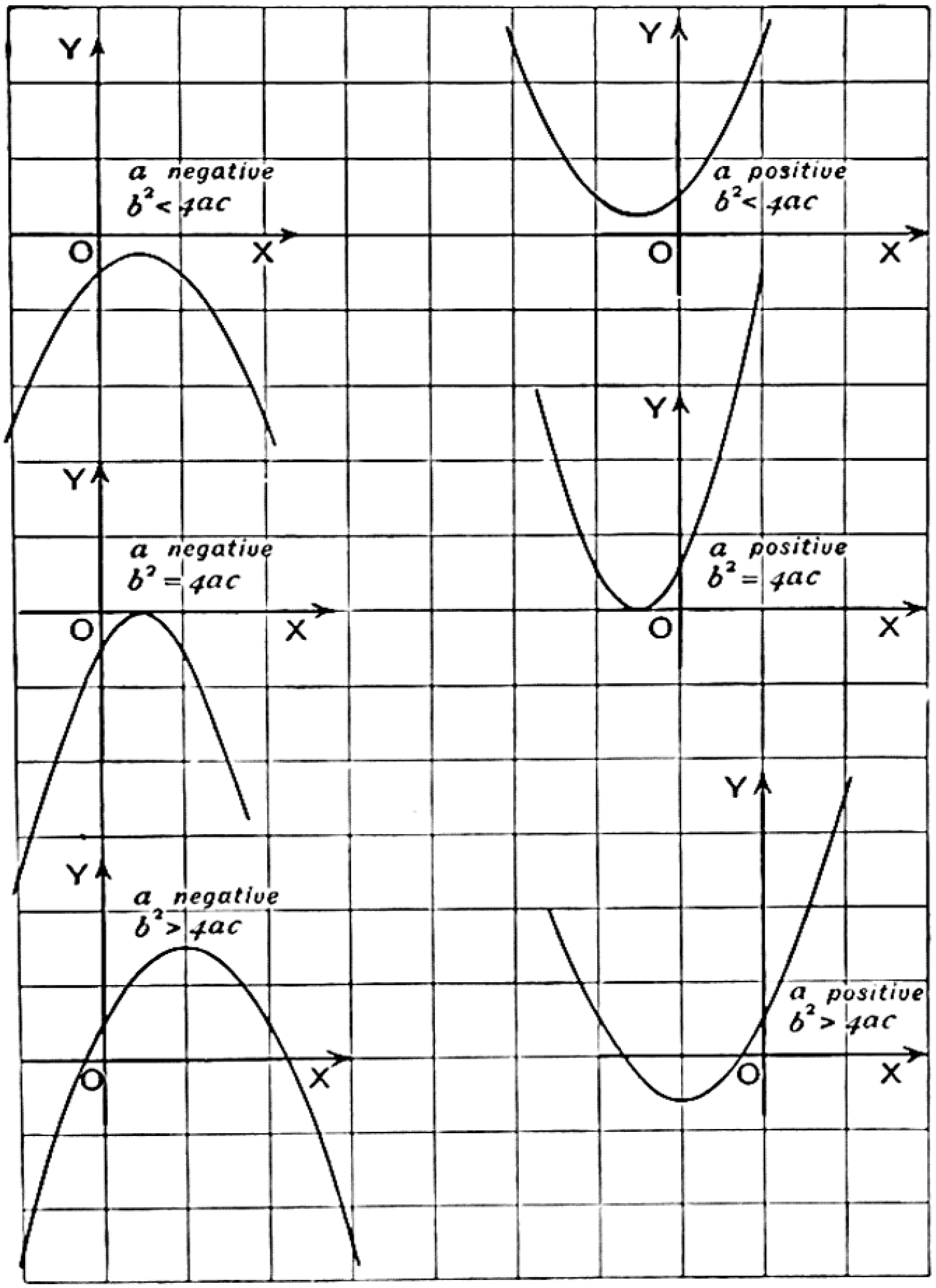
As before, $\left(x + \frac{b}{2a}\right)^2$ is positive or zero. Therefore, since a is negative, $a\left(x + \frac{b}{2a}\right)^2$ is negative or zero. Therefore, from (1), the maximum value of y is $\frac{4ac - b^2}{4a}$, and y takes this maximum value when $x = -\frac{b}{2a}$.

Fig. 43 shews the graph of $ax^2 + bx + c$, where b is positive for the following cases :

a	positive	negative	positive	negative	positive	negative
$b^2 - 4ac$	negative	negative	zero	zero	positive	positive

When a is negative, the expression $ax^2 + bx + c$ always has a maximum value of $\frac{4ac - b^2}{4a}$ at the point where $x = -\frac{b}{2a}$.

When a is positive, the expression $ax^2 + bx + c$ always has a minimum value of $\frac{4ac - b^2}{4a}$ at the point where $x = -\frac{b}{2a}$.



In Graphs on left: a is negative
The maximum value of ax^2+bx+c
is $\frac{4ac-b^2}{4a}$ where $x = -\frac{b}{2a}$

In Graphs on right: a is positive
The minimum value of ax^2+bx+c
is $\frac{4ac-b^2}{4a}$ where $x = -\frac{b}{2a}$

FIG. 43.

This characteristic curve $y = ax^2 + bx + c$ is called a *parabola*.

For another method of finding the maximum or minimum value of a quadratic expression, see Ex. 28, p. 80.

Ex. (i) *What conditions must a, b, c satisfy in order that the maximum value of ax^2+bx+c may be negative when x is positive.*

(ii) *Find the maximum and minimum values of $4x+\frac{9}{x}$. Has this expression an absolutely greatest or absolutely least finite value?*

(i) Let $$y=ax^2+bx+c.$$

Then $$y=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \qquad \text{.................. (1)}$$

As y is to have a maximum value, a must be negative.

Therefore, one condition is : a is negative.

Again, this condition being satisfied, the maximum value of y is $\frac{4ac-b^2}{4a}$, as can be seen from (1). As this maximum value is to be negative and the denominator, $4a$, is negative, therefore the numerator, $4ac-b^2$, must be positive.

Therefore, another condition is :

$$b^2<4ac.$$

Finally, the maximum value of y occurs when $x=-\frac{b}{2a}$, and this value of x is to be positive. Therefore, since a is negative, b must be positive.

Therefore, another condition is : b is positive.

Thus, there are three conditions which a, b, c must satisfy, viz. :—

a negative,
b positive,
$b^2<4ac$.

The graphs in Fig. 43 should be studied in order to discover which of them corresponds to the particular case just considered.

(ii) Again, let $$4x+\frac{9}{x}=y.$$

Then $$4x^2-xy+9=0.$$

Since x is real, $$y^2-144>0,$$

i.e. $$(y-12)(y+12)>0.$$

Therefore y is either greater than 12 or less than -12.

The graph of $y=4x+9/x$ is obtained by first drawing the graphs of $y_1=4x$ and $y_2=9/x$, shewn by the lines in Fig. 44.

Then, for any value of x, it is clear that y_1+y_2 is the ordinate of the graph of $y=4x+9/x$ corresponding to the same value of x. The resultant graph is represented by the thick line in Fig. 44 and shews that 12 is a minimum value, while -12 is a maximum value.

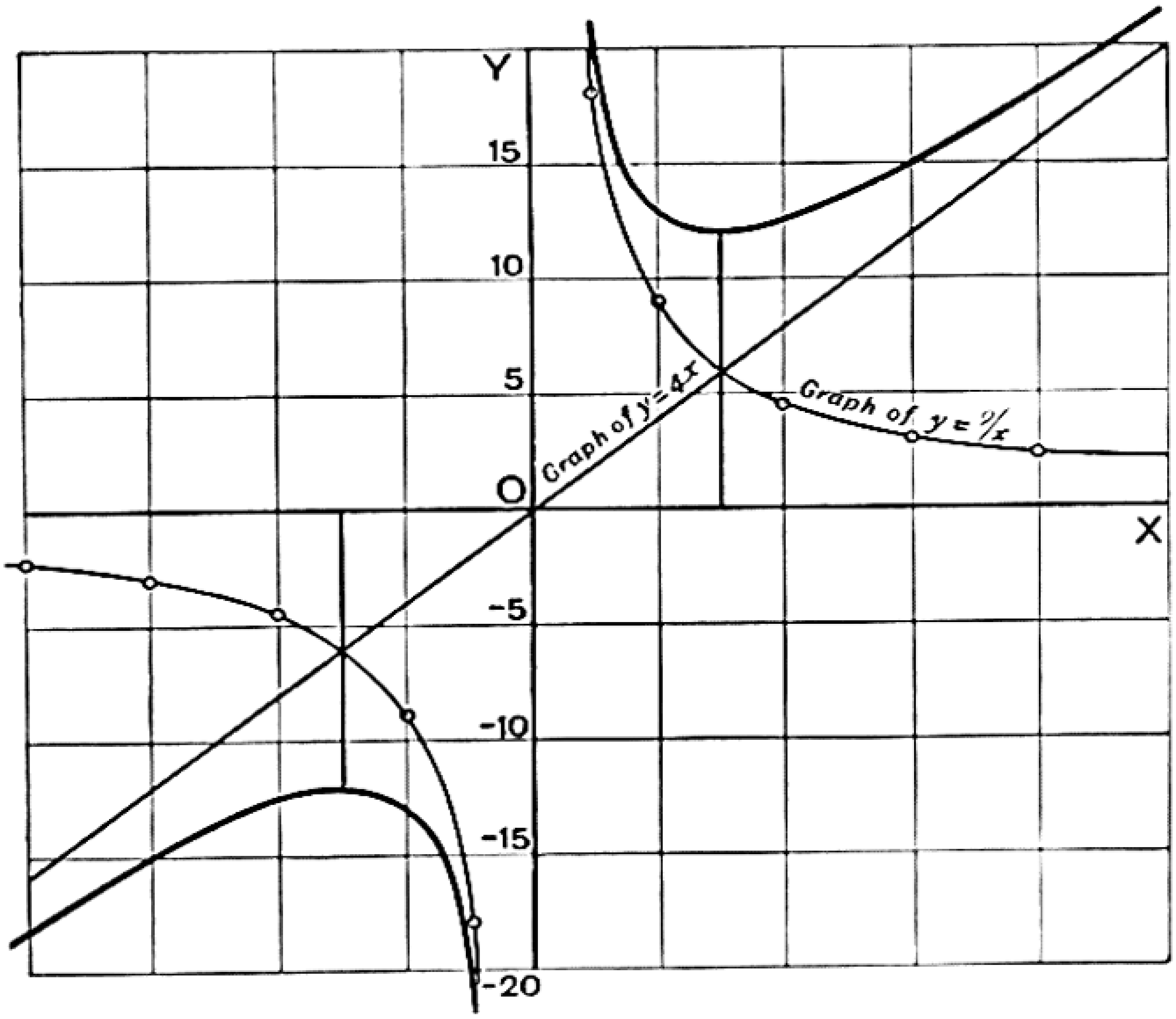


FIG. 44.

The graph also shews that the expression has no absolute maximum or absolute minimum values, for when x is negative and approaches zero, y is negative and approaches an infinitely great negative value. When x is positive and approaches zero, y is positive and approaches an infinitely great positive value. Further, x and y approach infinite values at the same time.

EXERCISES VI.

1. Find the coordinates of the centre and the radii of the circles of which the equations are:

(i) $x^2+y^2-4x=0$;

(ii) $x^2+y^2+2x-4y-4=0$;

(iii) $x^2+y^2-x+y=0$.

2. Find the values of a, b and c so that the graph of

$$y=ax^2+bx+c$$

may pass through the points given by

$$x=1,\ y=2;\quad x=2,\ y=3;\quad x=3,\ y=5.$$

Draw the graph and find the minimum value of y. [L.U.]

3. The cost C of a ship's voyage depends on the speed s, being given by the formula

$$C=\frac{12000}{s}+s^3.$$

Plot the relation between C and s, and determine the value of s for which C is as small as possible. [L.U.]

4. By considering the intersections of the graphs of $y=x^2$ and $y=4-\frac{2}{x}$, shew that the equation $x^3-4x+2=0$ has two positive roots and one negative root. By detailed plotting in the neighbourhood of the intersection which gives the greater of the two positive roots, determine this root to one place of decimals. [L.U.]

5. Draw graphs of $\frac{1}{x^2}$ and $\frac{1}{8}(23-4x-4x^2)$. Use these graphs to obtain (to two significant figures in each case) the roots of the equation $8-23x^2+4x^3+4x^4=0$. [L.U.]

6. Draw the graphs of $y=3x^2-6x-2$ and $xy=-16$ between $x=-2$ and $+3$, using the same axes for both.

From the first graph determine the minimum value of

$$3x^2-6x-2$$

and the roots of $3x^2-6x-2=0$. From the two graphs find a root of $3x^3-6x^2-2x+16=0$.

Do the graphs shew that there is no other real root? How could this be tested algebraically? [L.U.]

7. Trace the two graphs $9y=2x^2$ and $y(x-3)=1$ with the same axes; approximate graphically to a real root of $2x^2(x-3)=9$. [L.U.]

8. Find the roots of the equation $x^4 - 4x^3 + 5x^2 - 9 = 0$ by drawing the graphs of $x^2 + y^2 = 9$ and $y = (x-1)^2 - 1$.

9. To manufacture N like tools costs £$(8000 + 3\sqrt{N})$. The connection between the selling price of each tool and the probable number of sales is given by the following table :

Selling Price	2/-	3/-	4/-	5/-	6/6
No. Sold - -	160000	130000	90000	62500	40000

Calculate the estimated gross profit in each case, and by means of a graph find the price which would probably give the greatest gross profit. [L.U.]

10. Shew that the function $\dfrac{x^2 - 3x + 1}{x - 3}$ has no value between 1 and 5 for real values of x, and draw a rough graph of the function. [L.U.]

11. Find the condition that the roots of $ax^2 + bx + c = 0$ may be (1) real, (2) rational, (3) equal. Shew that, for all real values of x, the expression $\dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$ lies between 3 and $\frac{1}{3}$. [L.U.]

12. Find approximately the roots of the equation $x^3 - 20x + 9 = 0$.

13. Draw the graphs of $y = \dfrac{1}{x}$ and $y = x + \dfrac{2}{x^2}$, and thus find approximately the roots of the equation $x^3 - x + 2 = 0$.

14. By drawing the graphs of $y = x^2 + \dfrac{1}{x^2}$ and $y = \dfrac{3}{x} - 1$, solve the equation $x^4 + x^2 - 3x + 1 = 0$.

15. Solve the equation $(x^2 + 2x)^2 + x^2 = 4$ by drawing the graphs of $y = x^2 + 2x$ and $x^2 + y^2 = 4$.

16. Shew that the roots of $x^3 + px + q = 0$ are the abscissae of the points of intersection (other than the origin) of the graphs of

$$y = x^2 \quad \text{and} \quad x^2 + y^2 + qx + (p-1)y = 0.$$

Hence solve the following equations :

(*a*) $x^3 - 3x + 4 = 0$; (*b*) $x^3 + 3x + 4 = 0$; (*c*) $2x^3 - 5x + 1 = 0$.

17. Shew that the function $\dfrac{x - 2}{x(x+1)}$ cannot lie between $5 + 2\sqrt{6}$ and $5 - 2\sqrt{6}$, and draw a rough graph.

18. Shew that there are two values between which

$$\frac{(2x-1)(2x+5)}{(2x-3)}$$

cannot lie, and find them. Draw a rough graph of the function.

19. Draw a rough graph of $\dfrac{3x-2}{4x^2+1}$, and shew that this function must lie between $-2\frac{1}{4}$ and $\frac{1}{4}$.

20. Draw a rough graph of $\dfrac{x^2-x+1}{x^2+x+1}$, and shew that it must lie between two values. What are these values ?

21. Shew that the following functions can take all values, and draw a rough graph of each :

(a) $\dfrac{(2x-1)(x-2)}{x-1}$; (b) $\dfrac{(x+1)(x-2)}{x(x+3)}$; (c) $\dfrac{(x-4)(x+2)}{(x+1)(x-5)}$.

22. Find approximately all the roots of $x^3+3x^2-4x-5=0$ by considering the intersections of the graphs of the functions x^2+3x and $4+\dfrac{5}{x}$. [L.U.]

23. Prove that for real values of x the value of the expression $\dfrac{3x-4}{x^2+1}$ cannot lie outside certain limits, and find these limits. Illustrate by a rough graph of the function. [L.U.]

24. Prove that, x being positive, the least value of $(x^2+a^2)/x$ is $2a$.

A vessel makes a round trip in which part of the expenses vary as the time and the remainder as the velocity. When the voyage takes 15 days, the cost of the trip is £549 : when it takes 27 days the cost is £585. Find the least cost of a trip and its duration. [L.U.]

25. If x^3+ax^2+bx+c is exactly divisible by x^2+px+q, shew that $q^3-bq^2+acq-c^2=0$.

If this condition is satisfied, shew that x^3+ax^2+bx+c is identically equal to $(xq+c)\{x^2q+x(qa-c)+q^2\}/q^2$. [L.U.]

26. Taking 1 inch as unit, make careful graphs of the functions $y=2x^2+x-2$ and $2x=y^2+2y-1$.

From your drawing, state where they meet, and verify your statement by actual substitution in the equation resulting from the elimination of x. [L.U.]

27. The volume of a rectangular block is 240 cubic inches, its surface is 248 sq. in., and the sum of two of its edges is 14 in. Find its dimensions. [L.U.]

28. Write down the condition that $ax^2+bx+c=y$ should give real values for x, and thus obtain the conditions for a maximum or minimum value of ax^2+bx+c (see page 74).

29. If the quadratic function ax^2+bx+c has the values 2, 3, 1 when x is 1, 2, 3 respectively, find the values of a, b, c, and draw a graph of the function from $x=0$ to $x=4$.

Shew that $\frac{73}{24}$ is the greatest value of the function. [S.W.]

30. What conditions must a, b and c satisfy so that the graph of ax^2+bx+c may have a positive maximum ordinate for a negative value of x? Find the maximum and minimum values of $7x+4/x$.

Has the expression an absolutely greatest or absolutely least finite value? [L.U.]

31. Explain the difference between an equation and an identity. If $(x-2)^2(ax^2+bx+c) \equiv 3x^4-x^3-36x^2+60x-16$, find a, b and c, and with these values find A and B, α and β so that

$$\frac{1}{ax^2+bx+c}=\frac{A}{x-\alpha}+\frac{B}{x-\beta}.$$

Solve the equation $3x^4-x^3-36x^2+60x-16=0$. [L.U.]

32. If α, β be the roots of the equation $ax^2+bx+c=0$, prove that $\alpha+\beta=-b/a$, $\alpha\beta=c/a$.

Find the condition that the roots of the equation may be in the ratio $3:2$, and find the equation whose roots are halves of the roots of the given equation. [Durham.]

33. Shew that the expression $\dfrac{x^2-3x+4}{x-3}$ has no values between -1 and 7 for real values of x. Sketch roughly the graph of the expression. [L.U.]

34. Express the sum and product of the roots of the equation $ax^2+bx+c=0$ in terms of the coefficients.

If a is determined so that $x+\dfrac{a}{x-2}$ is equal to $3{\cdot}73$ when $x=5{\cdot}41$, find the other value of x which makes the expression take this value. [L.U.]

35. Find the maximum and minimum values, if any, of the expression

$$\frac{x^2-3x+4}{x-2}.$$

Sketch the graph of this expression. [Man.]

36. Prove that for real values of x the expression $\dfrac{x^2-8x+22}{2x-3}$ must be greater than 1 or less than -6.

If x may only be given values between $+1$ and -1, prove that the value of the expression must lie between -15 and $-6\frac{1}{5}$. [B.U.]

37. If α and β are the roots of the equation $ax^2+bx+c=0$, find the value of $\alpha^4-\beta^4$ in terms of a, b and c.

Shew that $\dfrac{x^2+4x-8}{x-2}$ cannot have values between 4 and 12 if x be real. Draw a graph of this function, and by means of it obtain an approximate solution of the equation $x^2+3x-6=0$.

[Durham.]

38. Solve the equations

$$\text{(i)}\quad (x+3)^3=(2x-3)^2(3-x);$$

$$\text{(ii)}\quad x^2+y^2+2x-2y-8=0,\qquad 3x+y=6.$$

Verify the solution of (ii) graphically. [Durham.]

39. In a hemispherical vessel of radius 4 ft., the volume V cubic ft. of water in the vessel when the water is D ft. deep is given by the formula $V=\pi\left(4D^2-\dfrac{D^3}{3}\right)$. Plot V against D on squared paper, and hence find as accurately as you can the depth of water in the vessel when it contains half the greatest quantity it will hold. [L.U.]

40. Find the limits between which the fraction

$$\frac{x^2+7x+16}{x^2-5x+16}$$

must lie if x is real.

Give a rough graph of the fraction for values of x between -10 and $+10$. [L.U.]

CHAPTER IV.

SURDS, INDICES AND LOGARITHMS. LAWS OTHER THAN LINEAR.

Rational and Irrational Numbers. A number which can be expressed in the form p/q, where p and q are positive or negative integers, is called a **rational number**; *e.g.* $\frac{3}{7}$, -2, $\sqrt{100}$ are rational numbers.

But numbers like $\sqrt{2}$, π, which *cannot* be expressed in the form p/q are called **irrational numbers.**

A **surd** is an irrational number of the form $\sqrt[n]{a}$, where a is not a perfect n^{th} power of a rational number; *e.g.* $\sqrt{2}$, $\sqrt[3]{7}$, $\sqrt[4]{\frac{3}{11}}$ are surds.

Irrational numbers can be shewn to obey the fundamental laws of Algebra. The laws of special importance for work with surds are:

$$\sqrt[n]{a}\,.\,\sqrt[n]{b}=\sqrt[n]{ab},\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}},\quad \sqrt[n]{(\sqrt[m]{a})}=\sqrt[mn]{a}=\sqrt[m]{(\sqrt[n]{a})}.$$

Ex. 1. *Simplify*

$$\sqrt{21}\times\sqrt{3},\quad \sqrt{12}-2\sqrt{8}+2\sqrt{18}-3\sqrt{3},\quad (3\sqrt{2}-4\sqrt{3})(\sqrt{2}+\sqrt{3}).$$

We have $\sqrt{21}\times\sqrt{3}=\sqrt{3}\times\sqrt{7}\times\sqrt{3}=3\sqrt{7}.$

$$\sqrt{12}-2\sqrt{8}+2\sqrt{18}-3\sqrt{3}=2\sqrt{3}-4\sqrt{2}+6\sqrt{2}-3\sqrt{3}=2\sqrt{2}-\sqrt{3}.$$

$$(3\sqrt{2}-4\sqrt{3})(\sqrt{2}+\sqrt{3})=6-4\sqrt{6}+3\sqrt{6}-12=-\sqrt{6}-6.$$

Ex. 2. *Express with rational denominators*

$$\frac{3}{\sqrt{5}},\quad \frac{5}{\sqrt{7}+\sqrt{2}},\quad \frac{x}{\sqrt{y}}\quad \textit{and}\quad \frac{c}{\sqrt{a}-\sqrt{b}}.$$

We have $\dfrac{3}{\sqrt{5}}=\dfrac{3\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\dfrac{3\sqrt{5}}{5}.$

$$\frac{5}{\sqrt{7}+\sqrt{2}}=\frac{5(\sqrt{7}-\sqrt{2})}{(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})}=\frac{5(\sqrt{7}-\sqrt{2})}{7-2}=\sqrt{7}-\sqrt{2}.$$

$$\frac{x}{\sqrt{y}}=\frac{x\sqrt{y}}{\sqrt{y}\times\sqrt{y}}=\frac{x\sqrt{y}}{y}.$$

$$\frac{c}{\sqrt{a}-\sqrt{b}}=\frac{c(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}$$

$$=\frac{c(\sqrt{a}+\sqrt{b})}{a-b}$$

Ex. 3. *Find the square roots of* $8+2\sqrt{15}$ and $12-\sqrt{140}$.

Let $$\sqrt{8+2\sqrt{15}}=\sqrt{x}+\sqrt{y}.$$

Squaring both sides,

$$8+2\sqrt{15}=x+y+2\sqrt{xy}.$$

Now, if numbers x, y can be found such that

$$\left.\begin{aligned} x+y&=8 \\ \text{and}\qquad xy&=15,\end{aligned}\right\} \qquad \text{..............................(1)}$$

the problem of finding the square root has been solved.

Equations (1) can be solved in the usual way as simultaneous equations, but in practice inspection is often sufficient.

Evidently 3 and 5 are two numbers whose sum is 8 and product is 15;

$$\therefore\ \sqrt{8+2\sqrt{15}}=\sqrt{5}+\sqrt{3}.$$

Again, let $$\sqrt{12-\sqrt{140}}=\sqrt{x}-\sqrt{y}.$$

Squaring, $$12-\sqrt{140}=x+y-2\sqrt{xy}.$$

Hence x and y have to be so chosen that

$$\left.\begin{aligned} x+y&=12, \\ 2\sqrt{xy}&=\sqrt{140}.\end{aligned}\right\} \qquad \text{..........................(1)}$$

Now $$\sqrt{140}=\sqrt{4\times 35}=2\sqrt{35}.$$

Therefore, from (1), $x+y=12$ and $xy=35$, whence $x=7, y=5$;

$$\therefore\ \sqrt{12-\sqrt{140}}=\sqrt{7}-\sqrt{5}.$$

Ex. 4. *Solve the equation*

$$\sqrt{x+2}+\sqrt{4x+1}=\sqrt{12x+1}. \qquad \text{[L.U.]}$$

Squaring both sides,

$$5x+3+2\sqrt{(x+2)(4x+1)}=12x+1\,;$$

$$\therefore\ 2\sqrt{(x+2)(4x+1)}=7x-2.$$

Squaring again, $4(4x^2+9x+2)=49x^2-28x+4$;

$$\therefore\ 33x^2-64x-4=0\,;\ \dots\dots\dots\dots\dots\dots(1)$$

$$\therefore\ (33x+2)(x-2)=0\,;$$

$$\therefore\ x=2 \quad\text{or}\quad -\tfrac{2}{33}.$$

Verification. Using $x=2$, the left-hand side of the equation becomes $\sqrt{4}+\sqrt{9}$, *i.e.* 5, and the right-hand side is also 5.

Using $x=-2/33$, the left-hand side of the equation becomes

$$\sqrt{\tfrac{64}{33}}+\sqrt{\tfrac{25}{33}},\quad i.e.\ 13/\sqrt{33},$$

whereas the right-hand side is $3/\sqrt{33}$.

$\therefore$ the only solution satisfying the given equation is $x=2$.

Note that both roots of (1) do not necessarily satisfy the original equation. It can easily be shown that (1) would have been obtained by the same process of squaring, if the original equation had been

$$\sqrt{x+2}-\sqrt{4x+1}=\sqrt{12x+1},$$

and it is this equation that $x=-\tfrac{2}{33}$ satisfies.

Ex. 5. *Solve the equation*

$$x^2-3x+12-5\sqrt{x^2-3x+6}=0.$$

Let $\sqrt{x^2-3x+6}=y$. Then $x^2-3x+6=y^2$, and the equation may be written

$$y^2+6-5y=0\,;$$

$$\therefore\ (y-3)(y-2)=0\,;$$

$$\therefore\ y=3 \quad\text{or}\quad 2.$$

I. If $y=2$,

$$x^2-3x+6=4\,;$$

$$\therefore\ x^2-3x+2=0\,;$$

$$\therefore\ (x-1)(x-2)=0\,;$$

$$\therefore\ x=1 \quad\text{or}\quad 2.$$

II. If $y=3$,
$$x^2-3x+6=9\,;$$
$$\therefore\ x^2-3x-3=0\,;$$
$$\therefore\ x=\frac{3\pm\sqrt{9+12}}{2}\,;$$
$$\therefore\ x=\frac{3+\sqrt{21}}{2}\ \text{ or }\ \frac{3-\sqrt{21}}{2}.$$

All these solutions will be found to satisfy the given equation.

Ex. 6. *Which is the greatest and which the least of* $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[6]{5}$?

We have
$$\sqrt{2}=\sqrt[6]{2^3}=\sqrt[6]{8},$$
and
$$\sqrt[3]{3}=\sqrt[6]{3^2}=\sqrt[6]{9}\,;$$
$\therefore$ $\sqrt[3]{3}$ is the greatest and $\sqrt[6]{5}$ is the least.

Ex. 7. *Simplify*
$$\frac{2}{\sqrt{34-\sqrt{931}}}+\frac{1}{\sqrt{14+\sqrt{171}}}-\frac{1}{\sqrt{22-\sqrt{475}}}.$$

Let $\sqrt{34-\sqrt{931}}=\sqrt{x}-\sqrt{y}$, where $x>y$.

Then, as in Ex. 3,
$$x+y=34, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$4xy=931. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Squaring (1) and subtracting (2),
$$x^2-2xy+y^2=1156-931\,;$$
$$\therefore\ (x-y)^2=225\,;$$
$$\therefore\ x-y=15,\ \text{since } x>y.$$

Using (1), we have $x=\frac{49}{2}$, $y=\frac{19}{2}$;
$$\therefore\ \sqrt{34-\sqrt{931}}=\sqrt{\tfrac{49}{2}}-\sqrt{\tfrac{19}{2}}.$$

In the same way $\sqrt{14+\sqrt{171}}=\sqrt{\tfrac{19}{2}}+\sqrt{\tfrac{9}{2}}$

and $\sqrt{22-\sqrt{475}}=\sqrt{\tfrac{25}{2}}-\sqrt{\tfrac{19}{2}}$.

The given expression is, therefore,
$$\frac{2}{\frac{7}{\sqrt{2}}-\sqrt{\frac{19}{2}}}+\frac{1}{\sqrt{\frac{19}{2}}+\frac{3}{\sqrt{2}}}-\frac{1}{\frac{5}{\sqrt{2}}-\sqrt{\frac{19}{2}}}$$

Rationalising each denominator, we obtain

$$\frac{2\left(\dfrac{7}{\sqrt{2}}+\dfrac{\sqrt{19}}{\sqrt{2}}\right)}{\dfrac{49}{2}-\dfrac{19}{2}}+\frac{\dfrac{\sqrt{19}}{\sqrt{2}}-\dfrac{3}{\sqrt{2}}}{\dfrac{19}{2}-\dfrac{9}{2}}-\frac{\dfrac{5}{\sqrt{2}}+\dfrac{\sqrt{19}}{\sqrt{2}}}{\dfrac{25}{2}-\dfrac{19}{2}},$$

that is $$\frac{2}{15}\left(\frac{7}{\sqrt{2}}+\frac{\sqrt{19}}{\sqrt{2}}\right)+\frac{1}{5}\left(\frac{\sqrt{19}}{\sqrt{2}}-\frac{3}{\sqrt{2}}\right)-\frac{1}{3}\left(\frac{5}{\sqrt{2}}+\frac{\sqrt{19}}{\sqrt{2}}\right),$$

that is $$\frac{7\sqrt{2}}{15}-\frac{3\sqrt{2}}{10}-\frac{5\sqrt{2}}{6} \quad\text{or}\quad -\frac{2\sqrt{2}}{3}.$$

Ex. 8. *If* $\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}=0$, *prove that*

$$a^2+b^2+c^2-2ab-2bc-2ca=0.$$

We have $$\sqrt{a}\pm\sqrt{b}=\mp\sqrt{c}.$$

Squaring, $$a+b\pm 2\sqrt{ab}=c\ ;$$

$$\therefore\ a+b-c=\mp 2\sqrt{ab}.$$

Squaring again, $$a^2+b^2+c^2+2ab-2bc-2ca=4ab\ ;$$

$$\therefore\ a^2+b^2+c^2-2ab-2bc-2ca=0.$$

EXERCISES VII.

1. Simplify $2\sqrt{18}$, $3\sqrt{12}$, $\sqrt[3]{16}$, $\sqrt[3]{250}$.

2. Simplify $(2\sqrt{7}-\sqrt{5})(3\sqrt{7}+\sqrt{5})$, $(2\sqrt{3}-\sqrt{6})(2\sqrt{2}+1)$.

3. Simplify

$2\sqrt{24}+\sqrt{54}-2\sqrt{96}+3\sqrt{6}$, $\quad 2\sqrt{75}-\sqrt{147}+\sqrt{50}+2\sqrt{12}-\sqrt{18}$.

4. Which is the greater of each of the following pairs of surds ?

$4\sqrt{7}$, $6\sqrt{3}$; $5\sqrt[3]{2}$, $3\sqrt[3]{9}$; $3\sqrt[4]{3}$, $\sqrt{15}$.

5. Express with rational denominators

$$\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{6}+5},\quad \frac{3\sqrt{5}+4\sqrt{2}}{3\sqrt{2}+2\sqrt{5}},\quad \frac{1}{\sqrt{6}-\sqrt{3}+\sqrt{2}},\quad \frac{1}{a+\sqrt{b}},\quad \frac{c+\sqrt{d}}{\sqrt{a}-\sqrt{b}}.$$

6. Simplify $\dfrac{2\sqrt{3}}{\sqrt{3}+\sqrt{6}}+\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{2}}+\dfrac{1}{\sqrt{3}-\sqrt{2}}$.

7. Find the square root of

$7+2\sqrt{10}$, $49+20\sqrt{6}$, $9+\sqrt{77}$, $14+\sqrt{171}$, $22-\sqrt{475}$.

8. Solve the equations and test the solutions as in Ex. 4, p. 85.

$$\sqrt{x+2}-\sqrt{x-3}=\sqrt{3x-20},$$

$$\sqrt{3x+4}+\sqrt{2x+3}=\sqrt{3-x},$$

$$\sqrt{2x+3}-2\sqrt{4-x}=\sqrt{10-3x},$$

$$\sqrt{x+p}+\sqrt{x+q}=\sqrt{p-q}.$$

9. Solve the equations and test the solutions.

$$x^2-9x+29-5\sqrt{x^2-9x+23}=0,$$

$$x^2-x-9\sqrt{x^2-x+10}+30=0,$$

$$2x^2-4x+\sqrt{x^2-2x-4}=9.$$

10. Solve the equation $\sqrt{x+2}-\sqrt{2x+5}+1=0$. [I.M.E.]

11. If $\sqrt{P}+\sqrt{Q}+\sqrt{R}=0$, prove that

$$P^2+Q^2+R^2=2(QR+RP+PQ).$$

But if $P^2+Q^2+R^2-2QR-2RP-2PQ=0$,

then $\sqrt{P}\pm\sqrt{Q}\pm\sqrt{R}=0$. [L.U.]

12. If $a=p+\sqrt{2p^2+b}$, find possible values for p if $a=9{\cdot}5$ and $b=100$.

If $a=kb$, find the greatest and least possible values of k that p may have a real value. [L.U.]

13. Simplify and express with rational denominators where necessary:

(a) $(\sqrt{a}-2\sqrt{b})(\sqrt{ab})$, (b) $\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)$,

(c) $\sqrt{x^4+x^2y^2}+\sqrt{x^2y^2+y^4}$, (d) $\dfrac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$.

14. Solve the equation

$$2\sqrt{x+1}-3\sqrt{2x-5}=\sqrt{x-2}.$$ [L.U.]

15. Solve the equation

$$\sqrt{x+1}+\sqrt{2x+6}=3\sqrt{3x-1}.$$

Test all the values you find for this equation, and explain why they are not all solutions. [L.U.]

16. Prove that if $a+\sqrt{p}=b+\sqrt{q}$, where a, b, p, q are rational numbers, and p, q are not perfect squares, then

$$a=b, \quad p=q.$$

Find in its simplest form the mean proportional between

$$2\sqrt{7}+3\sqrt{3} \quad \text{and} \quad \sqrt{7}-\sqrt{3}. \qquad \text{[S.W.]}$$

17. How can we rationalise the denominator of the fraction

$$\frac{1}{a+\sqrt{b}+\sqrt{c}}\ ?$$

Find the numerical value of this fraction to three decimal places when $a=2$, $b=3$, $c=5$. [S.W.]

18. Solve the equation $y^4-10y^2+1=0$.

The Laws of Indices. The fundamental law is

$$a^m \times a^n = a^{m+n} \text{..........................I.}$$

This is intelligible when m and n are positive integers.

By assuming I. to hold generally whatever the values of m and n may be, meanings may be assigned to a^0, $a^{\frac{p}{q}}$ and a^{-n}, where p, q and n are positive integers as follows :

Ex. 1. *Find the meaning of* a^0.

We have $a^m \times a^n = a^{m+n}$ always.

Putting $m=0$, we have

$$a^0 \times a^n = a^{0+n} = a^n.$$

So that a^0 is such that if it is multiplied by a^n, the result is still a^n ;

$$\therefore\ a^0 = 1.$$

This is true if a is any number.

Thus $\qquad 10^0=1, \quad (-2)^0=1, \quad \left(\frac{1}{\sqrt{3}}\right)^0=1.$

Ex. 2. *Find the meaning of* $a^{\frac{p}{q}}$ *where* p *and* q *are positive integers.*

We have $a^{m_1} \times a^{m_2} = a^{m_1+m_2}$ always ;

$$\therefore\ a^{m_1} \times a^{m_2} \times a^{m_3} = a^{m_1+m_2} \times a^{m_3} = a^{m_1+m_2+m_3}.$$

and so on.

Clearly $a^{m_1} \times a^{m_2} \times a^{m_3} \ldots$ to q factors $= a^{m_1+m_2+m_3+\ \ldots \text{ to } q \text{ terms}}$.

Let $m_1 = m_2 = m_3 = \ldots = \dfrac{p}{q}$.

Then $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots$ to q factors $= a^{\frac{p}{q}+\frac{p}{q}+\ldots \text{ to } q \text{ terms}}$

$$= a^{q \cdot \frac{p}{q}}$$

$$= a^p;$$

$$\therefore \left(a^{\frac{p}{q}}\right)^q = a^p.$$

Taking the q^{th} root of each side, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Thus

$$10^{\frac{2}{3}} = \sqrt[3]{10^2}, \quad (-2)^{\frac{4}{5}} = \sqrt[5]{(-2)^4}, \quad (-\tfrac{1}{3})^{\frac{4}{3}} = \sqrt[3]{(-\tfrac{1}{3})^4},$$

and conversely.

Ex. 3. *Find the meaning of a^{-n} when n is positive.*

We have $a^m \times a^n = a^{m+n}$ always.

Put $m = -n$, then $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$;

$$\therefore a^{-n} = 1/a^n.$$

Thus
$$(2)^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}},$$

$$(x)^{-0\cdot 7} = \frac{1}{x^{0\cdot 7}} = \frac{1}{\sqrt[10]{x^7}},$$

and conversely.

Other important laws of indices in constant use are:

$$(a^m)^n = a^{mn}, \ldots\ldots\ldots\ldots \text{II.}$$

and $$(ab)^n = a^n b^n \ldots\ldots\ldots\ldots \text{III.}$$

Although Law I. has been used to find the meanings of fractional and negative indices, it is important to notice that such meanings also follow from Law II.

For example, using Law II., $(a^{\frac{1}{2}})^{-3}$ should be $(a)^{-\frac{3}{2}}$.

Now, by the preceding examples, we have

$$(a^{\frac{1}{2}})^{-3} = \frac{1}{(a^{\frac{1}{2}})^3} = \frac{1}{a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}}} = \frac{1}{a^{\frac{3}{2}}} = a^{-\frac{3}{2}}.$$

Ex. 4. *Simplify* $\sqrt{p^2q^{-3}} \times (p^{-5}q^2x^{-1})^{-\frac{1}{2}} \div (x^{-2}p^{-2}q^4)^{-\frac{1}{4}}$.

Using the index laws, the expression is

$$pq^{-\frac{3}{2}} \times p^{\frac{5}{2}}q^{-1}x^{\frac{1}{2}} \div x^{\frac{1}{2}}p^{\frac{1}{2}}q^{-1}\,;$$

that is, $$p^{\frac{7}{2}}q^{-\frac{5}{2}}x^{\frac{1}{2}} \div x^{\frac{1}{2}}p^{\frac{1}{2}}q^{-1}\,;$$

that is, $$p^3q^{-\frac{3}{2}}$$

or $$\frac{p^3}{q\sqrt{q}}.$$

Ex. 5. *If* $x^a \times x^b = x^{a+b}$ *for all values of* a, b *and* x, *deduce meanings for* $x^{1\cdot 7}$ *and* $(x^{1\cdot 7})^3$.

Write in symbols the result of the following operations on x *and* y; *multiply* x *by the cube root of* y *and divide the product by the sum of the fourth powers of the cube roots of* x *and* y, *diminished by the product of the cube roots of the squares of* x *and* y.

Multiply the numerator and denominator by the sum of the cube roots of the squares of x *and* y *and simplify the result, expressing it in words. Obtain the value of the fraction when* $x = 8y$. [L.U.]

Since $x^a \times x^b = x^{a+b}$ for all values of a and b,

$$x^{1\cdot 7} \times x^{1\cdot 7} \times x^{1\cdot 7} \times \ldots \text{ to 10 factors} = x^{1\cdot 7+1\cdot 7+\ldots \text{ to 10 terms}}$$
$$= x^{17}\,;$$

therefore, taking the 10th root of each side,

$$x^{1\cdot 7} = \sqrt[10]{x^{17}}.$$

Again $$(x^{1\cdot 7})^3 = x^{1\cdot 7} \times x^{1\cdot 7} \times x^{1\cdot 7}$$
$$= x^{5\cdot 1}$$
$$= x^{\frac{51}{10}}$$
$$= \sqrt[10]{x^{51}}.$$

The first operation is expressed symbolically as

$$x \times \sqrt[3]{y} \quad \text{or} \quad xy^{\frac{1}{3}}.$$

This has to be divided by

$$(\sqrt[3]{x})^4 + (\sqrt[3]{y})^4 - \sqrt[3]{x^2} \times \sqrt[3]{y^2} \quad \text{or} \quad x^{\frac{4}{3}} + y^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}}.$$

The result of this operation is

$$\frac{xy^{\frac{1}{3}}}{x^{\frac{4}{3}}+y^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}}. \quad \text{........................(1)}$$

The sum of the cube roots of the squares of x and y is $x^{\frac{2}{3}}+y^{\frac{2}{3}}$. Multiplying numerator and denominator of (1) by this, we have

$$\frac{xy^{\frac{1}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})}{(x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})(x^{\frac{2}{3}}+y^{\frac{2}{3}})},$$

$$i.e. \qquad \frac{x^{\frac{5}{3}}y^{\frac{1}{3}}+xy}{x^{\frac{6}{3}}+y^{\frac{6}{3}}}$$

$$\text{or} \qquad \frac{x^{\frac{5}{3}}y^{\frac{1}{3}}+xy}{x^2+y^2}.$$

In words this is, add the product of x and y to the product of the cube root of y and the cube root of x to the power of 5, and divide the sum by the sum of the squares of x and y.

When $x=8y$, we have $x^{\frac{5}{3}}=(8y)^{\frac{5}{3}}=8^{\frac{5}{3}}y^{\frac{5}{3}}$

$$=2^5y^{\frac{5}{3}}.$$

The expression becomes

$$\frac{32y^2+8y^2}{64y^2+y^2}, \quad i.e. \ \frac{40}{65} \ \text{or} \ \frac{8}{13}.$$

Ex. 6. *Divide* $\sqrt[3]{x^2}+\sqrt[3]{y^2}-\sqrt[3]{z^2}+2\sqrt[3]{xy}$ *by* $\sqrt[3]{x}+\sqrt[3]{y}-\sqrt[3]{z}$.

Using indices, the dividend $=x^{\frac{2}{3}}+y^{\frac{2}{3}}-z^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}$

$$=(x^{\frac{1}{3}}+y^{\frac{1}{3}})^2-z^{\frac{2}{3}}$$

$$=(x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}}-z^{\frac{1}{3}}).$$

The divisor is $x^{\frac{1}{3}}+y^{\frac{1}{3}}-z^{\frac{1}{3}}$;

$\therefore$ the quotient is $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

Ex. 7. *Find the square root of*

$$x^{\frac{3}{4}}+6x^{\frac{1}{2}}+7x^{\frac{1}{4}}+3+x^{-\frac{1}{4}}-2x^{-\frac{1}{8}}-6x^{\frac{1}{8}}-4x^{\frac{5}{8}}-6x^{\frac{3}{8}}.$$

Arranging in descending powers of x and finding the square root in the ordinary way, the work is shown as follows:

$$\begin{array}{r|l}
 & x^{\frac{3}{8}} - 2x^{\frac{1}{4}} + x^{\frac{1}{8}} - 1 + x^{-\frac{1}{8}} \\
\hline
 & x^{\frac{3}{4}} - 4x^{\frac{5}{8}} + 6x^{\frac{1}{2}} - 6x^{\frac{3}{8}} + 7x^{\frac{1}{4}} - 6x^{\frac{1}{8}} + 3 - 2x^{-\frac{1}{8}} + x^{-\frac{1}{4}} \\
 & x^{\frac{3}{4}} \\
\hline
2x^{\frac{3}{8}} - 2x^{\frac{1}{4}} & -4x^{\frac{5}{8}} + 6x^{\frac{1}{2}} \\
 & -4x^{\frac{5}{8}} + 4x^{\frac{1}{2}} \\
\hline
2x^{\frac{3}{8}} - 4x^{\frac{1}{4}} + x^{\frac{1}{8}} & 2x^{\frac{1}{2}} - 6x^{\frac{3}{8}} + 7x^{\frac{1}{4}} \\
 & 2x^{\frac{1}{2}} - 4x^{\frac{3}{8}} + x^{\frac{1}{4}} \\
\hline
2x^{\frac{3}{8}} - 4x^{\frac{1}{4}} + 2x^{\frac{1}{8}} - 1 & -2x^{\frac{3}{8}} + 6x^{\frac{1}{4}} - 6x^{\frac{1}{8}} + 3 \\
 & -2x^{\frac{3}{8}} + 4x^{\frac{1}{4}} - 2x^{\frac{1}{8}} + 1 \\
\hline
2x^{\frac{3}{8}} - 4x^{\frac{1}{4}} + 2x^{\frac{1}{8}} - 2 + x^{-\frac{1}{8}} & 2x^{\frac{1}{4}} - 4x^{\frac{1}{8}} + 2 - 2x^{-\frac{1}{8}} + x^{-\frac{1}{4}} \\
 & 2x^{\frac{1}{4}} - 4x^{\frac{1}{8}} + 2 - 2x^{-\frac{1}{8}} + x^{-\frac{1}{4}} \\
\hline
\end{array}$$

The square root is $x^{\frac{3}{8}} - 2x^{\frac{1}{4}} + x^{\frac{1}{8}} - 1 + x^{-\frac{1}{8}}$.

EXERCISES VIII.

1. Simplify and express with positive indices

$$a^{\frac{5}{8}} \times a^{-\frac{1}{4}}, \quad a \times a^{-\frac{2}{3}} \times a^{-\frac{1}{2}}, \quad 4a^{\frac{5}{3}} \div 2a^{-\frac{2}{3}}, \quad 3x^{\frac{2}{3}}y^{-\frac{1}{2}} \div 6x^{-\frac{1}{3}}y^{-\frac{3}{2}}.$$

2. Simplify and express with positive indices

$$\left(\frac{a^4}{b^3}\right)^{-\frac{1}{2}} \times \left(\frac{c^{-\frac{3}{2}}}{a^{-\frac{1}{2}}}\right)^4 \div (a^{-1}b^2c^{-3})^2, \quad \sqrt{x^{-\frac{1}{2}}y^{-1}} \,.\, (x^{\frac{5}{6}}y^{\frac{1}{2}}) \,.\, \sqrt{x^{-1}y^{\frac{2}{3}}},$$

$$(p^2 - q^2)^{\frac{1}{2}}(p - q)^{\frac{3}{2}}(p + q)^{-\frac{1}{2}}, \quad \frac{a^{x(y-z)}}{a^{(x-z)y}} \cdot \left[\frac{a^{x(1-y)}}{a^{y(1-x)}}\right]^z$$

3. Simplify and express with positive indices

$$\left[\frac{a^{\frac{1}{p}}\, b^p}{a^{\frac{1}{q}}\, b^q}\right]^{\frac{1}{p-q}} \times \sqrt[pq]{a}, \quad \left[\sqrt[3]{\frac{x^{-\frac{1}{2}}y^2}{xy^{\frac{1}{2}}}} \times \sqrt{\frac{yx^{-1}}{\sqrt{xy^2}}}\right]^2.$$

4. Expand the following and arrange each result in descending powers of x:

$$(3x^{\frac{2}{3}}+2)(3x^{\frac{1}{3}}-2x^{-\frac{1}{3}}), \quad (x^a-x^{-a})^3, \quad (x+1+x^{-1})^2,$$
$$(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{4}}+y^{\frac{1}{4}})(x^{\frac{1}{4}}-y^{\frac{1}{4}}), \quad (x^{\frac{4}{3}}+2^{\frac{1}{3}}x^{\frac{2}{3}}+4^{\frac{1}{3}})(x^{\frac{2}{3}}-2^{\frac{1}{3}}).$$

5. Multiply $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$ by $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{1}{3}}a^{\frac{1}{3}}$.

6. Simplify $(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}+c^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}})(b^{\frac{1}{2}}+c^{\frac{1}{2}}-a^{\frac{1}{2}})$.

7. Write down the values of

$$\frac{x-y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}, \quad \frac{a-1}{a^{\frac{1}{3}}-1}, \quad \frac{x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}{x-y}.$$

8. Divide $p^{\frac{5}{2}}-p^{\frac{3}{2}}q+pq^{\frac{3}{2}}-2p^{\frac{1}{2}}q^2+q^{\frac{5}{2}}$ by $p^{\frac{3}{2}}-pq^{\frac{1}{2}}+p^{\frac{1}{2}}q-q^{\frac{3}{2}}$.

9. Divide

$$a^{\frac{5}{2}}-5a^2b^{\frac{1}{3}}+10a^{\frac{3}{2}}b^{\frac{2}{3}}-10ab+5a^{\frac{1}{2}}b^{\frac{4}{3}}-b^{\frac{5}{3}} \text{ by } a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.$$

10. Find the square roots of

$$4a+5+a^{-1}-12\sqrt{a}+6a^{-\frac{1}{2}},$$
$$4x^2y^{-2}-12xy^{-1}+25-24x^{-1}y+16x^{-2}y^2,$$
$$4x^{\frac{2}{3}}y^{-\frac{1}{2}}+12x^{\frac{1}{3}}y^{-\frac{1}{4}}+29+30x^{-\frac{1}{3}}y^{\frac{1}{4}}+25x^{-\frac{2}{3}}y^{\frac{1}{2}}.$$

11. Express in its simplest form

$$\sqrt{\left\{\frac{9a^2-12a^{\frac{3}{2}}-2a+4a^{\frac{1}{2}}+1}{1-4a^{\frac{1}{2}}+6a-4a^{\frac{3}{2}}+a^2}\right\}}.$$

12. Express the following with rational denominators:

$$\frac{1}{\sqrt[3]{2}-1}, \quad \frac{1}{\sqrt[3]{9}+\sqrt[3]{3}+1}, \quad \frac{a}{b^{\frac{1}{3}}+c^{\frac{1}{3}}}, \quad \frac{x}{y^{\frac{1}{4}}-z^{\frac{1}{4}}}, \quad \frac{1}{a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}}$$

13. (1) Find the square root of

$$(x+1)^2(x+2)^2+(x+2)^2(2x+3)^2+(2x+3)^2(x+1)^2.$$

(2) Simplify the expression

$$\sqrt[3]{ab^2c^4}\times\sqrt{a^3b^{-1}c^3}\div ab^{\frac{1}{2}}c^{\frac{5}{6}}.$$

[I.C.E.]

14. If $x^{2\cdot4}=a^{0\cdot6}y^{1\cdot8}$ and $y^3=az^2$, find the simplest relation between x, z and a only.

[I.C.E.]

15. Starting from the fundamental index law for positive integral indices, namely $x^m \times x^n = x^{m+n}$, shew how to find meanings for negative and fractional indices. Illustrate by finding meanings for $x^{\frac{4}{5}}$, x^0 and x^{-5}.

Extract the square root of

$$x^{\frac{7}{5}} - 4x^{\frac{6}{5}} + 10x - 12x^{\frac{4}{5}} + 9x^{\frac{3}{5}}. \qquad \text{[L.U.]}$$

Common Logarithms. If x is a positive number, the common logarithm of x (written $\log x$) is the power to which 10 is raised to produce x, *i.e.*

$$10^{\log x} = x.$$

Similarly, if y is a positive number,

$$10^{\log y} = y.$$

And

$$\begin{aligned} 10^{\log xy} &= xy \\ &= 10^{\log x} \times 10^{\log y} \\ &= 10^{\log x + \log y}. \quad \text{(By the laws of indices.)} \end{aligned}$$

$$\therefore \ \log xy = \log x + \log y. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Again,

$$\begin{aligned} 10^{\log \frac{x}{y}} &= \frac{x}{y} \\ &= \frac{10^{\log x}}{10^{\log y}} \\ &= 10^{\log x - \log y}. \quad \text{(By the laws of indices.)} \end{aligned}$$

$$\therefore \ \log \frac{x}{y} = \log x - \log y. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Also, if n is any number,

$$\begin{aligned} 10^{\log x^n} &= x^n \\ &= (10^{\log x})^n \\ &= 10^{n \log x}. \quad \text{(By the laws of indices.)} \end{aligned}$$

$$\therefore \ \log x^n = n \log x. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

A detailed introduction to the use of logarithms will be found in the authors' *Practical Mathematics for Technical Students*, Pt. I., Ch. VII.

A logarithm consists of two parts: the decimal part, or **mantissa**; the integral part, or **characteristic**. The tables give the mantissae; the characteristics are written down by inspection.

Study the following:

Number.	Which may be written.	The logarithm is	The most convenient form for the logarithm is
490700	$4{\cdot}907 \times 10^{5}$	$0{\cdot}6908 + 5$	$5{\cdot}6908$
4907	$4{\cdot}907 \times 10^{3}$	$0{\cdot}6908 + 3$	$3{\cdot}6908$
49·07	$4{\cdot}907 \times 10^{1}$	$0{\cdot}6908 + 1$	$1{\cdot}6908$
0·4907	$4{\cdot}907 \times 10^{-1}$	$0{\cdot}6908 - 1$	$\bar{1}{\cdot}6908$
0·04907	$4{\cdot}907 \times 10^{-2}$	$0{\cdot}6908 - 2$	$\bar{2}{\cdot}6908$
0·0004907	$4{\cdot}907 \times 10^{-4}$	$0{\cdot}6908 - 4$	$\bar{4}{\cdot}6908$
etc.	etc.	etc.	etc.

Logarithm of number.	Significant figures of number.	Result as number between 1 and 10 multiplied by power of 10.	Alternative form of result.
$7{\cdot}2576$	1809	$1{\cdot}809 \times 10^{7}$	18090000
$4{\cdot}2576$		$1{\cdot}809 \times 10^{4}$	18090
$1{\cdot}2576$		$1{\cdot}809 \times 10^{1}$	18·09
$0{\cdot}2576$		$1{\cdot}809 \times 10^{0}$	1·809
$\bar{1}{\cdot}2576$		$1{\cdot}809 \times 10^{-1}$	0·1809
$\bar{3}{\cdot}2576$		$1{\cdot}809 \times 10^{-3}$	0·001809
etc.		etc.	etc.

The characteristic of the logarithm of a number N is evidently found as follows:

If $N > 1$, the characteristic is positive, and is one less than the number of digits before the decimal point.

If $N < 1$, the characteristic is negative, and is numerically one more than the number of zeros between the decimal point and the first significant figure.

N.B.—The mantissa is always positive.

After using 4-figure tables, the result cannot be relied on as accurate beyond 3 significant figures.

Ex. 1. *Evaluate* 89·08 × 0·0003771.

Approximate result is 100 × 0·0004 or 0·04.

$$\begin{aligned} \log 89{\cdot}08 &= 1{\cdot}9498 \\ \log\ 0{\cdot}0003771 &= \bar{4}{\cdot}5764 \end{aligned}$$

Adding, we get

$$\log \text{product} = \bar{2}{\cdot}5262$$
$$\therefore\ \text{product} = 0{\cdot}0336$$

The whole-number part of the sum of the logarithms is 1 − 4 + 1 (carried), *i.e.* − 2 : the answer therefore lies between 0·01 and 0·1.

Ex. 2. *Evaluate* 3771 ÷ 8·987.

Result is, roughly, 4000/10, *i.e.* 400.

$$\begin{aligned} \log 3771 &= 3{\cdot}5764 \\ \log 8{\cdot}987 &= 0{\cdot}9536 \end{aligned}$$

Subtracting, we get

$$\log \text{quotient} = 2{\cdot}6228$$
$$\therefore\ \text{quotient} = 420.$$

The whole-number part of the logarithm is here 2. Therefore the quotient required lies between 100 and 1000.

Ex. 3. *Evaluate* 4·001 ÷ 778·3.

Result ≑ 4/800, *i.e.* $\frac{1}{200}$ or 0·005.

$$\begin{aligned} \log 4{\cdot}001 &= 0{\cdot}6022 \\ \log 778{\cdot}3 &= 2{\cdot}8913 \end{aligned}$$

Subtracting, we get

$$\log \text{quotient} = \bar{3}{\cdot}7109$$
$$\therefore\ \text{quotient} = 0{\cdot}00514.$$

Ex. 4. *Evaluate* 0·08724 × 36·85 ÷ 791·4.

Result ≑ 0·09 × 37 ÷ 800, *i.e.* 3·33 ÷ 800 ≑ 0·004.

$$\begin{aligned} \log 0{\cdot}08724 &= \bar{2}{\cdot}9407 \\ \log 36{\cdot}85 &= 1{\cdot}5664 \\ \hline \log \text{product} &= 0{\cdot}5071 \\ \log 791{\cdot}4 &= 2{\cdot}8984 \\ \hline \log \text{result} &= \bar{3}{\cdot}6087 \\ \text{Result} &= 0{\cdot}00406. \end{aligned}$$

Ex. 5. *Evaluate* $(31{\cdot}17)^3$.

Result ≑ $(30)^3$ or 27000.

We have $\log (31{\cdot}17)^3 = 3 \times (1{\cdot}4938) = 4{\cdot}4814 = \log 30330$;

$$\therefore\ (31{\cdot}17)^3 = 30300.$$

Ex. 6. *Find the values of*

$$\sqrt{2407},\quad \sqrt{2\cdot407},\quad \sqrt{0\cdot2407}\quad \textit{and}\quad \sqrt{0\cdot0002407}.$$

Number.	Logarithm.	Half the logarithm.	Square root of numbers.
2407	$3\cdot3814$	$1\cdot6907$	$49\cdot1$
$2\cdot407$	$0\cdot3814$	$0\cdot1907$	$1\cdot55$
$0\cdot2407$	$\bar{1}\cdot3814$	$*\bar{1}\cdot6907$	$0\cdot491$
$0\cdot0002407$	$\bar{4}\cdot3814$	$\bar{2}\cdot1907$	$0\cdot0155$

Ex. 7. *Evaluate* $(0\cdot08005)^{1\cdot7}$.

The result is approximately $(0\cdot1)^2$ or $0\cdot01$.

$$\log 0\cdot08005=\bar{2}\cdot9034$$

and

$$1\cdot7\times\bar{2}\cdot9034=1\cdot7\times0\cdot9034+1\cdot7\times\bar{2}$$
$$=0\cdot9034+0\cdot6324-3\cdot4.$$

Here we evaluate 1 times $0\cdot9034$ plus $0\cdot7$ times $0\cdot9034$ plus $1\cdot7$ times -2, obtaining $1\cdot5358-3\cdot4$;

$$\therefore\ \log\text{ result}=\bar{2}\cdot1358=\log(0\cdot01368);$$
$$\therefore\ \text{result}=0\cdot0137.$$

Ex. 8. *Compute* $97\cdot43\div(0\cdot3524\times6\cdot321)^{2\cdot56}$.

Result is approximately the same as

$$100\div(0\cdot4\times6)^3,$$
$$i.e.\quad 100\div(2\cdot4)^3,$$
$$\doteqdot 100\div10\quad\text{or}\quad 10.$$

$$\log\text{ result}=1\cdot9887-2\cdot56(\bar{1}\cdot5470+0\cdot8008)$$
$$=1\cdot9887-2\cdot56\times0\cdot3478$$
$$=1\cdot9887-0\cdot8903$$
$$=1\cdot0984;$$
$$\therefore\ \text{result}=12\cdot5.$$

$$\begin{array}{r} 34\cancel{7}\cancel{8} \\ 256 \\ \hline 6956 \\ 1739 \\ 208 \\ \hline 8903 \end{array}$$

Ex. 9. *Evaluate* $(0\cdot002371)^{-1\cdot47}$.

$$\log\text{ result}=-1\cdot47\times\log 0\cdot002371$$
$$=-1\cdot47\times\bar{3}\cdot3749$$
$$=4\cdot41-1\cdot47\times0\cdot3749$$
$$=4\cdot41-0\cdot5511$$
$$=3\cdot8589;$$
$$\therefore\ \text{result}=7230.$$

$$\begin{array}{r} 37\cancel{4}\cancel{9} \\ 147 \\ \hline 3749 \\ 1500 \\ 262 \\ \hline 5511 \end{array}$$

* *Note.*—$\frac{1}{2}(\bar{1}\cdot3814)=\frac{1}{2}(\bar{2}+1\cdot3814)$
$=\bar{1}+0\cdot6907$
$=\bar{1}\cdot6907.$

EXERCISES IX.

1. Using logarithms, find the value of

(i) $1{\cdot}234 \times 6{\cdot}278$; (ii) $0{\cdot}0827 \times 0{\cdot}008362$;
(iii) $5{\cdot}678 \div 1{\cdot}879$; (iv) $0{\cdot}6287 \div 0{\cdot}8274$;
(v) $2{\cdot}821 \times 0{\cdot}699 \div 24{\cdot}3$.

2. Using logarithms, find the value of

(*a*) $\dfrac{9{\cdot}75 \times 10{\cdot}3 \times 9{\cdot}2}{(1{\cdot}45)^3}$; (*b*) $\sqrt{7{\cdot}123}$; (*c*) $(7{\cdot}6)^{1{\cdot}41}$;
(*d*) $(2{\cdot}354)^3 - (2{\cdot}354)^2 + 1$.

3. Compute, using logarithms,

$\sqrt{2{\cdot}607}$ and $26{\cdot}07^{1{\cdot}13}$. [B.E.]

4. Compute, using logarithms,

$\sqrt{84{\cdot}05 \times 0{\cdot}1357 \div 1{\cdot}163}$. [B.E.]

5. Extract the square root of 17840, 1·784, 0·0001784. [B.E.]

6. What is the $\frac{3}{2}$ power of 1·62 ? *i.e.* raise 1·62 to the power of $\frac{3}{2}$. [B.E.]

7. Compute, using logarithms,

$\sqrt{12{\cdot}13 \times 5{\cdot}621}$ and $\sqrt[3]{18{\cdot}92 \div 2{\cdot}753}$. [B.E.]

8. Compute, using logarithms,

$\sqrt{8{\cdot}405 \times 22{\cdot}67 \div 8{\cdot}967}$. [B.E.]

9. Using logarithms, find the cube root of $8782 \div 3{\cdot}152$. [B.E.]

10. Compute

$30{\cdot}56 \div 4{\cdot}105$, $0{\cdot}03056 \times 0{\cdot}4105$, $(4{\cdot}105)^{1{\cdot}23}$, $(0{\cdot}04105)^{-2{\cdot}3}$.

The answers must be right to three significant figures.
Why do we multiply log a by b to obtain the logarithm of a^b ? [B.E.]

11. Compute $2{\cdot}307^{0{\cdot}65}$ and $23{\cdot}07^{-1{\cdot}25}$, the answers to consist of three significant figures.
Why do we add logarithms to obtain the logarithm of a product ? [B.E.]

12. Compute, using logarithms,

$\sqrt[3]{37{\cdot}24}$, $\sqrt[2]{3{\cdot}724}$, $372{\cdot}4^{2{\cdot}43}$, $0{\cdot}3724^{-2{\cdot}43}$. [B.E.]

13. Evaluate $\sqrt[3]{2{\cdot}354 \times 1{\cdot}607}$ and $(32{\cdot}15)^{1{\cdot}52}$. [B.E.]

14. Find, using logarithms, the value of

(i) $\sqrt{1{\cdot}234}$; (ii) $\dfrac{2{\cdot}32 \times 1{\cdot}27 \times (0{\cdot}642)^2}{(0{\cdot}913)^2;}$;

(iii) $(2{\cdot}124)^{2{\cdot}16}$.

15. Find the value of x from the following:

$$x = \sqrt{\frac{(0{\cdot}25)^5 \times 10{\cdot}14 \times 0{\cdot}308}{250 \times 0{\cdot}000414}}.$$

16. Using logarithms, compute

$$(1{\cdot}342 \times 0{\cdot}01731 \div 0{\cdot}0274)^{0{\cdot}317}.$$

Explain why we multiply a logarithm by 3 when we wish to find the cube of a number. [B.E.]

17. Compute, using logarithms,

$$\sqrt{0{\cdot}2354 \times 16{\cdot}07},\ (32{\cdot}15)^{0{\cdot}152},\ (32{\cdot}15)^{-0{\cdot}152}.$$

Explain why we add logarithms when we wish to multiply numbers. [B.E.]

Illustrations of the Use of Common Logarithms. It should now be easy to write formulae in logarithmic form, that is, in a form adapted for logarithmic calculation. The only operations which can be carried out by logarithms are those of multiplication, division, involution and evolution (*i.e.* taking roots and raising to powers). The operations of addition and subtraction cannot be performed in this way. Thus, if $\dfrac{a+b}{c}$ has to be evaluated, the sum of a and b must be found first and the logarithm of c subtracted from the logarithm of this sum to obtain the logarithm of the expression.

Ex. 1. *Evaluate* $ax^{1{\cdot}46} + bx^{2{\cdot}5}$ *when* a *is* 3·712 *and* b *is* 0·2178 *for the following values of* x: 5, 6. 7, 8, 9, 10.

Each term of the expression $3{\cdot}712x^{1{\cdot}46} + 0{\cdot}2178x^{2{\cdot}5}$ must be calculated by the use of logarithms and the results added.

It will be convenient to make a table giving the values of x, $\log x$, $1{\cdot}46 \log x$ and $2{\cdot}5 \log x$.

x	log x	1·46 log x		2·5 log x	
5	0·6990	0·6990 1·46 6990 27960 4194 1·02054	1·0205	0·6990 2·5 1·3980 34950 1·74750	1·7475
6	0·7782	0·7782 1·46 7782 31128 4669 1·13617	1·1362	0·7782 2·5 1·5564 38910 1·94550	1·9455
7	0·8451	0·8451 1·46 8451 33804 5071 1·23385	1·2339	0·8451 2·5 1·6902 42255 2·11275	2·1128
8	0·9031	0·9031 1·46 9031 36124 5419 1·31853	1·3185	0·9031 2·5 1·8062 45155 2·25775	2·2578
9	0·9542	0·9542 1·46 9542 38168 5725 1·39313	1·3931	0·9542 2·5 1·9084 47710 2·38550	2·3855
10	1		1·46		2·5

Adding 0·5696, *i.e.* log 3·712 to each number in column 3, and $\bar{1}$·3381, *i.e.* log 0·2178 to each number in column 4, the following table is obtained :

x	log 1st term	log 2nd term	1st term $=3{\cdot}712x^{1{\cdot}46}$	2nd term $=0{\cdot}2178x^{2{\cdot}5}$	v
5	1·5901	1·0856	38·9	12·2	51·1
6	1·7058	1·2836	50·8	19·2	70·0
7	1·8035	1·4509	63·6	28·2	91·8
8	1·8881	1·5959	77·3	39·4	116·7
9	1·9627	1·7236	91·8	52·9	144·7
10	2·0296	1·8381	107·0	68·9	175·9

Ex. 2. *In any class of turbine, if P is the power of the waterfall and H the height of the fall and n the rate of revolution, then it is known that for any particular class of turbines of all sizes,*

$$n \propto H^{1\cdot25}P^{-0\cdot5}.$$

In the list of a particular maker I take a turbine at random for a fall of 6 feet, 100 horse-power, 50 revolutions per minute. By means of this I find I can calculate n for all the other turbines of the list. Find n for a fall of 20 feet and 75 horse-power. [B.E.]

Since $n \propto H^{1\cdot25}P^{-0\cdot5}$, we may write

$$n = k \,.\, \mathrm{H}^{1\cdot25}P^{-0\cdot5}, \text{ where } k \text{ does not vary.}$$

Now, when $\qquad H=6,\ P=100,\ n=50;$

$$\therefore\ 50 = k \,.\, 6^{1\cdot25}100^{-0\cdot5} \quad \text{.............................(1)}$$

We have to find n when $H=20$ and $P=75$.

Hence $$n = k \,.\, 20^{1\cdot25}75^{-0\cdot5}. \quad \text{..................(2)}$$

Dividing (2) by (1), $$\frac{n}{50} = \frac{20^{1\cdot25}75^{-0\cdot5}}{6^{1\cdot25}100^{-0\cdot5}}.$$

Taking logarithms,

$$\log n - \log 50 = 1\cdot25 \log 20 - 0\cdot5 \log 75 - [1\cdot25 \log 6 - 0\cdot5 \log 100],$$

$$\begin{aligned} \textit{i.e. } \log n - 1\cdot6990 &= 1\cdot25(\log 20 - \log 6) - 0\cdot5(\log 75 - \log 100) \\ &= 1\cdot25(1\cdot3010 - 0\cdot7782) - 0\cdot5(1\cdot8751 - 2) \\ &= 1\cdot25(0\cdot5228) - 0\cdot5(\bar{2} + 1\cdot8751) \\ &= 0\cdot5228 + 0\cdot1307 - (\bar{1}\cdot9376) \\ &= 0\cdot6535 - \bar{1}\cdot9376 \\ &= 0\cdot7159; \end{aligned}$$

$$\therefore\ \log n = 0\cdot7159 + 1\cdot6990 = 2\cdot4149 = \log 259\cdot9;$$

$$\therefore\ n = 260 \text{ approximately.}$$

Ex. 3. *What is the idea on which compound interest is calculated? Explain, as if to a beginner, how it is that*

$$A = P\left(1 + \frac{r}{100}\right)^n,$$

where P is the money lent, and A is what it amounts to in n years at r per cent. per annum.

If A is 130, and P is 100, and n is 7, find r.

r per cent. per annum means that £100 amounts to £$(100 + r)$ in 1 year, *i.e.* that £1 amounts to £$\left(1 + \frac{r}{100}\right)$ in 1 year;

$$\therefore \ \text{£}P \text{ amounts to } \text{£}P\left(1 + \frac{r}{100}\right) \text{ in 1 year.}$$

This amount earns interest at the rate of r % per annum for the second year.

$\therefore$ £$P\left(1 + \frac{r}{100}\right)$ amounts to £$P\left\{1 + \frac{r}{100}\right\}\left\{1 + \frac{r}{100}\right\}$ in 1 year,

i.e. £P amounts at compound interest to £$P\left(1 + \frac{r}{100}\right)^2$ in 2 years.

Now £$P\left(1 + \frac{r}{100}\right)^2$ earns interest at r% per annum for the third year, and amounts at the end of the third year to

$$\text{£}P\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right);$$

$\therefore$ £P amounts at compound interest to £$P\left(1 + \frac{r}{100}\right)^3$ in 3 years.

Hence we have

Principal	£P
Amount at end of 1 year	£$P\left(1 + \frac{r}{100}\right)$
,, ,, 2 years	£$P\left(1 + \frac{r}{100}\right)^2$
,, ,, 3 ,,	£$P\left(1 + \frac{r}{100}\right)^3$
and generally ,, n ,,	£$P\left(1 + \frac{r}{100}\right)^n$

If A is 130, $P = 100$ and $n = 7$, we have

$$130 = 100\left(1 + \frac{r}{100}\right)^7;$$

$$\therefore\ \log 130 = \log 100 + 7\log\left(1+\frac{r}{100}\right);$$

$$\therefore\ 2{\cdot}1139 = 2{\cdot}0 + 7\log\left(1+\frac{r}{100}\right);$$

$$\therefore\ \log\left(1+\frac{r}{100}\right) = \frac{0{\cdot}1139}{7} \doteqdot 0{\cdot}0163;$$

$$\therefore\ 1+\frac{r}{100} = 1{\cdot}039;$$

$$\therefore\ \frac{r}{100} = 0{\cdot}039 \quad\text{or}\quad r \doteqdot 4.$$

Ex. 4. *If* $F = EI\pi^2 \div 4L^2$ *and* $I = bt^3 \div 12$; *if* $E = 3\times 10^7$, $\pi = 3{\cdot}142$, $L = 62$, $b = 2$, $t = 0{\cdot}5$, *find* F. [B.E.]

$$\log F = \log E + \log I + 2\log\pi - \log 4 - 2\log L$$

$$\text{and } \log I = \log b + 3\log t - \log 12.$$

Substituting for $\log I$ in the first expression, we have

$$\begin{aligned}\log F &= \log E + \log b + 3\log t - \log 12 + 2\log\pi - \log 4 - 2\log L\\ &= 7{\cdot}4771 + 0{\cdot}3010 + 3(\bar{1}{\cdot}6990) - 1{\cdot}0792\\ &\qquad + 2(0{\cdot}4972) - 0{\cdot}6021 - 2(1{\cdot}7924)\\ &= \quad 7{\cdot}4771 - 1{\cdot}0792\\ &\quad + 0{\cdot}3010 - 0{\cdot}6021\\ &\quad + \bar{1}{\cdot}0970 - 3{\cdot}5848\\ &\quad + 0{\cdot}9944\\ &= \quad 7{\cdot}8695 - 5{\cdot}2661, \quad \textit{i.e. } 2{\cdot}6034;\end{aligned}$$

$\therefore\ F = 401.$

Ex. 5. *Because of a centrifugal force due to want of balance of a wheel, the stress* y *in a certain shaft when rotating* n *times per second is*

$$y = \frac{0{\cdot}026n^2}{1 - 0{\cdot}012n^2}$$

What is the critical speed—that is, the speed for which y *becomes indefinitely great? For speeds 20 per cent. less and more than the critical speed, find the stresses.*

The smaller the divisor, the larger is the quotient. **The quotient becomes indefinitely great when the divisor is indefinitely small.**

The critical speed is the speed which makes y indefinitely great, and this occurs when $1-0{\cdot}012n^2$ is indefinitely small.

The critical speed therefore occurs when

$$1-0{\cdot}012n^2=0,$$

i.e. when $$0{\cdot}012n^2=1,$$

or when $$\log 0{\cdot}012+2\log n=0,$$

i.e. when $$\bar{2}{\cdot}0792=-2\log n,$$

i.e. when $$2-0{\cdot}0792=2\log n;$$

$$\therefore\ \log n=1-0{\cdot}0396$$
$$=0{\cdot}9604;$$
$$\therefore\ n \doteqdot 9{\cdot}128.$$

The critical speed therefore occurs when the shaft makes about 9·13 revolutions per second.

20 % more than 9·128 is $\frac{120}{100}\times 9{\cdot}128 \doteqdot 10{\cdot}95$,

20 % less than 9·128 is $\frac{80}{100}\times 9{\cdot}128 \doteqdot 7{\cdot}302$.

Calculating $0{\cdot}012\times n^2$ for these values of n, we have

n	$\log n$	$\log 0{\cdot}012$	$2\log n$	$\log 0{\cdot}012+2\log n$	Result
10·95	1·0396	$\bar{2}{\cdot}0792$	2·0792	0·1584	1·440
7·302	0·8634	$\bar{2}{\cdot}0792$	1·7268	$\bar{1}{\cdot}8060$	0·6397

For speeds 20 % greater than critical speed,

$$y=\frac{0{\cdot}026\times(10{\cdot}95)^2}{1-0{\cdot}012\times(10{\cdot}95)^2}$$
$$=\frac{0{\cdot}026\times(10{\cdot}95)^2}{-0{\cdot}440}$$

Hence we have a reversed stress, $-y$, of magnitude given by

$$0{\cdot}026\times(10{\cdot}95)^2\div 0{\cdot}440.$$

$$\begin{aligned}\log 0{\cdot}026 &= \bar{2}{\cdot}4150\\ \log(10{\cdot}95)^2 &= 2{\cdot}0792\\ &\overline{0{\cdot}4942}\\ \log 0{\cdot}440 &= \underline{\bar{1}{\cdot}6435}\\ \therefore\ \log(-y) &= 0{\cdot}8507\\ \therefore\ -y &= 7{\cdot}09.\end{aligned}$$

For speeds 20% less than the critical speed,

$$y=\frac{0\cdot026\times(7\cdot302)^2}{1-0\cdot012(7\cdot302)^2}$$

$$=\frac{0\cdot026\times(7\cdot302)^2}{0\cdot3603}.$$

$$\begin{aligned}\log 0\cdot026&=\bar{2}\cdot4150\\ \log (7\cdot302)^2&=\underline{1\cdot7268}\\ &0\cdot1418\\ \log 0\cdot3603&=\underline{\bar{1}\cdot5567}\\ \therefore\ \log y&=0\cdot5851\\ \therefore\ y&=3\cdot85.\end{aligned}$$

EXERCISES X.

1. If $xy^{2\cdot5}=350$, find x if $y=12$. [B.E.]

2. If $x^2y^{1\cdot5}=100$, find y if x is 8. [B.E.]

3. The wheel of a carriage is 4 ft. diameter. How many revolutions does it make per mile? [B.E.]

4. If $pv^{1\cdot4}=\text{constant}$, and $p=21\cdot4$ when $v=10\cdot8$, find p when $v=9$. [I.C.E.]

5. Solve the equation $(1\cdot25)^x=5\cdot372$, and evaluate $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$ when $a=53\cdot72$ and $b=36\cdot37$. [I.M.E.]

6. (i) Find, as accurately as the logarithm tables permit, the value of the seventh root of 5·18643.

(ii) Find, to two places of decimals, the value of μ in the formula $T_1=T_2\,.\,e^{\mu\theta}$, if $T_1:T_2=5:2$ when $\theta=3\cdot1416$ and $e=2\cdot7183$. [I.C.E.]

7. Evaluate, by means of logarithms, when $a=37\cdot35$ and $b=23\cdot28$,

$$\text{(i)}\ \sqrt{a+\sqrt{a^2-b^2}};\quad \text{(ii)}\ a^{\frac{b}{10}}.$$ [I.M.E.]

8. If pv^k is constant; and if $p=1$ when $v=1$, find for what value of v, p is 0·2. Do this for the following values of k: 0·8, 0·9, 1·0, 1·1. Tabulate your answers. [B.E.]

9. If $A=P\left(1+\frac{r}{100}\right)^n$, find A when $P=200$, $r=4$, $n=12$. [B.E.]

10. $y = ax^2 + bx^3$. When $x = 1$, $y = 4{\cdot}3$, and when $x = 2$, $y = 30$. Find a and b. What is y when $x = 1{\cdot}5$? [B.E.]

11. If $y = ax^{\frac{3}{2}} - bx^{\frac{5}{2}}$, if $y = 5{\cdot}82$ when x is $0{\cdot}51$, and if y is $32{\cdot}10$ when x is $0{\cdot}98$, find a and b. [B.E.]

12. The electrical resistance of a wire $R \propto l/d^2$, where l is length and d is diameter. Its weight $W \propto ld^2$. Shew that the resistance of a wire $R \propto W/d^4$.

If a pound of wire, diameter $d = 0{\cdot}06$ in., has a resistance of $2{\cdot}25$ ohm, what is the resistance of a pound of wire of the same material, the diameter being $0{\cdot}01$ in. ? [B.E.]

13. A disc varies in thickness, so that when running at a certain speed the radial and hoop stresses may be the same, and constant everywhere. The thickness x at the radius r is such that

$$ar^2 = \log \frac{x_1}{x}.$$

If a is $0{\cdot}04$ and x_1 is $0{\cdot}3$, find r for the following values of x :

$$0{\cdot}2, \quad 0{\cdot}1, \quad 0{\cdot}05,$$

and draw a section of the disc. The logarithms are common logarithms. [B.E.]

14. If u is usefulness of flywheels,

$$u \propto d^5 n^2,$$

if d is the linear size (say diameter) and n the speed. We assume all flywheels to be similar in shape. I wish to have the usefulness one hundred times as great, the speed being trebled ; what is the ratio of the new diameter to the old one ? [B.E.]

15. If

$$\frac{x}{y} = e^{a\theta},$$

where $e = 2{\cdot}718$; if $a = 0{\cdot}3$ and $\theta = 2{\cdot}85$; and if $x - y = 550$, find x. [B.E.]

16. If $\theta = 0{\cdot}8\pi$, $\mu = 0{\cdot}3$ and $n = me^{\mu\theta}$; if $(n - m)V = 33000P$; if $P = 30$ and $V = 250$, find n. (Note $e = 2{\cdot}718$: $\pi = 3{\cdot}142$.) [B.E.]

17. Because of a centrifugal force due to want of balance of a wheel, the stress in a certain shaft when rotating n times per second is

$$y = \frac{0{\cdot}05n^2}{1 - 0{\cdot}01n^2}.$$

Calculate and tabulate y for the following values of n :

$$0, \quad 50, \quad 90, \quad 95, \quad 105, \quad 110, \quad 200, \quad 1000.$$

What is the critical speed, *i.e.* the speed at which y becomes indefinitely great ? [B.E.]

18. If $t=6931\frac{\omega}{d^2}\left\{\frac{1}{v}-\frac{1}{v_1}\right\}$ and $s=6931\frac{\omega}{d^2}\left\{\log\frac{v_1}{v}\right\}\times 2\cdot303$, find s and t when $v_1=2000$, $v=1500$, $\omega=12\cdot63$ and $d=3\cdot27$.

19. Evaluate $\frac{sr^{-1}-r^{-s}}{s-1}$ when $r=1\cdot75$ and $s=(a)\ 0\cdot95$, $(b)\ 1\cdot15$.

20. If $Q=\frac{2}{3}\sqrt{2g}b(h_2^{1\cdot5}-h_1^{1\cdot5})$, find Q when $g=32\cdot2$, $h_1=1\cdot42$, $h_2=1\cdot50$ and $b=0\cdot084$.

21. The area of a triangle is given by the formula

$$A=\sqrt{\{s(s-a)(s-b)(s-c)\}},\ \text{where } 2s=a+b+c.$$

What is the area of a triangle whose sides a, b and c are 11·37, 28·42 and 22·31 ft.?

22.* If a Savings Certificate costs 16/-, and is worth 26/- in 10 years' time, what rate per cent. per annum compound interest is this equivalent to?

23. In how many years will a sum of money double itself at 5% compound interest?

24. A man borrows a sum of money and repays it by paying £100 at the end of the first year and £200 at the end of the second year. Allowing compound interest at 4% per annum, what was the sum he borrowed?

25. A man borrows £1000 and repays it by three equal sums at the end of 1, 2 and 3 years. What is the amount of each instalment, compound interest at 4½% being allowed?

26. A man gets 5% compound interest on his money. Which would be cheaper, to pay £300 at once, or to pay £105 now, £105 at the end of one year, and £105 at the end of two years?

27. A sum of £500 is to be repaid by instalments, compound interest at 5% being reckoned on what is owed at any time. £150 is paid off (including interest) at the end of each year, until a year comes when a sum less than £150 pays off the debt. What is this sum?

28. In how many years will £500 amount to £800 @ 5% per annum, interest being added to the capital quarterly? [L.U.]

29. A manufacturer can buy for £1000 a machine which would probably last 8 years and be replaceable for another £1000. For £1600 he can buy a better machine which would probably last 16 years. If he can invest money at 6½%, which machine had he better buy? [L.U.]

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* Further exercises similar to Exs. 22-29 will be found on p. 260.

Logarithms to the base a. Let a be any positive number. Then, if a and M are numbers which are so related that $a^x = M$, x is called the **logarithm of M to base a**, or, briefly, $\mathbf{log_a M}$.

Thus $a^x = M$ and $x = \log_a M$ are merely two ways of saying the same thing, and further

$$M \equiv a^{\log_a M}$$

(*read* M is identically a to the power of $\log_a M$).

The chief properties of logarithms are :

I. $$\mathbf{\log_a MN = \log_a M + \log_a N}.$$

For let $\log_a M = x$ and $\log_a N = y$.

Then $a^x = M$ and $a^y = N$; ....(1)

$$\therefore\ a^{x+y} = MN,$$

i.e. $$x + y = \log_a MN$$

or $$\log_a M + \log_a N = \log_a MN.$$

II. $$\mathbf{\log_a \frac{M}{N} = \log_a M - \log_a N}.$$

For, from (1), $$\frac{a^x}{a^y} = \frac{M}{N},$$

i.e. $$a^{x-y} = \frac{M}{N},$$

i.e. $$x - y = \log_a \frac{M}{N}$$

or $$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

III. $$\mathbf{\log_a M^p = p \log_a M}.$$

Let $\log_a M = x$.

Then $a^x = M$;

$$\therefore\ (a^x)^p = M^p;$$

$$\therefore\ a^{xp} = M^p,$$

i.e. $$xp = \log_a M^p$$

or $$p \log_a M = \log_a M^p.$$

IV. $$\mathbf{\log_b M = \log_a M/\log_a b.}$$

Let $\log_b M = x$;

$$\therefore\ b^x = M;$$

$$\therefore\ \log_a b^x = \log_a M,$$

i.e. $x \log_a b = \log_a M$;

$$\therefore\ x = \log_a M/\log_a b$$

or $\log_b M = \log_a M/\log_a b$.

This formula can also be written $\log_a b \times \log_b M = \log_a M$.

Ex. 1. *Show that* $\log_a \sqrt[n]{\frac{x}{y}} = \frac{1}{n}(\log_a x - \log_a y)$.

$$\sqrt[n]{\frac{x}{y}} = \left(\frac{x}{y}\right)^{\frac{1}{n}};$$

$$\therefore\ \log_a \sqrt[n]{\frac{x}{y}} = \frac{1}{n}\log_a \frac{x}{y}, \text{ using III.,}$$

$$= \frac{1}{n}(\log_a x - \log_a y), \text{ using II.}$$

Ex. 2. *Calculate* $\log_3 5$ *to three decimal places.*

From the tables, $\log_{10} 5 = 0{\cdot}6990$,

$\log_{10} 3 = 0{\cdot}4771$.

Using IV., $$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{0{\cdot}6990}{0{\cdot}4771} \doteqdot 1{\cdot}465.$$

Ex. 3. *Show that* $\dfrac{\log_a(\log_b a)}{\log_b(\log_a b)} = -\log_a b$.

Let $\log_b a = x$;

$$\therefore\ a = b^x \quad \text{and} \quad b = a^{\frac{1}{x}};$$

$$\therefore\ \log_a b = \frac{1}{x}.$$

$$\frac{\log_a(\log_b a)}{\log_b(\log_a b)} = \frac{\log_a x}{\log_b\left(\frac{1}{x}\right)} = \frac{\log_a x}{-\log_b x}, \text{ using II.,}$$

$$= \frac{\log_b x/\log_b a}{-\log_b x}, \text{ using IV.,}$$

$$= \frac{1}{-\log_b a}$$

$$= -\log_a b, \text{ using IV.}$$

Ex. 4. *Explain the advantage of common logarithms, that is logarithms to base* 10.

Let x, y be two numbers having the same digits, *i.e.*

$$x = 10^p \,.\, y, \text{ where } p \text{ is an integer.} \quad \dots\dots\dots\dots\dots (1)$$

Taking logarithms of both sides of (1) to base 10,

$$\log_{10} x = p + \log_{10} y. \quad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Taking logarithms of both sides of (1) to base a,

$$\log_a x = p \log_a 10 + \log_a y. \quad \dots\dots\dots\dots\dots\dots\dots (3)$$

Equation (2) shews that the common logarithms of x and y differ only by an integer, *i.e.* they have the same decimal part, or **mantissa.** Hence tables of common logarithms need only give the logarithms of numbers from 1 to 10.

Equation (3) shews that the logarithms of x and y to base a do not differ by an integer. Thus if a table of logarithms to base a were given, the logarithms of *all* numbers would have to be shewn.

Ex. *Compare* $\log_3 0{\cdot}5$, $\log_3 5$ *and* $\log_3 50$ *with* $\log_{10} 0{\cdot}5$, $\log_{10} 5$, $\log_{10} 50$.

$$\log_3 0{\cdot}5 = \log_{10} 0{\cdot}5 / \log_{10} 3 = \frac{\bar{1}{\cdot}6990}{0{\cdot}4771} = -\frac{0{\cdot}3010}{0{\cdot}4771} \doteqdot -0{\cdot}631.$$

$$\log_3 5 \doteqdot 1{\cdot}465, \text{ as in Ex. 2 above.}$$

$$\log_3 50 = \log_{10} 50 / \log_{10} 3 = \frac{1{\cdot}6990}{0{\cdot}4771} \doteqdot 3{\cdot}561.$$

Whereas

$$\log_{10} 0{\cdot}5 = \bar{1}{\cdot}6990\,;$$

$$\log_{10} 5 = 0{\cdot}6990\,;$$

$$\log_{10} 50 = 1{\cdot}6990.$$

EXERCISES XI.

1. Use indices to express the following statements :

$$\log_3 9 = 2, \quad \log_e 1 = 0, \quad \log_a \left(\frac{1}{a^4}\right) = -4.$$

2. Without using tables, find the values of

$$\log_2 8, \quad \log_3 (\tfrac{1}{9}), \quad \log_5 (0{\cdot}04).$$

3. Use the tables to find the approximate values of $\log_2 6$, $\log_3 1{\cdot}2$, $\log_5 10$, $\log_e 3{\cdot}7$, $\log_e 0{\cdot}37$. [$e=2{\cdot}718$.]

4. Prove that $\log_x y \,.\, \log_y x=1$, $\log_x y.\ \log_y z \,.\, \log_z x=1$.

5. If $\log_a(p+q)=\log_a p-\log_a q$, shew that $p(1-q)=q^2$.

6. Solve the equations :

(i) $2^{3x-2}=50$; (ii) $4^x \,.\, 7^{x+1}=12^{2x-1}$;

(iii) $5^{2x}-3.5^x+2=0$; (iv) $5^{2x}-5^{x+1}+6=0$;

(v) $e^x+e^{-x}=5{\cdot}2$, where $e=2{\cdot}718$.

7. With the help of tables, find the values of

(i) $(51{\cdot}89)^{-\frac{3}{7}}+(0{\cdot}05189)^{\frac{3}{5}}$; (ii) $\log_4 7{\cdot}914$. [B.U.]

8. Prove that $\log a^m=m\log a$.

If $x^2+y^2=6xy$, prove that $2\log(x+y)=\log x+\log y+3\log 2$. [B.U.]

9. Explain what is meant by the logarithm of a number x to the base y, and prove that

$$\log_x y=\frac{\log_z y}{\log_z x}.$$

Calculate from your tables the value of

$$(0{\cdot}013)^{2\cdot5}\div(0{\cdot}187)^{4\cdot73}.$$ [B.U.]

10. Solve the equation

$$2\log x=\log(5x-6).$$

Calculate from your tables the value of

$$\frac{(0{\cdot}11)^{2\cdot3}+(1{\cdot}32)^{0\cdot4}}{(2{\cdot}3)^{0\cdot41}+(0{\cdot}4)^{13\cdot2}}.$$

11. Calculate the values of

$$(25{\cdot}7)^{3\cdot75},\ (2{\cdot}57)^{37\cdot5}.$$

Which is the greater ? Is it the one which you expected to be greater ? If so, give your reason. [S.W.]

12. How is a^x defined when x is a vulgar fraction, and why ? Prove that $a^x\times a^y=a^{x+y}$ in the case where $x=\frac{2}{3}$, $y=\frac{1}{2}$. Verify for the case of $a=64$.

Calculate, by any method, the value of $(0{\cdot}00284)^{-\frac{3}{4}}$ to three significant figures. [S.W.]

Examples 13 *to* 27 *are from papers set at the Inter. B.Sc.* [L.U.]

13. Prove from the definition of a logarithm that $\log_3 10=\dfrac{1}{\log_{10}3}$, and find the value of $\log_3 30$.

If $y=Ae^{mx}+C$, where $e=2{\cdot}71828185\ldots$, determine the values of A, m and C, given that when $x=0, 1, 3$, $y=10, 20, 30$.

14. The volume of a double convex lens whose faces are equally curved to a radius R is $\frac{1}{3}\pi h(3r^2+h^2)$, where $2r$ is the aperture (diameter of rim) and h (the semi-thickness) is equal to

$$R-\sqrt{R^2-r^2}.$$

How many lenses of 3 in. aperture, the radii of curvature of whose faces are 9·326 in., can be made out of a cubic foot of glass, allowing 10% for waste of material in manufacture? $(\pi = 3{\cdot}142.)$

15. A rectangular block of length h ft. has square ends. Shew that if its total superficial area is S sq. ft. and its volume is V cubic ft.,

$$(2V-hS)^2=16h^3V.$$

Find the volume when the length is 3 ft. and the total superficial area is 10 sq. ft.

16. Solve the equation $e^x-7e^{-x}=2$.

17. By giving x positive integral values up to 10, calculate the corresponding values of y when $0{\cdot}1y=1-(0{\cdot}9)^x$, and plot a graph. Shew that y can never exceed 10, and find the least integral value of x so that y may exceed 9·998.

18. If $x=y^2$, then $y=x^{\frac{1}{2}}$.

Express these in logarithmic form, and deduce the relation between $\log_x y$ and $\log_y x$.

Calculate the values of

$$(1)\ 0{\cdot}03142^{-0{\cdot}002718} \text{ and of } (2)\ \sqrt{\frac{a^2-b^2}{10a-11b}},$$

when $a=0{\cdot}6954$ and $b=0{\cdot}5027$.

19. Solve $4^x-6\times 2^x-2=0$.

If $y=a+bx^n$ is satisfied by the values

x	2	4	8
y	10	15	27

shew that $n=\dfrac{\log 2{\cdot}4}{\log 2}$, and deduce the values of a and b.

20. Define the logarithm of a number, and explain shortly the advantage of base 10. Prove that

$$\frac{\log_{10}(\log_e 10)}{\log_e(\log_{10}e)}=-\log_{10}e.$$

Find the value of $\log_7(\log_e 1738)$.

21. Shew how you can find a meaning for $a^{\frac{p}{q}}$ and a^0, and prove that $\log_a b \times \log_b c = \log_a c$.

If $\log_x 20{\cdot}3 = y$ and $\log_y 784 = 2{\cdot}37$, calculate the values of y and x.

22. Define the logarithm of a number to a given base, and prove that $\log_a x = (\log_{10} x)/(\log_{10} a)$.

The tension of a rope wound round a rough cylinder increases with the angle (θ) turned through in the ratio $\epsilon^{\mu\theta}:1$, where μ is a constant and $\epsilon = 2{\cdot}7183$. Find the least value of μ necessary for the tension to increase more than a thousandfold for each complete turn. (For one complete turn $\theta = 6{\cdot}283$.)

23. Calculate the values of

(i) $(1{\cdot}936)^{\frac{1}{2}} \times (0{\cdot}03625)^{\frac{1}{5}}$; (ii) $(0{\cdot}1936)^{\frac{1}{2}} - (0{\cdot}3625)^{\frac{2}{5}}$.

A quantity x is known to vary as $\dfrac{l^3}{bd^3}$; if $x = 1$ in. when $l = 21$ ft., $b = 6{\cdot}35$ in. and $d = 9{\cdot}47$ in., calculate the value of x in inches when $l = 33$ ft., $b = 4{\cdot}23$ in. and $d = 6{\cdot}32$ in.

24. What is meant by the "base" of a system of logarithms? Having given the logarithm of a number to base 10, how is the logarithm to base 2·718 found?

Evaluate
$$\frac{x^2}{12y^2}\left(\log_e \frac{x^{1\cdot5}}{y^{0\cdot5}} - 1{\cdot}882\right),$$
where $e = 2{\cdot}718$, $x = 12{\cdot}37$, $y = 0{\cdot}0788$.

25. Explain how to find a meaning for a^0, $a^{\frac{p}{q}}$ and a^{-n}.

Find i from the formula $i = i_0 e^{-\frac{Rt}{L}}$, where $i_0 = 7{\cdot}68$, $e = 2{\cdot}718$, $R = 0{\cdot}56$, $t = 3{\cdot}22$ and $L = 0{\cdot}026$.

26. Define the logarithm of a number with respect to a given base, and shew from definition that $\log_a x^m = \log_b x$ if $a = b^m$.

Two variables x and y are known to be connected by an equation of form $y = ax^n$, where a and n are constants. When $x = 19{\cdot}3$, $y = 38700$, and when $x = 14{\cdot}8$, $y = 27400$. Find by aid of logarithmic tables the values of a and n.

27. Show that if the base of a system of logarithms be changed to its cube root, all the logarithms are trebled.

Find the value of

(i) $0{\cdot}009876^{-2\cdot3}$, (ii) $0{\cdot}03765 \log_e(\log_{10} 9876)$, where $e = 2{\cdot}718$.

The Determination of Laws other than Linear. When two sets of experimental numbers are plotted, the resulting graph may not be linear. In these circumstances, certain modifications of the observed numbers are plotted, with the view of obtaining a straight-line graph. Theoretical considerations generally give some indication of the necessary modification, and the general method of treatment is deferred.

A fuller treatment of this important subject will be found in the authors' *Practical Mathematics for Technical Students*, Pt. II., Ch. V.

Ex. 1. *In the following numbers there is reason to suspect that there is a simple connection between y and the reciprocal of x. Test this, and find the law if possible.*

x	0·2	0·25	0·4	0·8	1·0	1·5	2·0
y	10·5	8·4	5·4	3·2	2·5	2·0	1·6

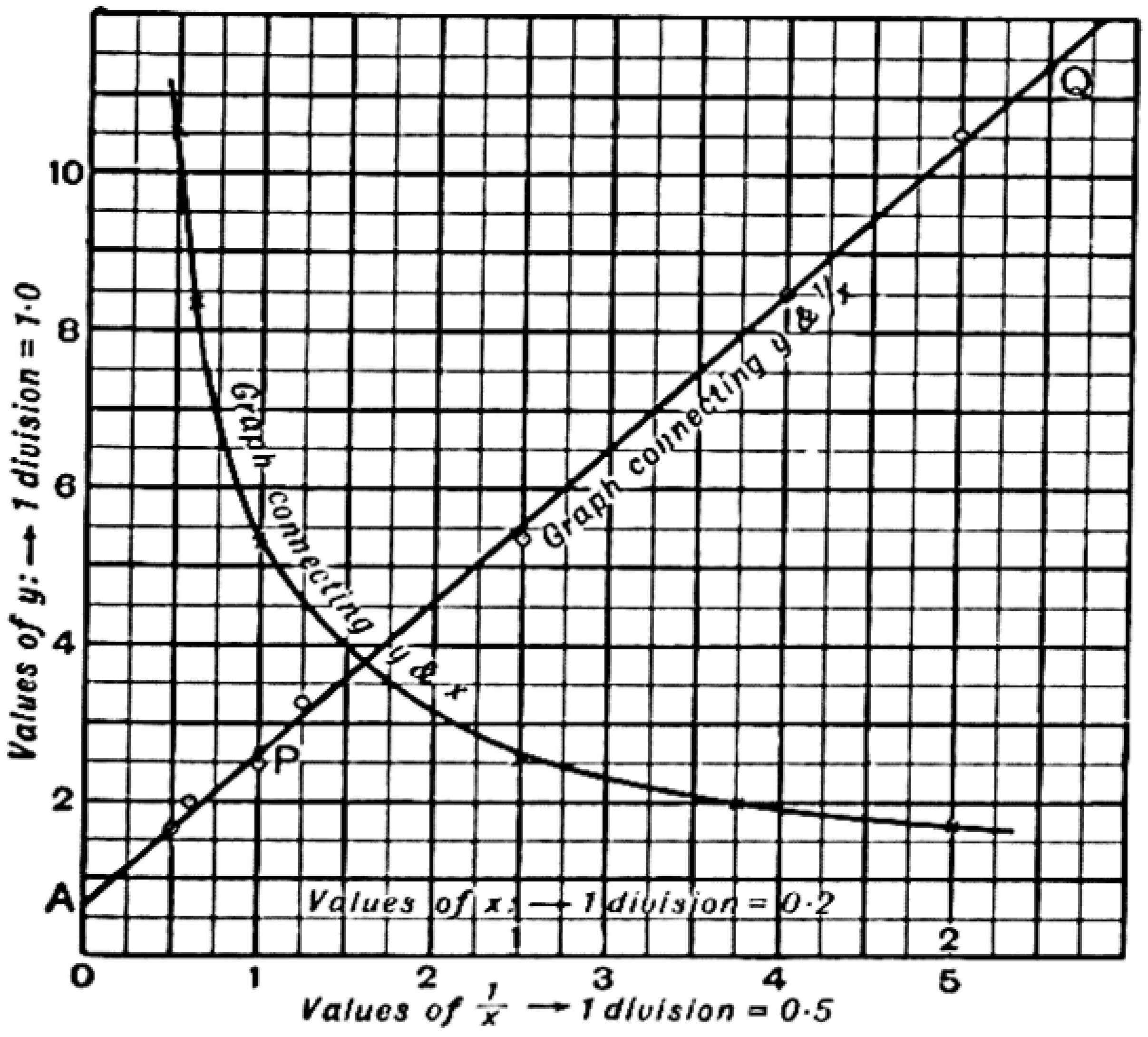


FIG. 45.

Note x increases and y diminishes. Evaluating the reciprocals of x: $1/x = 5$, 4, 2·5, 1·25, 1, 0·667, 0·5.

Note that y and $1/x$ decrease together.

The plotted values of y and $1/x$, indicated by circles, are shewn in Fig. 45. A black thread gives the straight line drawn, which shews that there is a linear connection between y and $1/x$, say $y=a+b\,.\,\frac{1}{x}$.

The intercept on the axis of y is 0·65, as shewn by OA. Therefore $a=0{\cdot}65$.

$$\text{At } P,\quad 1/x=1,\qquad y=2{\cdot}6.$$

$$\text{At } Q,\quad 1/x=5{\cdot}5,\quad y=11{\cdot}4;$$

$$\therefore\ b=\frac{\Delta y}{\Delta(1/x)}=\frac{11{\cdot}4-2{\cdot}6}{5{\cdot}5-1}=\frac{8{\cdot}8}{4{\cdot}5}\doteqdot 1{\cdot}96.$$

Hence the law connecting y and x is approximately

$$y=0{\cdot}65+\frac{1{\cdot}96}{x}.$$

In Fig. 45 the curved line obtained by plotting y and x is given for comparison.

Ex. 2. *There is reason to suppose that in the following experimental values there is a simple linear connection between* $\log p$ *and* $\log h$. *Test if this is so, and find the law of connection.*

h	5	15	25	35	50
p	13	20	24	27	31

Tabulating logarithms to four significant figures, we have

$\log h$	0·700	1·176	1·398	1·544	1·700
$\log p$	1·114	1·301	1·380	1·431	1·491

Fig. 46 shews the graph obtained by the black thread.

$$\text{At } P,\quad \log h=0{\cdot}65,\quad \log p=1{\cdot}1.$$

$$\text{At } Q,\quad \log h=1{\cdot}7,\qquad \log p=1{\cdot}5.$$

Therefore, assuming the law is $\log p=a+b\log h$,

$$b=\frac{\Delta(\log p)}{\Delta(\log h)}=\frac{1{\cdot}5-1{\cdot}1}{1{\cdot}7-0{\cdot}65}=\frac{0{\cdot}4}{1{\cdot}05}\doteqdot 0{\cdot}38.$$

Hence, to find 'a,' we have from the point Q, using the equation

$$\log p = a + b \log h,$$

$$1{\cdot}5 \doteqdot a + 0{\cdot}38 \times 1{\cdot}7;$$

$$\therefore\ a \doteqdot 1{\cdot}5 - 0{\cdot}65 \text{ or } 0{\cdot}85.$$

Hence $\log p \doteqdot 0{\cdot}85 + 0{\cdot}38 \log h.$

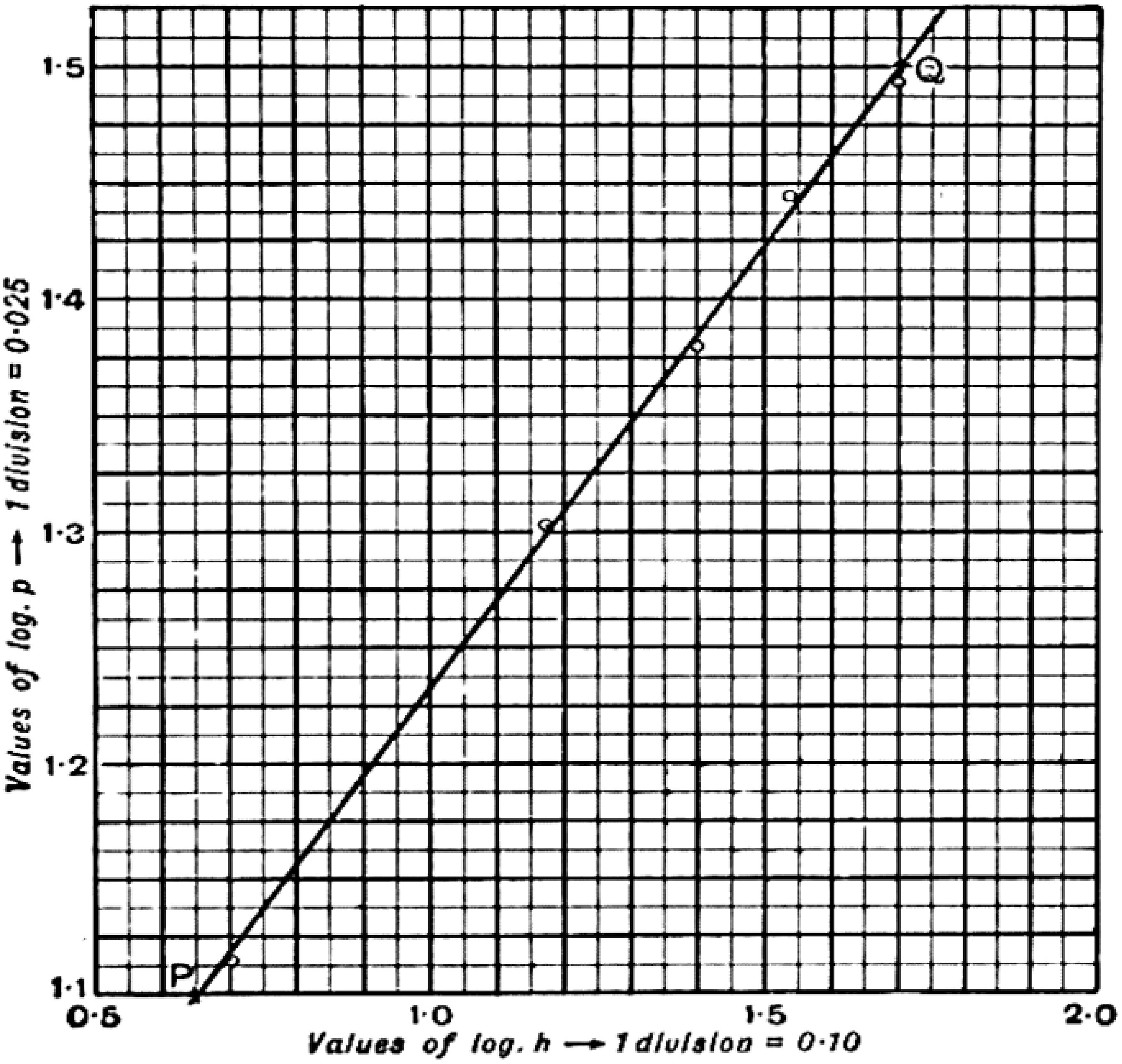


FIG. 46.

Note, since $0{\cdot}85 \doteqdot \log 7{\cdot}08$, this equation may be put in the form

$$\log p \doteqdot \log 7{\cdot}08 + \log h^{0\cdot38},$$

i.e. $\log p \doteqdot \log (7{\cdot}08 \times h^{0\cdot38}),$

and therefore $p \doteqdot 7{\cdot}08 h^{0\cdot38}$.

EXERCISES XII.

1. The following numbers are thought to follow a law like

$$y = ax^2 + b:$$

y	1·68	2·26	2·48	3·18	3·52	4·72	6·20
x	0·5	1·2	1·5	2	2·2	2·8	3·4

Test if this is so, and find the law connecting y and x.

2. The following numbers are thought to follow a law like

$$y = a + \frac{b}{x}:$$

y	3·45	3·24	3·17	3·09	3·03	3·00	2·98	2·97	2·96
x	2·2	3·4	4·1	5·6	7·2	9·0	10·5	11·0	12·0

Test if this is so, and find the most probable values of a and b.

3. The horse-power H, required to drive a certain vessel at V knots, and V are supposed to be connected by an equation of the form

$$H = a + kV^3.$$

Test the following numbers to see if this is so, and find the best values of a and k:

H	290	560	1040	1810	2300
V	5	7	9	11	12

4. Mrs. Ayrton found the following numbers for the potential difference V volts and the current A amperes in the electric arc:

V	50·3	48·7	47·9	47·5	46·8	45·7	45·1	44·0	43·6	43·5
A	1·96	2·46	2·98	3·45	3·96	4·97	5·96	6·97	7·97	9·00

Plot V and A. Then plot watts, (*i.e.* VA) and A, and hence find the law connecting V and A.

5. The coefficient of friction, μ, of steel tyres on steel rails is supposed to be inversely proportional to the speed, V, in miles per hour. Test if this is so from the following results obtained by Westinghouse and Galton :

V	25	38	45	50
μ	0·080	0·051	0·047	0·040

6. The following data are taken from the price list of Clyburn's adjustable spanners, where p is the price in shillings and s is the size in inches :

s	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	$2\frac{3}{4}$	3
p	4·5	6	9·5	15	21	24·5	28·5

Try if these numbers obey a law like $p = a + bs^2$, and find it.

7. The following data are taken from a table of Whitworth's standard watch screws : d is the diameter in inches, n the number of threads per inch. Test if there is a law of the form

$$\log n = a + b \log d,$$

and find the best values of a and b.

d	0·010	0·020	0·026	0·032	0·040	0·060
n	250	180	150	120	100	80

Find the number of threads per inch if the diameter is 0·015 inch.

8. The following data are taken from the wiring rules of the Institution of Electrical Engineers for rubber-covered wires, cross-sectional area A sq. in., exposed to high external temperatures. C is the maximum current in amperes :

C	3·2	5·9	22·0	42·0	84·0	102·0
A	0·00181	0·00407	0·0223	0·050	0·1250	0·160

Test if these numbers obey a law like $\log C = a + b \log A$, and find the maximum current when the cross-sectional area is 0·007 sq. in., and the cross-sectional area if the maximum current is 68 amperes.

9. For Whitworth standard triangular threads we have:

d, diameter in inches	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$
n, number of threads per inch - -	20	18	16	14	12	11	10	9	8	7

See if there is a simple linear law connecting the pitch p of the thread with d, the diameter. If so, find it.

10. The following data are taken from the *Proceedings of the Institution of Mechanical Engineers.*

For a split bearing using gas-engine oil, when v is the velocity in feet per second, and t the rise of temperature in degrees Fahrenheit:

v	3·93	5·38	6·52	7·95	9·44	11·02	12·85
t	26·4	32·6	39·3	46·5	52·5	58·4	63·2

Test if these numbers follow a law like $\log t = a + b \log v$, and find it.

11. The table connects values of p, the pressure of saturated steam in lb. per sq. in., and u, the volume of a pound of steam in cubic feet.

Plot $\log p$ with $\log u$, and deduce a formula of the form

$$\log p = a + b \log u.$$

Shew that the formula you have deduced is approximately equivalent to $pu^{1\cdot065} = 479$.

p	10	15	20	25	30	35	40	45	50
u	37·8	25·9	19·7	15·6	13·5	11·7	10·3	9·21	8·34

12. For the numbers given in the last exercise there is an approximate law of the form $\frac{1}{u} = a + bp$. Plot p with the reciprocal of u, and find the best values of a and b.

13. Shew that the following values of p and v are connected by a law of the form $pv^n = c$, and find the probable values of n and c:

p	100	63·2	45·7	35·8	28·9	24·3	20·9
v	1	1·5	2	2·5	3	3·5	4

14. It is thought that the following observed quantities, in which there are probably errors of observation, follow a law like

$$y=ae^{bx}.$$

Test if this is so, and find the most probable values of a and b.

x	2·30	3·10	4·00	4·92	5·91	7·20
y	33·0	39·1	50·3	67·2	85·6	125·0

15. It is said that the following values of w, the number of pounds of steam per hour per horse-power for a perfect condensing engine using the Rankine cycle, may be calculated from a formula

$$w=a+\frac{b}{p+50},$$

where p is the pressure in lb. per sq. inch. Test this, and find the best values for a and b:

p	50	55	60	65	70	80	90	100
w	10·3	10·1	9·87	9·68	9·52	9·23	8·99	8·77

16. See if the following values of u and p follow a law like

$$\frac{1}{u}=a+bp,$$

and find the best values for a and b:

p	80	120	140	180	220	280
u	5·37	3·67	3·18	2·51	2·09	1·65

17. If H is the horse-power supplied at one end of a line of pipes in a system of hydraulic transmission and U is the useful power delivered at the other end, there is reason to suppose that the useful power delivered per unit power supplied is connected by a linear law with the square of the power supplied. Test the following numbers to see if this is so, and find the law:

H	100	150	200	250	300
U	96·5	138	172	196	206

18. In some experiments in towing a canal boat the following observations were made; P being the pull in pounds and v the speed of the boat in miles per hour:

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Plot $\log v$ and $\log P$ upon squared paper, and give an approximate formula connecting P and v. [B.E.]

19. The following quantities are thought to follow a law like $pv^n = \text{constant}$. Try if they do so; find the most probable value of n.

v	1	2	3	4	5
p	205	114	80	63	52

[B.E.]

20. Prove that if two quantities x and y are connected by an equation of the form $y = Cx^n$, where C and n are constants, the graph resulting from plotting values of $\log x$ against the corresponding values of $\log y$ is a straight line.

x	3	3·5	4	4·5	5
y	6·19	6·79	7·35	7·89	8·40

The above are simultaneously observed values of x and y which are believed related by such a law. Verify that this is approximately correct, and find the best values for C and n. [L.U.]

21. Prove that $\log a^m = m \log a$ for all real values of m. Shew that the series of observations

A	3	4	5	6	7
B	21·2	14·2	10·3	8	6·4

is approximately consistent with an assumed law of the form $BA^{\gamma} = \text{constant}$; and shew that $\gamma = 1{\cdot}4$ approximately. [L.U.]

CHAPTER V.

ELEMENTARY TRIGONOMETRY.

The Circular Functions of an Acute Angle. Let XOY (Fig. 47) be any acute angle. Take any point P on either arm of the angle, and let N be the foot of the perpendicular let fall from P on the other arm. Evidently, wherever P is taken, the angle at O is the same and the angle at N is always 90°. Therefore the triangle OPN has two, and consequently, all three of its angles always of the same magnitudes. Thus, by the properties of similar triangles, the triangle OPN has a definite shape, so that the ratio of any pair of its sides is independent of the position of P.

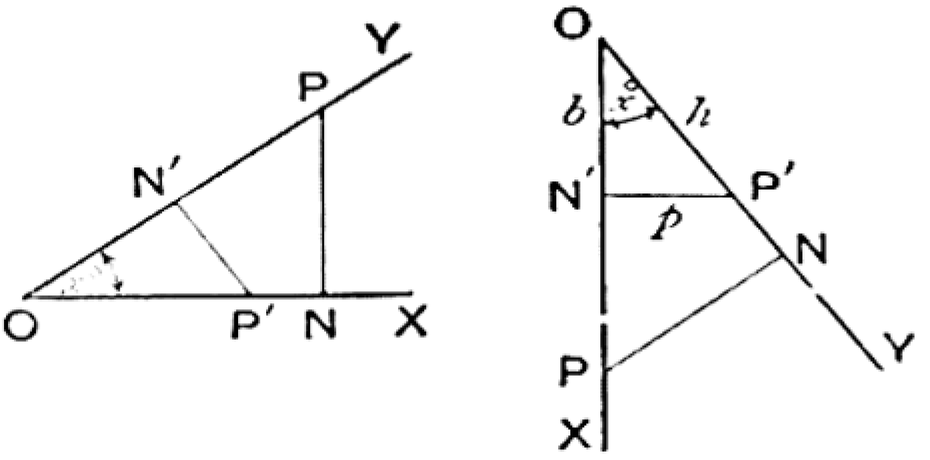


FIG. 47.

PN may be called the *side opposite* to the angle,
ON may be called the *side adjacent* to the angle,
and OP is the *hypotenuse*.

The **sine** of the angle XOY is the ratio $\frac{\text{side opposite}}{\text{hypotenuse}}$, *i.e.* $\frac{PN}{OP}$.

The **cosine** of the angle XOY is the ratio $\frac{\text{side adjacent}}{\text{hypotenuse}}$, *i.e.* $\frac{ON}{OP}$.

The **tangent** of the angle XOY is the ratio $\frac{\text{side opposite}}{\text{side adjacent}}$, *i.e.* $\frac{PN}{ON}$

If the angle XOY contains x degrees and the letters p, b, h stand for side opposite, side adjacent and hypotenuse, respectively, these definitions may be written briefly:

$$\sin x = \frac{p}{h}, \quad \cos x = \frac{b}{h}, \quad \tan x = \frac{p}{b}.$$

The sine, cosine and tangent of an angle are **circular functions** of the angle. There are three other circular functions which are the reciprocals of these, viz. those called the **cosecant**, the **secant** and the **cotangent**.

Thus we have, briefly,

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{h}{p},$$

$$\sec x = \frac{1}{\cos x} = \frac{h}{b},$$

$$\cot x = \frac{1}{\tan x} = \frac{b}{p}.$$

Ex. 1. *Draw an angle of 35° and find the approximate values of its circular functions by measurement.*

Make the angle XOY equal to 35°. From OY cut off $OP = 10$ cm., and draw PN perpendicular to OX.

By measurement $PN = 5{\cdot}74$ cm., $ON = 8{\cdot}19$ cm. (Fig. 48).

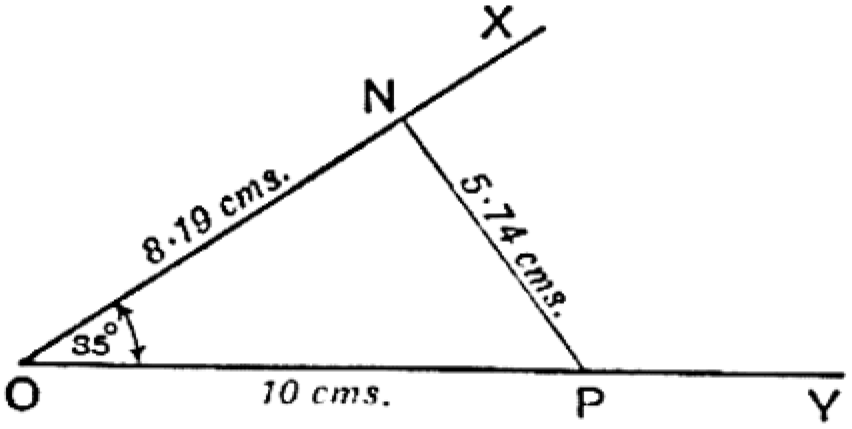


FIG. 48.

$$\therefore \sin 35° = \frac{PN}{OP} = \frac{5{\cdot}74}{10} = 0{\cdot}574,$$

$$\cos 35° = \frac{ON}{OP} = \frac{8{\cdot}19}{10} = 0{\cdot}819,$$

$$\tan 35° = \frac{PN}{ON} = \frac{5{\cdot}74}{8{\cdot}19} = 0{\cdot}70,$$

$$\operatorname{cosec} 35° = \frac{OP}{PN} = \frac{10}{5{\cdot}74} = 1{\cdot}74,$$

$$\sec 35° = \frac{OP}{ON} = \frac{10}{8{\cdot}19} = 1{\cdot}22,$$

$$\cot 35° = \frac{ON}{PN} = \frac{8{\cdot}19}{5{\cdot}74} = 1{\cdot}43.$$

Ex. 2. *Find the circular functions of* 45°.

Make the angle XOY equal to 45°. Take any point P on OY and draw PN perpendicular to OX.

Clearly the angle NPO is 45° and $ON = NP = a$, say (Fig. 49).

Then $OP = \sqrt{\{a^2 + a^2\}} = a\sqrt{2}$;

$$\therefore \ \sin 45^\circ = \frac{PN}{OP} = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{ON}{OP} = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = \frac{PN}{ON} = 1,$$

$$\operatorname{cosec} 45^\circ = \frac{OP}{PN} = \sqrt{2}, \quad \sec 45^\circ = \frac{OP}{ON} = \sqrt{2}, \quad \cot 45^\circ = \frac{ON}{PN} = 1.$$

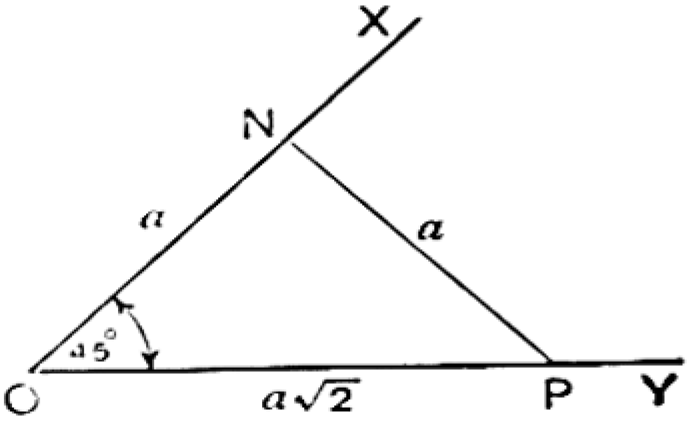


FIG. 49.

Ex. 3. *Find the circular functions of* 30° *and* 60°.

Make $XOY = 60^\circ$. Take any point P on OX and draw PN perpendicular to OY.

In NY take a point Q so that $NQ = ON$ and join PQ.

Clearly the triangle POQ is equilateral (Fig. 50).

Let each side equal $2a$.

Then $ON = a$ and $PN = \sqrt{4a^2 - a^2} = a\sqrt{3}$;

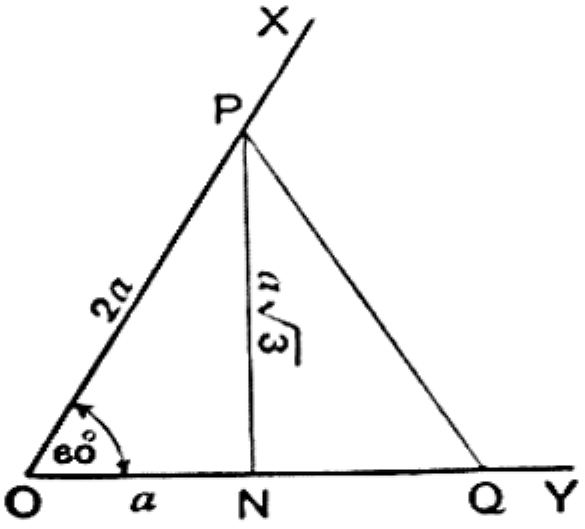


FIG. 50.

$$\therefore \ \sin 60^\circ = \frac{PN}{OP} = \frac{\sqrt{3}}{2}; \quad \therefore \ \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$$

$$\cos 60^\circ = \frac{ON}{OP} = \frac{1}{2}; \quad \therefore \ \sec 60^\circ = 2.$$

$$\tan 60^\circ = \frac{PN}{ON} = \sqrt{3}; \quad \therefore \ \cot 60^\circ = \frac{1}{\sqrt{3}}.$$

From the figure, the angle $OPN = 30^\circ$;

$$\therefore \ \sin 30^\circ = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{ON}{OP} = \frac{1}{2}; \quad \therefore \ \operatorname{cosec} 30^\circ = 2.$$

$$\cos 30^\circ = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{PN}{OP} = \frac{\sqrt{3}}{2}; \quad \therefore \ \sec 30^\circ = \frac{2}{\sqrt{3}}.$$

$$\tan 30^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{ON}{PN} = \frac{1}{\sqrt{3}}; \quad \therefore \ \cot 30^\circ = \sqrt{3}.$$

Ex. 4. *Construct an angle whose sine is* 0·7.

Draw a line NP, 7 cm. long, and through N draw a line NX perpendicular to PN. With P as centre and 10 cm. radius describe an arc cutting NX in O (Fig. 51).

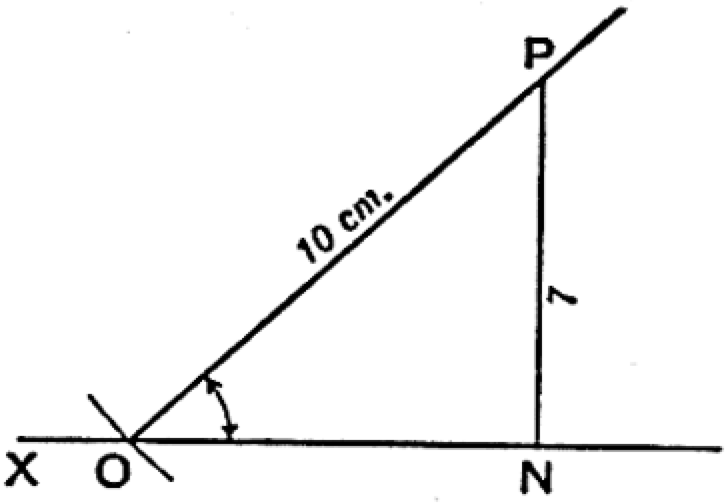


FIG. 51.

Then clearly PON is an angle whose sine is 0·7.

By measurement $P\hat{O}N = 44{\cdot}5°$ approximately.

Fundamental Relations between the Circular Functions. From the relations on p. 124, we have, evidently,

$$\tan x = \frac{p}{b} = \frac{p}{h} \div \frac{b}{h} = \frac{\sin x}{\cos x} \quad \text{.....................(1)}$$

and $$\cot x = \frac{b}{p} = \frac{b}{h} \div \frac{p}{h} = \frac{\cos x}{\sin x}. \quad \text{.....................(2)}$$

Again, in Fig. 47, using Pythagoras' Theorem, we have

$$ON^2 + NP^2 = OP^2.$$

Dividing by OP^2, $(ON/OP)^2 + (NP/OP)^2 = 1$,

i.e. $(\sin x)^2 + (\cos x)^2 = 1.$

This is written as $\sin^2 x + \cos^2 x = 1.$ ....(3)

Dividing by ON^2, $1 + (NP/ON)^2 = (OP/ON)^2$;

i.e. $1 + (\tan x)^2 = (\sec x)^2,$

or $1 + \tan^2 x = \sec^2 x.$ ....(4)

Finally, dividing by NP^2,

$$(ON/NP)^2 + 1 = (OP/NP)^2;$$

i.e. $(\cot x)^2 + 1 = (\operatorname{cosec} x)^2,$

or $1 + \cot^2 x = \operatorname{cosec}^2 x.$ ....(5)

Relations (1), (2), (3), (4) and (5) are very important. The proof given is for acute angles, but on p. 155 it is shewn that these relations are always true.

Ex. 1. *Prove the identity*

$$(1+\cos x)^2+(1+\sin x)^2=2(\sin x+\cos x)+3.$$

The left-hand side is $1+2\cos x+\cos^2 x+1+2\sin x+\sin^2 x$,

i.e. $3+2\cos x+2\sin x$, since $\cos^2 x+\sin^2 x=1$,

i.e. $3+2(\sin x+\cos x)$.

Ex. 2. *Shew that*

$$\frac{\tan\theta}{\sec\theta-1}-\frac{\sin\theta}{1+\cos\theta}=2\cot\theta.$$

On multiplying numerator and denominator by $\cos\theta$,

$$\frac{\tan\theta}{\sec\theta-1}=\frac{\sin\theta}{1-\cos\theta}.$$

Therefore the left-hand side is

$$\sin\theta\left[\frac{1}{1-\cos\theta}-\frac{1}{1+\cos\theta}\right],$$

i.e. $\sin\theta\left[\frac{2\cos\theta}{1-\cos^2\theta}\right]$,

i.e. $\frac{2\sin\theta\cos\theta}{\sin^2\theta}$, since $\sin^2\theta=1-\cos^2\theta$,

i.e. $\frac{2\cos\theta}{\sin\theta}$,

i.e. $2\cot\theta$.

Ex. 3. *If $\tan x=p/q$, find $\sin x$ and $\cos x$, assuming x is a positive acute angle.*

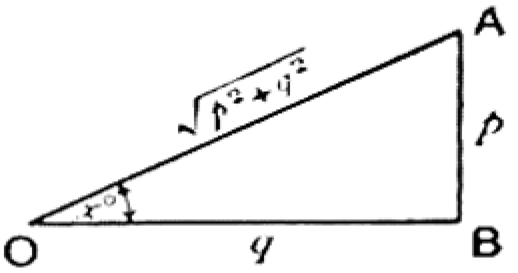


FIG. 52.

Fig. 52 shews an angle $x°$ whose tangent is p/q.

Since, by Pythagoras's Theorem,

$$OA=\sqrt{p^2+q^2},$$

$$\sin x=\frac{p}{\sqrt{p^2+q^2}}\quad\text{and}\quad\cos x=\frac{q}{\sqrt{p^2+q^2}}.$$

Ex. 4. *Express sin A in terms of sec A.*

Draw any right-angled triangle OPQ (Fig. 53).

Call the hypotenuse sec A, and let $OQ=1$, so that the angle POQ represents the angle A.

By Pythagoras's Theorem,

$$PQ=\sqrt{\sec^2 A-1},$$

and $$\sin A=\frac{PQ}{OP};$$

$$\therefore\ \sin A=\frac{\sqrt{\sec^2 A-1}}{\sec A}.$$

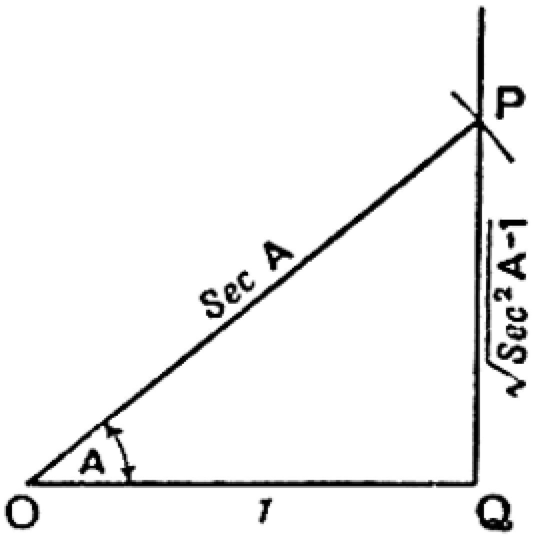


FIG. 53.

EXERCISES XIII.

1. Find, by drawing, the approximate values of the circular functions of 28°, 57°, 72° and 80°.

2. Construct the angle whose sine is 0·25, 0·5, 0·75;
the angle whose cosine is 0·3, 0·5, 0·8;
the angle whose tangent is 0·5, 1·5, 2·0;
the angle whose secant is 1·5, 2·0;
the angle whose cosecant is 1·5, 2·5;
the angle whose cotangent is 0·5, 1·4, 2·0.

3. Given $\sin A=\frac{5}{8}$, calculate $\sec A$ and $\tan A$ to two decimal places. Use your results to find by how much $\sec^2 A$ differs from $1+\tan^2 A$.

4. If $\sin x=0{\cdot}5$, find the value of $1+\tan^2 x$ and of $1-\cos^2 x$.

5. If $\sin\theta=\dfrac{p}{q}$, find $\sec\theta$, $\tan\theta$ and $\cot\theta$.

6. If $x=a\cos\theta$ and $y=b\sin\theta$, shew that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

7. If $x=a\sec\phi$ and $y=b\tan\phi$, shew that $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

8. If $\operatorname{cosec} A=x$, find $\sin A$, $\cos A$, $\tan A$, $\sec A$ and $\cot A$.

9. If $\cot A=y$, find $\sin A$, $\cos A$, $\sec A$, $\operatorname{cosec} A$ and $\tan A$.

10. Prove the following identities :

(i) $\sec^2 x - \operatorname{cosec}^2 x = \tan^2 x - \cot^2 x$;

(ii) $(1 + \tan B)^2 + (1 - \tan B)^2 = 2\sec^2 B$;

(iii) $\dfrac{1 - 2\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x} = \tan^2 x + \cot^2 x$;

(iv) $(\tan y + \sec y)^2 = \dfrac{1 + \sin y}{1 - \sin y}$;

(v) $(\tan^2 x + \tan x + 1)(1 - \cot x + \cot^2 x) = \tan^2 x + 1 + \cot^2 x$.

11. Simplify :

(i) $(\sin A + \cos A)^2 + (\sin A - \cos A)^2$;

(ii) $\dfrac{(1 + \tan x)^2 + (1 - \tan x)^2}{2\sec^2 x}$;

(iii) $(1 + \tan^2 B)\cos^2 B + \tan^2 B$;

(iv) $\tan x(1 - \cot^2 x) + \cot x(1 - \tan^2 x)$.

12. Shew that

$$\frac{\sin\theta + 2\sin\theta\cos\theta}{1 + \cos\theta + \cos^2\theta - \sin^2\theta} = \tan\theta.$$

13. Prove that $\operatorname{cosec}^4\theta(1 - \cos^4\theta) - 2\cot^2\theta = 1$, and that

$$\frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} + \frac{1 + \sin A + \cos A}{1 + \sin A - \cos A} = 2\operatorname{cosec} A.$$

14. If $\sin\theta = \dfrac{p^2 - q^2}{p^2 + q^2}$, find the other circular functions of θ.

15. If x is less than 90°, shew that $\sin x + \cos x$ is greater than 1.

16. The altitude of an equilateral triangle is p. Calculate the lengths of the sides.

17. The radius of the base of a cone 3 ft. high is 1 ft. 3 in. Calculate the vertical angle of the cone, using Tables.

18. The triangles in Fig. 54 are right angled at B. One side of each is x inches long and one angle contains $\theta°$ as shewn. Express the lengths of the other two sides of each triangle in terms of x and θ.

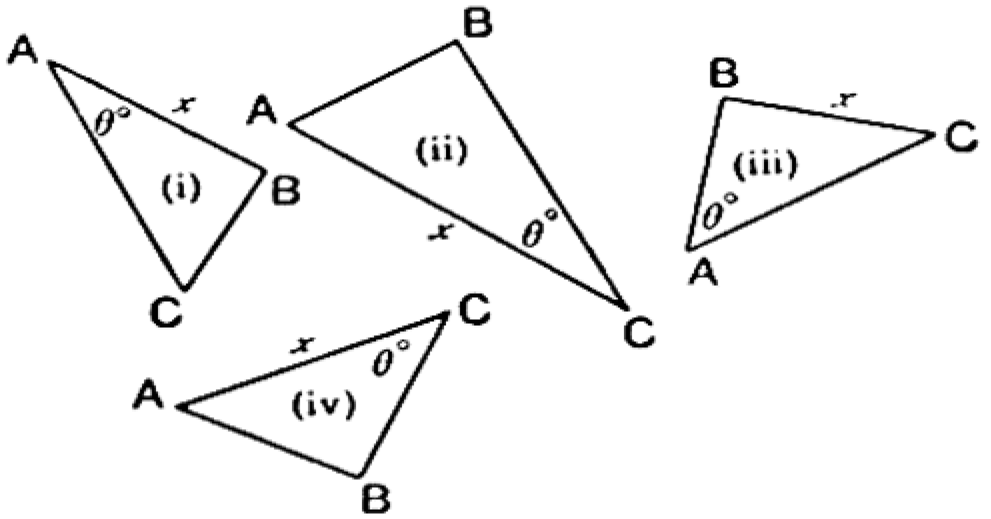


FIG. 54.

Simple Applications. The angle which a line joining a distant object to a point makes with its projection[*] on a horizontal plane through that point is called the *angle of elevation* if the object is above the point. If the object is below the point, this angle is called the *angle of depression.*

Ex. 1. *When the angle of elevation of the sun is 50°, what is the length of the shadow cast by a 10 ft. pole?*

Let the length of the shadow be x ft.
Using Fig. 55,

$$\tan 50^\circ = 10/x;$$

$$\therefore\ x = 10/\tan 50^\circ = 10 \cot 50^\circ.$$

From a 4-figure table of natural cotangents (or tangents),

$$\cot 50^\circ (\text{or } \tan 40^\circ) = 0{\cdot}8391;$$

$$\therefore\ x = 8{\cdot}39.$$

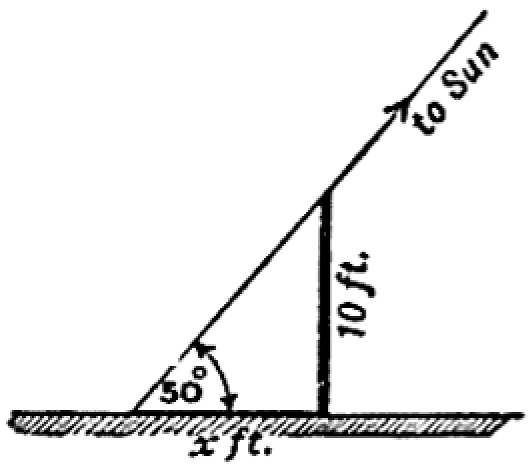


FIG. 55.

N.B.—After using 4-figure tables, the result cannot be relied on as accurate beyond three significant figures.

Ex. 2. *A boat, known to be 220 ft. from the foot of a cliff, is observed at an angle of depression of 25° 8′ from the top of the cliff. Find the height of the cliff.*

If B shews the position of the boat, the angle CBO (Fig. 56) is evidently equal to the angle marked as the angle of depression.

Let h ft. be the height of the cliff. Then evidently

$$h = 220 \tan 25^\circ 8'.$$

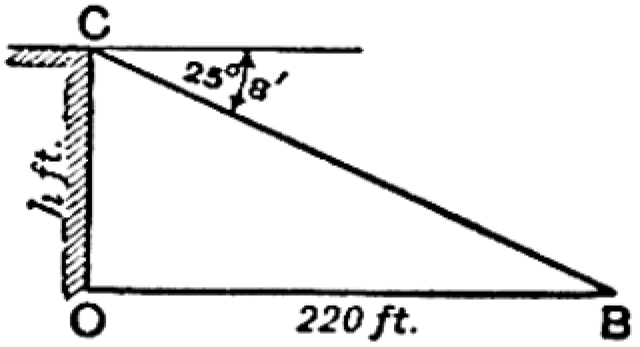


FIG. 56.

From the tables,

$$\tan 25^\circ 8' = 0{\cdot}4691;$$

$$\therefore\ h = 220 \times 0{\cdot}4691;$$

$$\therefore\ \log h = 2{\cdot}3424 + \bar{1}{\cdot}6713$$

$$= 2{\cdot}0137;$$

$$\therefore\ h = 103.$$

---

[*] See p. 139.

Ex. 3. *A ship steams* 10 *miles from* A *to* B *in a direction* 23° *North of East, and then for* 20 *miles from* B *to* C *in a direction* 51° *North of East. In what direction should the ship steam to go direct from* A *to* C, *and what would be the length of its course then?*

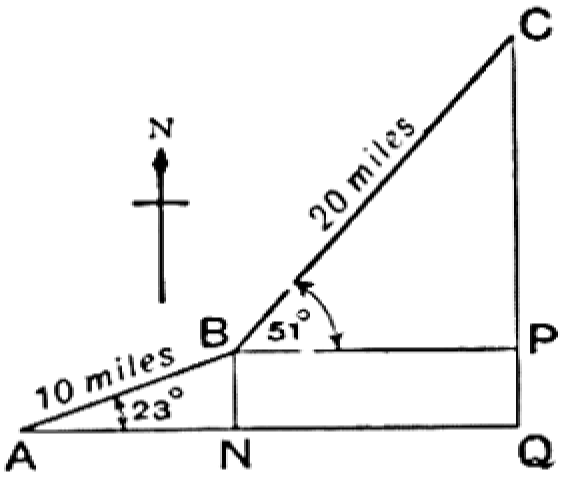


FIG. 57.

From Fig. 57,

$$AN = 10\cos 23^\circ = 9{\cdot}205. \ldots\ldots(1)$$
$$BN = 10\sin 23^\circ = 3{\cdot}907. \ldots\ldots(2)$$
$$BP = 20\cos 51^\circ = 12{\cdot}586. \ldots\ldots(3)$$
$$CP = 20\sin 51^\circ = 15{\cdot}542. \ldots\ldots(4)$$

From (1) and (3), $AQ = 21{\cdot}791$.

From (2) and (4), $CQ = 19{\cdot}449$;

$$\therefore\ \tan CAQ = \frac{19{\cdot}449}{21{\cdot}791} = 0{\cdot}8926, \text{ using logarithms;}$$

$\therefore$ the angle $CAQ = 41^\circ 45'$, from the tables.

$\therefore$ the ship must steam $41^\circ 45'$ North of East to go direct from A to C. This direction can be written E $41^\circ 45'$ N.

Again, $AC = AQ \sec CAQ = 21{\cdot}79 \times 1{\cdot}341$;

$\therefore$ $AC = 29{\cdot}2$ miles, using logarithms.

Ex. 4. *The angles of elevation of a mountain peak, taken from two points due West of it and* 150 *yards apart, are* $33^\circ 31'$ *and* $36^\circ 55'$. *Find the height of the mountain in feet.*

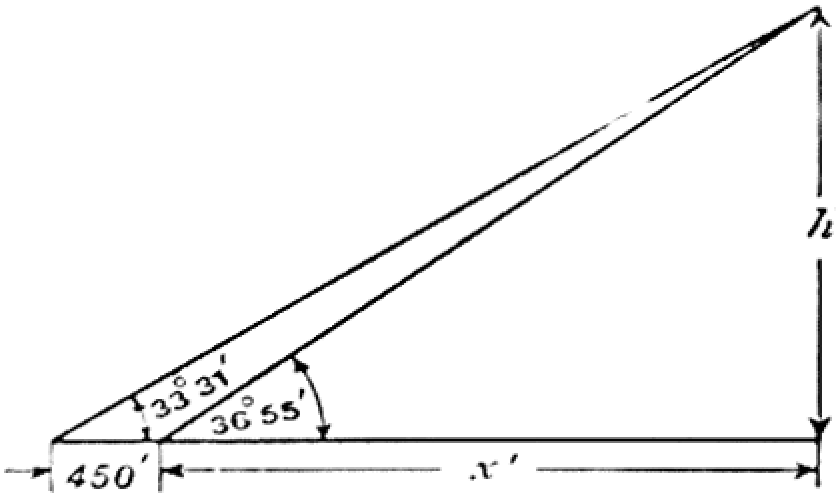


FIG. 58.

Let h feet be the height of the mountain (Fig. 58).

Using the figure, we have

$$\cot 33^\circ 31' = \frac{x+450}{h},$$

$$\cot 36^\circ 55' = \frac{x}{h};$$

$$\therefore\ h(\cot 33^\circ 31' - \cot 36^\circ 55') = (x+450) - x,$$

i.e. $h(1{\cdot}5098 - 1{\cdot}3311) = 450$;

$$\therefore\ h = \frac{450}{0{\cdot}1787} = 2519.$$

The mountain is about 2520 feet high.

Ex. 5. *Shew that the area of any triangle is one half the product of two sides multiplied by the sine of the included angle.*

Let ABC be any triangle (Fig. 59).

The side opposite $\hat{A}$, viz. BC, is generally denoted by a.

Similarly CA is b and AB is c.

Let p be the perpendicular from B to AC.

Then,

$$\begin{aligned}\text{area of } ABC &= \tfrac{1}{2}b \, . \, p \\ &= \tfrac{1}{2}b \, . \, a \sin C.\end{aligned}$$

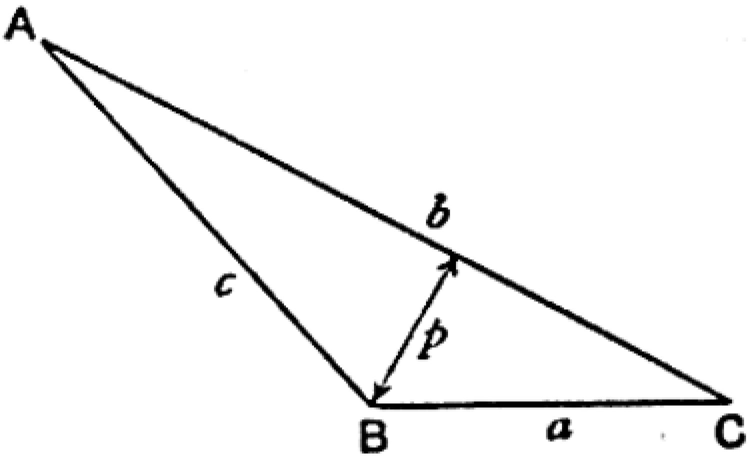


FIG. 59.

Similarly the area can be proved equal to $\tfrac{1}{2}bc \sin A$.

So far the circular functions of obtuse angles have not been defined, but in Ch. XI., p. 325, this formula will be found quite general, so that the area of the triangle ABC is also equal to $\tfrac{1}{2}ac \sin B$.

Ex. 6. *Shew that the length of any side of a triangle divided by the sine of the opposite angle is equal to the diameter of the circumscribing circle.*

Let ABC be any triangle (Fig. 60) and R the radius of its circumscribing circle, centre O.

Produce CO to meet the circumference again in D.

Then $C\hat{D}B = C\hat{A}B$, being in the same segment, and $C\hat{B}D$ is a right angle, being the angle in a semicircle;

$$\therefore \ \frac{a}{CD} = \sin CDB;$$

$$\therefore \ a = 2R \sin A$$

or $$\frac{a}{\sin A} = 2R.$$

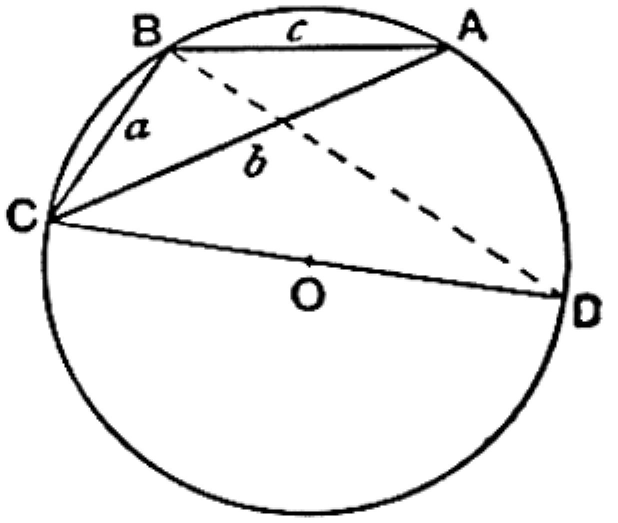


FIG. 60.

Similarly $$\frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

N.B.—The observation about obtuse angles made at the end of the last example applies in this case.

Ex. 7. *Find the area of a regular polygon of* 10 *sides which is* (i) *inscribed in,* (ii) *circumscribed about, a circle of radius* 2 *ft.*

The angle subtended at the centre of the circle by a side of the polygon is $\frac{360^\circ}{10}$ or 36° in each case.

(i) In Fig. 61 (i), the area of the triangle OA_1A_2 is

$$\tfrac{1}{2}OA_1 \,.\, OA_2 \,.\, \sin A_1OA_2 \quad \text{or} \quad \tfrac{1}{2} \,.\, 2^2 \sin 36^\circ, \text{ by Ex. 5.}$$

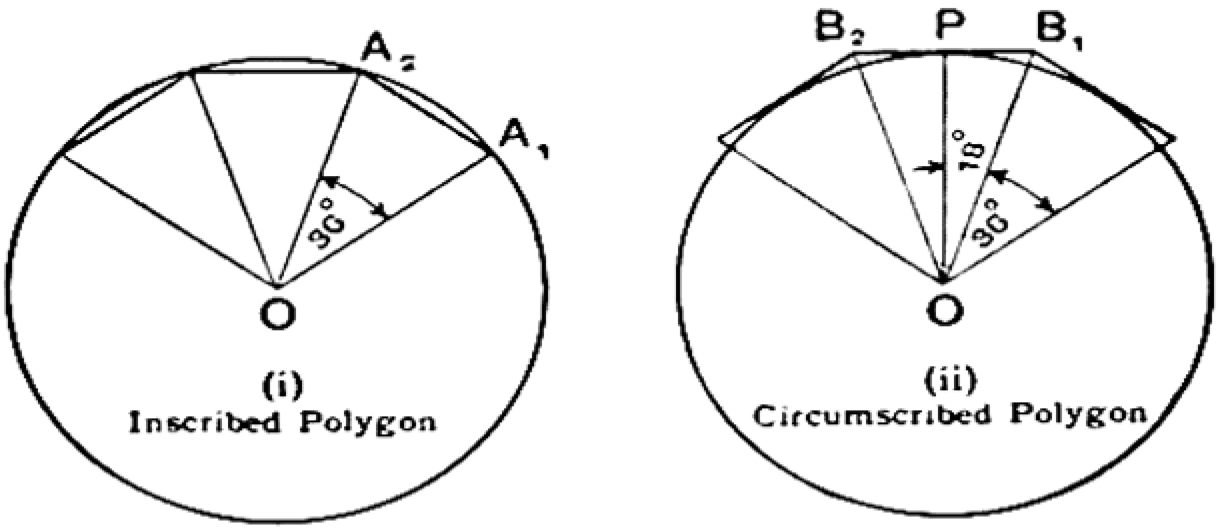


FIG. 61.

The area of the polygon, S sq. ft., is therefore,

$$10 \times \text{area of } OA_1A_2\,;$$

$$\therefore \; S = 20 \sin 36^\circ$$

$$= 20 \times 0{\cdot}5878\,;$$

$$\therefore \; S \doteqdot 11{\cdot}8.$$

(ii) In Fig. 61 (ii), the area of the triangle OB_1B_2 is

$$\tfrac{1}{2}OP \,.\, B_1B_2 \quad \text{or} \quad \tfrac{1}{2} \,.\, 2 \,.\, B_1B_2,$$

and $$B_1B_2 = 2PB_1 = 2\,(OP \tan 18^\circ) = 4 \tan 18^\circ\,;$$

$$\therefore \; \text{area of triangle } OB_1B_2 = 4 \tan 18^\circ\,;$$

$\therefore$ area of polygon, S' sq. ft., is $40 \tan 18^\circ$;

$$\therefore \; S' = 40 \times 0{\cdot}3249\,;$$

$$\therefore \; S' \doteqdot 13{\cdot}0.$$

Ex. 8. *In a triangle ABC, if $AB = 300$ feet, $CAB = 75° 15'$ and $CBA = 53° 28'$, find AC and BC.* [I.M.E.]

Since the sum of the three angles of a triangle is 180°, the angle $ACB = 51° 17'$.

Using the result of Ex. 6, viz.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we have
$$\frac{b}{\sin 53° 28'} = \frac{300}{\sin 51° 17'};$$

$$\begin{aligned} \therefore \log b &= \log 300 + \log \sin 53° 28' - \log \sin 51° 17' \\ &= 2{\cdot}4771 + \bar{1}{\cdot}9050 - \bar{1}{\cdot}8922 \\ &= 2{\cdot}4899; \end{aligned}$$

$$\therefore AC = 309 \text{ ft.}$$

Again,

$$\frac{a}{\sin 75° 15'} = \frac{300}{\sin 51° 17'};$$

$$\begin{aligned} \therefore \log a &= \log 300 + \log \sin 75° 15' - \log \sin 51° 17' \\ &= 2{\cdot}4771 + \bar{1}{\cdot}9854 - \bar{1}{\cdot}8922 \\ &= 2{\cdot}5703; \end{aligned}$$

$$\therefore BC = 372 \text{ ft.}$$

EXERCISES XIV.

1. A man, who is 250 feet from the base of a tower, sees the top of it at an elevation of 21° 32′. What is the height of the tower?

2. From a ship's mast, 140 feet above sea level, the angle of depression of a buoy is observed to be 34° 30′. How far is the buoy from the ship?

3. Find the perimeter and area of a regular octagon inscribed in a circle of radius 5 in.

4. What is the length of a chord which subtends an angle of 38° 27′ at the circumference of a circle of radius 3·28 in.?

5. A man who is 5 ft. 8 in. high stands 30 feet away from a window, the bottom of which is 12 ft. from the ground. If the window is $4\frac{1}{2}$ ft. high, what angle does it subtend at the man's eye?

6. Find the area of a parallelogram whose sides are 6·7 in., 7·3 in., and contain an angle of 53°.

7. Two observers a mile apart, who are in the vertical plane containing a balloon and on the same side of it, find that the angles of elevation of the balloon are 69° 15′ and 39° 23′ respectively. How high is the balloon ?

8. A flagstaff 20 ft. high is erected at the centre of the top of a circular column of diameter 40 ft. From a point on the ground, the top of the column and the top of the flagstaff are observed at elevations of 20° 47′ and 21° 30′. Find the height of the column.

9. A and B are the vertices of two isosceles triangles on the same base. The angles at A and B are 132° 30′ and 75° 48′. Find the length of the base, given that $AB=3$ inches. (1) when AB are on the same side of the base ; (2) when AB are on opposite sides of the base.

10. A man walks 3 miles from A to B in a direction 23° North of East ; 2 miles from B to C in a direction 50° North of East ; and then 2·5 miles from C to D in a direction 20° West of North. Find the bearing of D from A and the distance AD.

11. A ship is steaming on a course 22° South of West. At a certain instant it is 10 miles due North of a lighthouse. After steaming 10 miles a second lighthouse is observed to be due North-west. After steaming another two miles the second lighthouse is observed to be due North of the ship. How far apart are the lighthouses, and what is the bearing of one as seen from the other ?

12. A regular polygon of 15 sides is inscribed in a circle. Its area is 24·5 sq. in. Find the radius of the circle.

13. Fig. 62 illustrates a well-known construction for finding the semi-circumference of a circle. O is the centre, AOB a diameter and TBS the tangent at B. OT is a line at 30° to OB and TS is cut off equal to three times OB. Deduce an expression for AS in terms of the radius r, and thus shew that it is approximately equal to the semi-circumference of the circle.

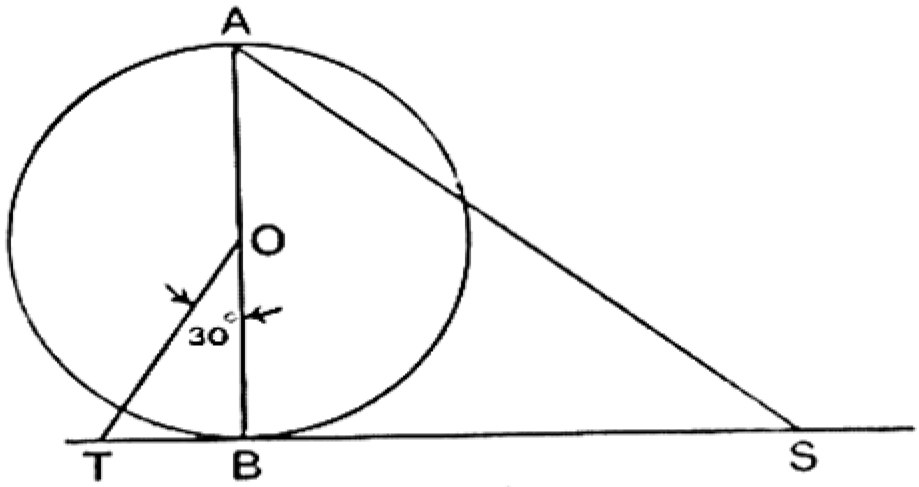


FIG. 62.

14. In a circle of radius 5·67 cm., a triangle is inscribed, two of whose angles are 37° 43′ and 82° 13′. Find the lengths of its sides.

15. Two angles of a triangle are $35^\circ\ 39'$ and $103^\circ\ 15'$, and the side between them is 18·7 inches long. Find the radius of the circumscribing circle.

16. Prove that the area of a triangle is $\frac{1}{2}c^2 \sin A \sin B \operatorname{cosec} C$.

17. Prove $a = b \cos C + c \cos B$.

18. In the triangle ABC, $\hat{A} < \hat{B}$. If X is the midpoint of AB, prove that $2 \cot C\hat{X}B = \cot A - \cot B$.

19. PQR is a triangle and X is a point in QR such that $Q\hat{P}X = \theta^\circ$ and $R\hat{P}X = \phi^\circ$. Shew that

$$\frac{QX}{XR} = \frac{\sin R \sin \theta}{\sin Q \sin \phi}.$$

20. Shew that the area of any quadrilateral is one-half the product of its diagonals multiplied by the sine of the angle between them.

21. In a triangle ABC, $a = 314$; $b = 245$; $A = 62^\circ\ 17'$. Find, by calculation, the other side and the other two angles to the nearest minute. [I.E.C.]

22. A surveyor measures a distance up a hill and finds it to be 473 feet, and also measures the inclination to the horizontal $12^\circ\ 33'$: he wishes to determine the height of one point above the other and also the horizontal projection of the distance. Calculate these values. [S.W.]

23. The top of a spire is observed to have an elevation of 56° as seen from the ground, and an elevation of 40° at a point 40 feet above. Calculate the height of the spire and its horizontal distance from the observer: and verify by making a drawing to scale on squared paper. [S.W.]

24. Find expressions for the areas of regular polygons of n sides inscribed and circumscribed to a circle of radius a.

Taking $n = 100$, shew that the value of π lies between 3·139 and 3·143. ($\sin 3^\circ\ 36' = 0{\cdot}06279$, $\tan 1^\circ\ 48' = 0{\cdot}03143$.)

25. From two consecutive milestones A, B on a straight road a distant object C is seen. The angles CAB, CBA are $67^\circ\ 15'$, $73^\circ\ 26'$ respectively. Determine the distance of C from the nearest point of the road. [I.M.E.]

26. In Fig. 63 find the perpendicular distance of A from the line CX.

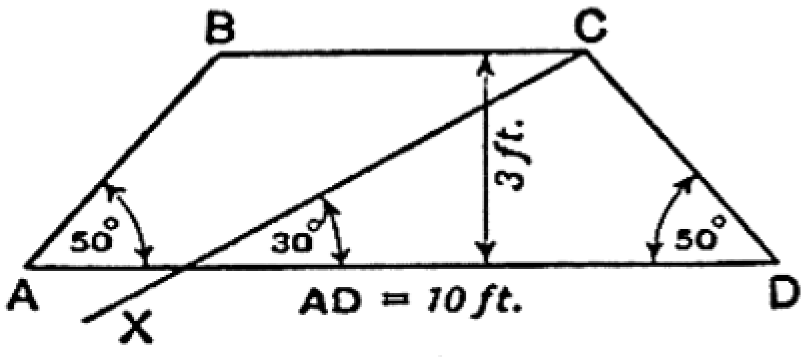


FIG. 63.

27. Work Exercise 26 when $AD=2a$, distance between AD and $BC=h$, angles at A and D are α°, and the line CX makes θ° with AD.

28. An aeroplane is observed, first when it is vertically above a point A, next when, one minute later, it is vertically above B. Its first angular elevation, viewed from B, is 20° ; and its distance from B is the same in each case. If AB be 500 yards, and level, and if the aeroplane travels straight, find its speed in feet per second, and the inclination of its line of flight to the horizon. [S.W.]

29. AB, BC, CD are straight lines, $AB=200$ feet, $BC=320$ feet and $CD=430$ feet. If B be N. 20° E. of A, C be S. 40° E. of B, and D be S. 25° W. of C, find the distance and bearing of D from A. [B.U.]

30. Find the radii of the circles circumscribing and inscribed in a regular polygon of n sides, the length of each side being $2a$.

In a regular octagon a circle is inscribed and a second regular octagon is inscribed within it. Prove that the ratio of the radii of the circumscribed circle of the first octagon and the inscribed circle of the second is $4-2\sqrt{2}:1$. [B.U.]

31. The route from the bottom of the shaft of a coal mine to a point of the coal face follows three level roads which are 600 yards N. 25° 12′ E., 400 yards N. 42° 35′ E., and 57 yards N. 33° E. Find how far N. and how far E. the point of the coal face is from the shaft. If the coal rights of the mine are limited to a plot of land, two sides of which run 1 mile E. and 1 mile N. of the shaft, find how much farther the third road can be taken. [Man.]

32. The distance and bearing of B from A are 5 miles and 75° East of North, of C from B they are 10 miles and 20° East of North, and of D from A they are 2 miles and 40° West of North. Calculate the distances of C East and North of D, and deduce the distance and bearing of C from D.

Verify your results by a carefully drawn figure. [L.U.]

33. The angle subtended (in a horizontal plane) by a cylindrical gasometer is 150° at a certain point ; at a point 10 feet further away the angle subtended is 130°. Find the diameter of the gasometer. [L.U.]

34. Why are tables of logarithms of sines, cosines, and tangents used in preference to the tables of the trigonometric ratios themselves in solving triangles ?

Two observers on the same level, one mile apart, observe an airship when it is vertically over the line (produced) joining them. The angles of elevation are 23° 12′ and 31° 18′. Find its height in feet. [S.W.]

35. A length 150 ft. is measured from A to B, and the angles CAB and ABD are each found to be 57° 30′ while the angles DAB and ABC are each 41° 24′. Calculate the distances AC, AD, CD. [S.W.]

36. A man making for a village which lies along a road at right angles to that in which he is walking, can keep the church tower in view. He observes its direction makes an angle of 57° with the road, and after walking another 400 feet along the same road the angle has increased to 60°. Shew that he still has rather more than a mile and a half to go to reach it. [Durham.]

Examples 37-41 *are taken from papers set at the Inter. B.Sc.* [L.U.]

37. A man whose height is h stands at the edge of a cliff and observes the angle of depression of an object in the sea to be β. He then steps back away from the object, for a distance d, and finds that the object is in a line with his eye and the edge of the cliff. Shew that the height of the cliff is

$$h^2 \cot \beta/(d - h \cot \beta).$$

38. A tower AB stands on a hill, which may be taken to be a plane inclined at an angle θ to the horizon. BCD is the line of greatest slope of the hill through the foot of the tower B. The angles ACB, ADB are found to be α and β respectively, and the length CD is x; prove that the height of the tower is

$$x \sin \alpha \sin \beta/\sin(\alpha - \beta) \cos \theta.$$

39. Prove that the diameter of the circle circumscribing a triangle ABC is equal to $a/\sin A$.

A spire BC stands on a tower CD; prove that the greatest angle which BC subtends at a point in a horizontal line through D is the angle BAC, where A is such that the circle BAC touches DA.

Find the height of the tower when that of the spire is 40 ft. and the angle BAC is 20°.

40. From a ship steaming at 10 miles an hour in a direction 78° East of North, the bearing of a lighthouse is observed to be 40° East of North at 10.0 p.m., and to be 35° West of North at 11.0 p.m. Find the distance of the lighthouse from the ship's path. (Neglect the curvature of the Earth.)

41. At a point of a straight road running due N. a tower bears α° W. of N., and the angle of elevation of its summit is θ°. At a point a ft. further on, the bearing changes to β° W. of N., and the angle of elevation to ϕ°.

Prove that $\cot \theta \sin \alpha = \cot \phi \sin \beta$; also that the height of the tower is $a \tan \theta \sin \beta/\sin(\beta - \alpha)$.

Projection. The **projection** of one line upon another line is the intercept between the feet of the perpendiculars let fall from the ends of the first line upon the second.

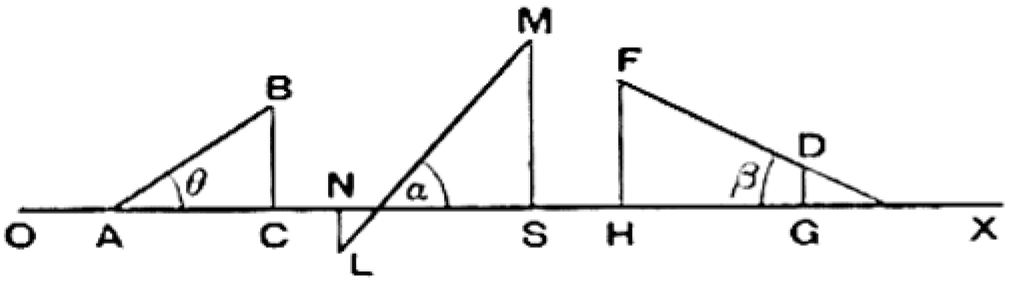


FIG. 64.

Thus in Fig. 64 the projection of AB on OX is AC,
the projection of LM on OX is NS,
the projection of DF on OX is GH.

Evidently, from the definition of the cosine of an angle,

$$AC = AB \cos\theta, \quad NS = LM \cos\alpha \text{ and } GH = DF \cos\beta.$$

That is, the projection upon OX of a line of length L inclined at any angle θ to OX is $L \cos\theta$.

The Projection of a Line on a Plane. The projection of a line on a plane is the straight line joining the feet of the perpendiculars let fall on the plane from the ends of the line. In Fig. 65, *ab* is the **plan** of AB, the line in space; that is to say, Aa, Bb are each perpendicular to the horizontal plane, and ab is the projection of AB on that plane. Similarly $a'b'$ is the projection of AB on the vertical plane, and is the **elevation** of the line on that plane. If a line is horizontal, the length of its plan is equal to the length of the line.

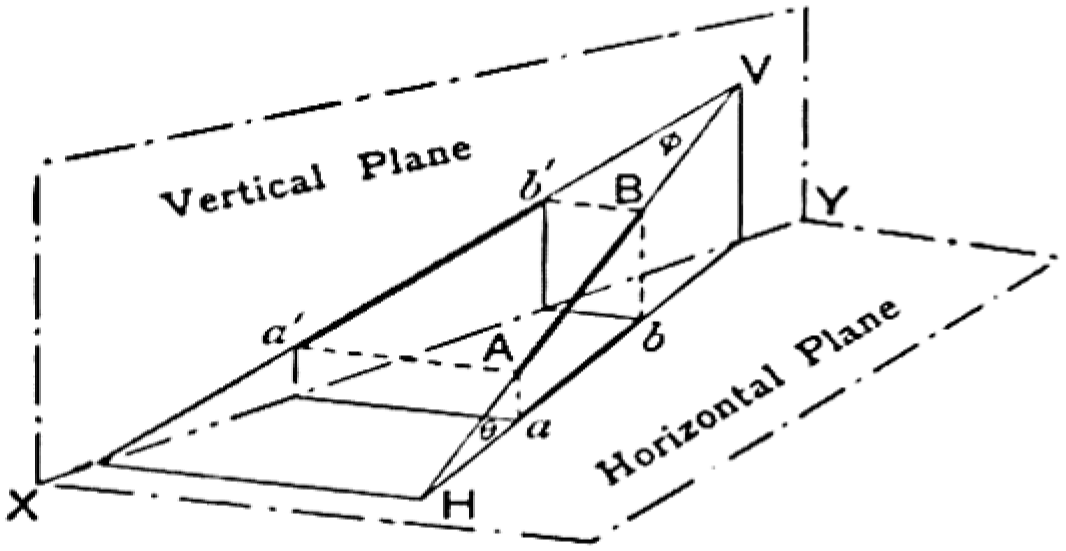


FIG. 65.

The angle between a line and a plane is the angle between the line and its projection on the plane. Thus the inclination of AB to the horizontal plane (Fig. 65) is the angle AHa, *i.e.* θ; the inclination of AB to the vertical plane, is the angle BVb', *i.e.* ϕ. **The trace of a line on a plane is the point where the line meets the plane.** Thus the trace of AB on the horizontal plane is the point H, the vertical trace of AB is V.

Angle between two Planes. In Fig. 66, two planes P and Q, intersecting in the line LM, are shewn in perspective. LM is called the **trace** of the plane P on the plane Q. Take any point H in the line of intersection and draw HA, in the plane P, perpendicular to LM; draw Ha, in the plane Q, also perpendicular to LM. The angle between the planes P and Q is defined to be the angle between the lines HA and Ha, *i.e.* the angle θ in the figure.

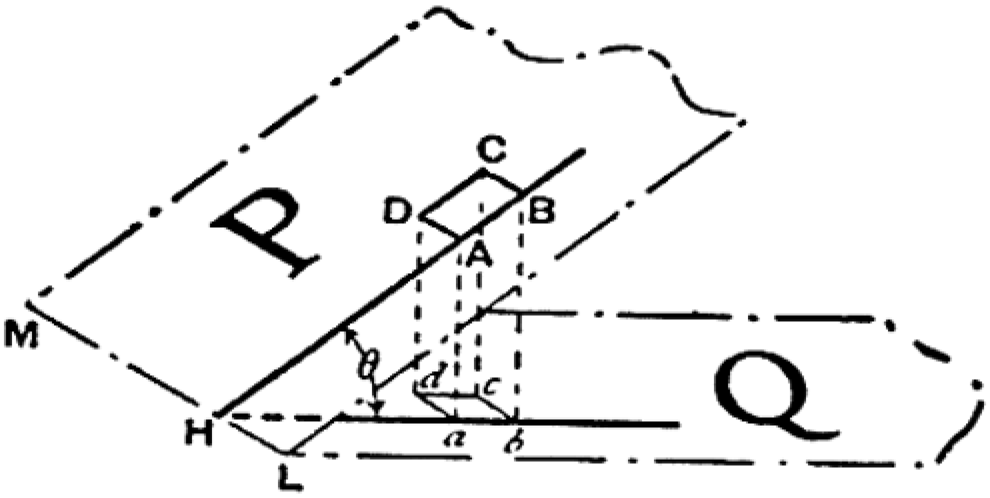


FIG. 66.

Line of Greatest Slope. A line of greatest slope in a plane is perpendicular to a horizontal line in it. Thus, in Fig. 66, if Q is a horizontal plane, LM is a horizontal line and HA is a line of greatest slope in the plane P, LHA being 90°.

The angle between two non-intersecting lines is defined as the angle between two lines parallel to them which do intersect.

The projection of a line AB on another line XY, which does not intersect it, equals $AB \, . \cos\theta$, where θ is the angle between the lines AB, XY. For suppose Q (Fig. 67) is the plane through A perpendicular to XY meeting it in C, and P the plane through B perpendicular to XY meeting it in E. Then the lines AC, BE are each perpendicular to XY, and hence CE is the projection of AB on XY.

Let CD be parallel to AB meeting P in D. Now $ACDB$ is a plane intersecting two parallel planes P, Q, and hence the traces AC, BD are parallel. Hence $ACDB$ is a parallelogram, and therefore $CD = AB$. Also the angle DCY equals the angle between AB and XY, *i.e.* θ.

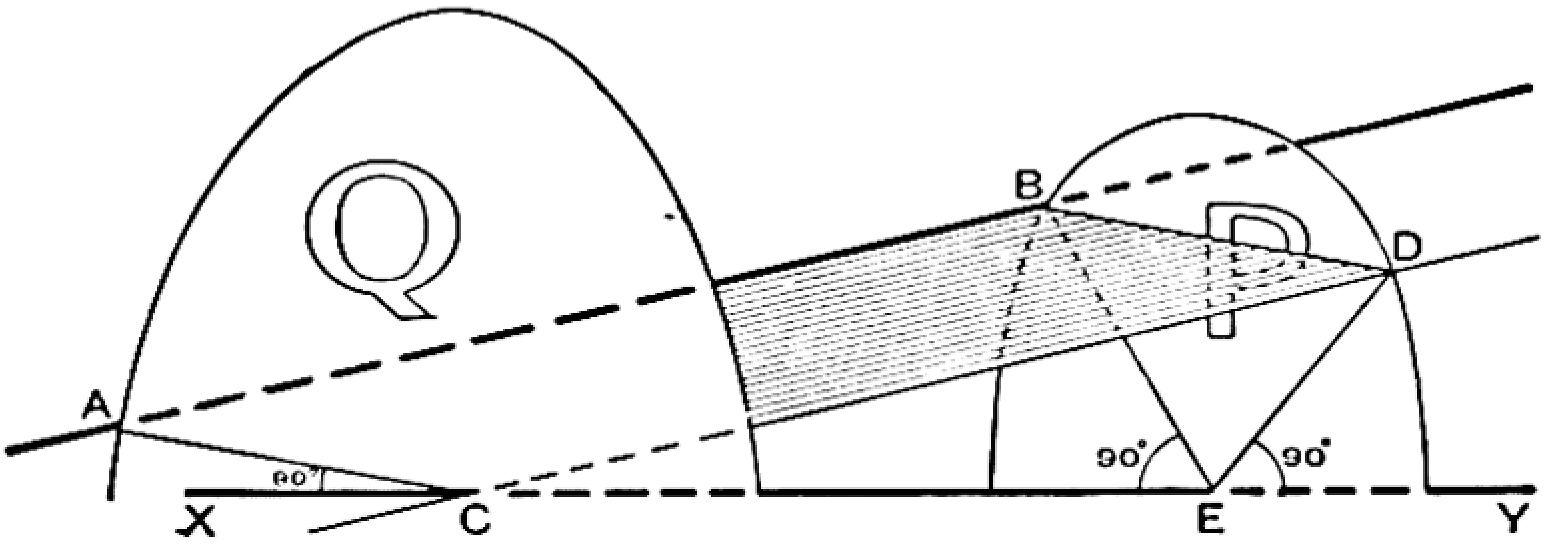


FIG. 67.—Projection of line AB on line XY.

Also DE, lying in the plane P, is at right angles to XY. Therefore

$$CE = CD \cos\theta,$$

$$\textit{i.e.} \qquad CE = AB \cos\theta,$$

i.e. the projection of AB on XY equals $AB \cos$ (angle between AB and XY).

Projection of an Area. Let $ABCD$ (Fig. 66) be a rectangle in plane P, having the sides AB, DC along lines of greatest slope and the sides AD, BC horizontal.

$abcd$ is the projection of $ABCD$ on plane Q.

Evidently $bc = BC$ (since BC is horizontal, and therefore parallel to the plane Q)

$$\text{and} \qquad ab = \text{the projection of } AB \text{ on } Ha$$
$$= AB\cos\theta \text{ (by definition)};$$
$$\therefore \text{ projected area} = ab \,.\, bc$$
$$= AB\cos\theta \,.\, BC$$
$$= \text{original area} \times \cos\theta.$$

In general, if θ is the angle between two planes, S is an area in one plane and S' the projected area on the other plane, it can be shown that

$$S' = S\cos\theta.$$

In words, **The area of the projection of a plane figure is equal to the area of the figure multiplied by the cosine of the angle of inclination of the figure to the plane upon which it is projected.**

Ex. 1. *A tower 200 ft. high is due West of an observer. The observer walks 100 yards due South and observes that the angle of elevation of the tower is 25°. How far was he from the tower originally?*

A (Fig. 68) denotes the first position of the observer, B his second position and D the foot of the tower. Let the top of the tower be H.

Then the angle HDB is 90°; and the angle HBD is 25°.

$$\therefore \quad \frac{HD}{BD} = \tan 25° ;$$

$$\therefore \quad BD = \frac{HD}{\tan 25°} = \frac{200}{0{\cdot}4663} \text{ ft.};$$

$$\begin{aligned}
\therefore \quad \log BD &= 2{\cdot}3010 - \bar{1}{\cdot}6687 \\
&= 2{\cdot}6323 ; \\
\therefore \quad BD &= 428{\cdot}8 \text{ ft.} \\
\text{and} \quad AB &= 300 \text{ ft.} \\
\text{Now} \quad BD^2 &= BA^2 + AD^2 ; \\
\therefore \quad AD &= \sqrt{\{428{\cdot}8^2 - 300^2\}} \text{ ft.} \\
&= \sqrt{\{728{\cdot}8 \times 128{\cdot}8\}} \text{ ft.}; \\
\therefore \quad \log AD &= \tfrac{1}{2}[2{\cdot}8626 + 2{\cdot}1099] \\
&= 2{\cdot}4863 ; \\
\therefore \quad AD &\doteqdot 306 \text{ ft.}
\end{aligned}$$

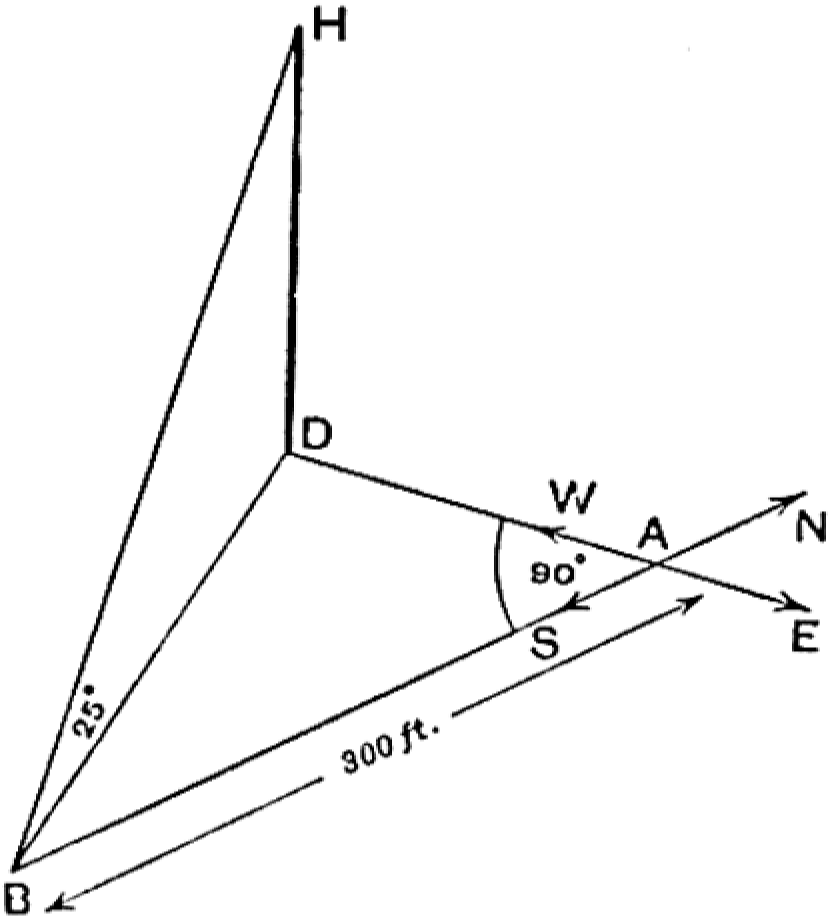


FIG. 68.

Ex. 2. *A hill 300 ft. high is inclined at 30° to the horizontal, and a path on the hill is inclined at 10° to a line of greatest slope. Find the inclination of the path to the horizontal and its length.*

In Fig. 69, O is the trace of the path on the horizontal plane H, OA is a line of greatest slope of the hill, so that

$$P\hat{O}A = 10°.$$

A is the projection of P on OA, so that $O\hat{A}P = 90°$; p, a are the projections of P, A on H, and therefore the angles PpO, AaO are each 90°.

AOa is a vertical plane, PA is a horizontal line on the hill, and therefore PA is perpendicular to the plane AOa. But pa is the plan of the horizontal line PA, and therefore equal and *parallel* to PA; hence pa is at right angles to the plane AOa, and therefore the angle $Oap = 90°$.

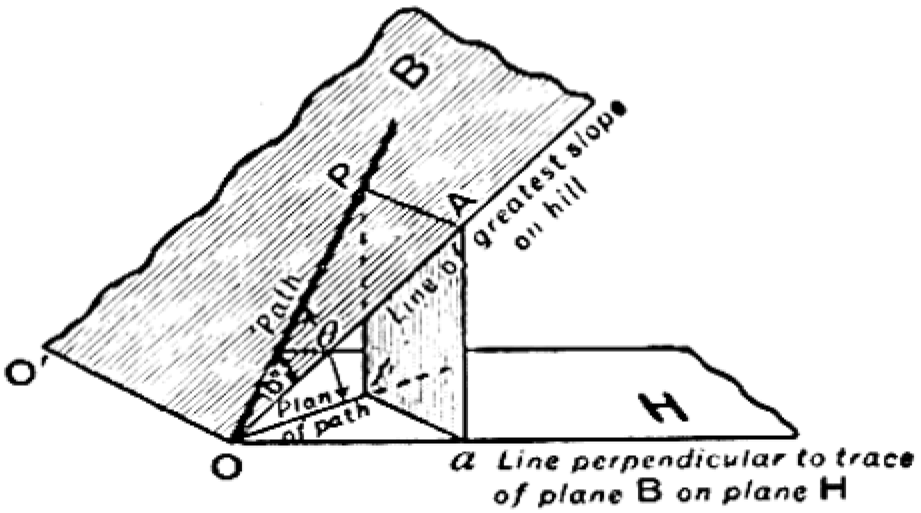

B
P
A
Line of greatest slope on hill
Path
10°
θ
Plan of path
p
O'
O
H
a Line perpendicular to trace of plane B on plane H

FIG. 69.

Now
$$\begin{aligned} OA &= \text{the projection of } OP \text{ on } OA \\ &= OP\cos 10° \text{ by definition;} \\ Aa &= \text{the projection of } OA \text{ on the vertical} \\ &= OA\sin 30° \\ &= OP\cos 10°\sin 30°. \qquad\qquad \text{......(1)} \\ Pp &= \text{the projection of } OP \text{ on the vertical} \\ &= OP\sin\theta. \qquad\qquad \text{......(2)} \end{aligned}$$

But $Pp = Aa$; therefore, using (1) and (2),

$$OP\sin\theta = OP\cos 10°\sin 30°;$$

$$\begin{aligned} \therefore\ \log\sin\theta &= \log\cos 10° + \log\sin 30° \\ &= \bar{1}\cdot9934 + \bar{1}\cdot6990 \\ &= \bar{1}\cdot6924; \end{aligned}$$

$$\therefore\ \theta = 29°\ 30'.$$

If the hill is 300 ft. high, the length of the path is

$$300 \operatorname{cosec} 29°\ 30'.$$

From the tables, $\operatorname{cosec} 29°\ 30' = 2\cdot0308$.
Therefore the path is about 609 ft. long.

Ex. 3. *A strut, A sq. in. cross-sectional area, carries a load of P tons. What is the load per sq. in. on a section making an angle of $\theta°$ with the cross-section? Resolve the total load normal* (i.e. *perpendicular*) *to and tangential* (i.e. *parallel*) *to the oblique section, and find how much per square inch it is each way.*

Let A' sq. in. be the area of the oblique section, then the cross-section is the projection of all sections of the bar on a plane perpendicular to its axis; therefore

$$A = A' \cos\theta; \qquad \therefore\ A' = A \sec\theta.$$

The load per sq. in. on the oblique section is consequently

$$\frac{P}{A\sec\theta} \text{ tons} \quad \text{or} \quad \frac{P\cos\theta}{A} \text{ tons}.$$

Now the axis of the bar is perpendicular to the cross-section, and the normal to the oblique section is perpendicular to the oblique section; so that the load P is inclined at $\theta°$ to the normal to the oblique section. Therefore the resolved part of the load normal to the oblique section is $P\cos\theta$.

Similarly the resolved part of the load tangential to the oblique section is $P\sin\theta$.

$$\therefore \text{ normal load per sq. in.} = \frac{P\cos\theta}{A\sec\theta} = \frac{P\cos^2\theta}{A}$$

$$\text{and} \quad \text{tangential load per sq. in.} = \frac{P\sin\theta}{A\sec\theta} = \frac{P\sin\theta\cos\theta}{A}.$$

Ex. 4. *V is the vertex of a right circular cone, height 10 in., radius of base 4 in. Points A, B, C are taken on the circumference of the base, so that*

$$\hat{A} = 30°, \ \hat{B} = 70°, \ \hat{C} = 80°.$$

Find (i) *the inclination of CV to the base of the cone,* (ii) *the side BC,* (iii) *the angle CVB, and* (iv) *the inclination of the plane VBC to the base of the cone.*

Fig. 70 shews the plan of the cone and pyramid $VABC$, and an elevation on a vertical plane perpendicular to BC.

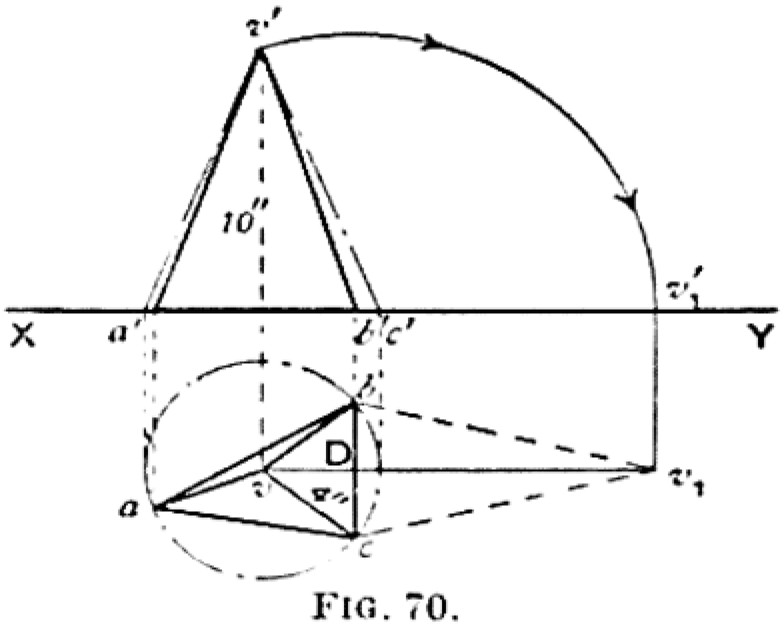


FIG. 70.

(i) cv is the plan of the line CV, of which the end V is at height 10 in. and the end C is on the horizontal.

$\therefore$ CV is inclined to the base at an angle θ°, such that

$$\tan\theta^\circ = \tfrac{10}{4} = 2{\cdot}5 \quad \text{or} \quad \theta^\circ = 68^\circ\,12'.$$

(ii) By Ex. 6, p. 132, $bc = 2R\sin a$

$$= 8\sin 30^\circ \quad \text{or} \quad 4.$$

Alternatively, $b\widehat{v}c$ is 60° (since it is $2b\widehat{a}c$) and $bv = vc$;

$\therefore$ the triangle bvc is equilateral and $bc = 4$ in.

(iii) By Pythagoras's Theorem $VC = VB = \sqrt{10^2 + 4^2} = 10{\cdot}77$ in. With centre c' and radius $c'v'$ describe an arc cutting the ground line XY in v_1'. Through v draw vv_1 perpendicular to bc to meet the projector through v_1' in v_1.

Then bv_1c is the *rabatment* on the horizontal plane of the face BVC: *i.e.* bv_1c is the triangle BVC swung about BC into the horizontal plane.

If vv_1 cuts bc in D, $cD = Db$ and Dv_1 bisects $b\widehat{v_1}c$, since the triangle bv_1c is isosceles.

$$\operatorname{Sin} cv_1D = \frac{cD}{cv_1} = \frac{2}{10{\cdot}77};$$

$$\therefore \quad \operatorname{cosec} cv_1D = 5{\cdot}385;$$

$$\therefore \quad c\widehat{v_1}D = 10^\circ\,42' \text{ approximately};$$

$$\therefore \quad B\widehat{V}C = 21^\circ\,24' \text{ approximately}.$$

(iv) Since the elevation shews the edge view of the plane VBC, the angle between VBC and the horizontal is a°, where

$$\tan a = \frac{10}{vD}.$$

Now
$$vD = vc\cos cvD$$
$$= 4\cos 30^\circ,$$

$$\therefore \quad \tan a = \frac{10}{4\cos 30^\circ};$$

$$\therefore \quad \log\tan a = 1 - 0{\cdot}6021 - \bar{1}{\cdot}9375, \quad \text{or} \quad 0{\cdot}4604;$$

$$\therefore \quad a = 70^\circ\,53' \text{ approximately}.$$

This example and Ex. 5 illustrate the use of plans and elevations in solving problems in three dimensions.

Ex. 5. *The end elevation of a roof and dormer window is given in Fig. 71; ABC is the roof and BDF the side of the dormer window. The front elevation of the window is an equilateral triangle, df being the length of a side. If*

$$AB = BC = 18 \text{ ft.}, \quad AC = 30 \text{ ft.} \quad \text{and} \quad BF = 2FC,$$

find the angles which the line of intersection of the roof and sloping side of the dormer window make with the horizontal and with a side of the equilateral triangle. Calculate the area of the sloping face of the dormer window. [L.U.]

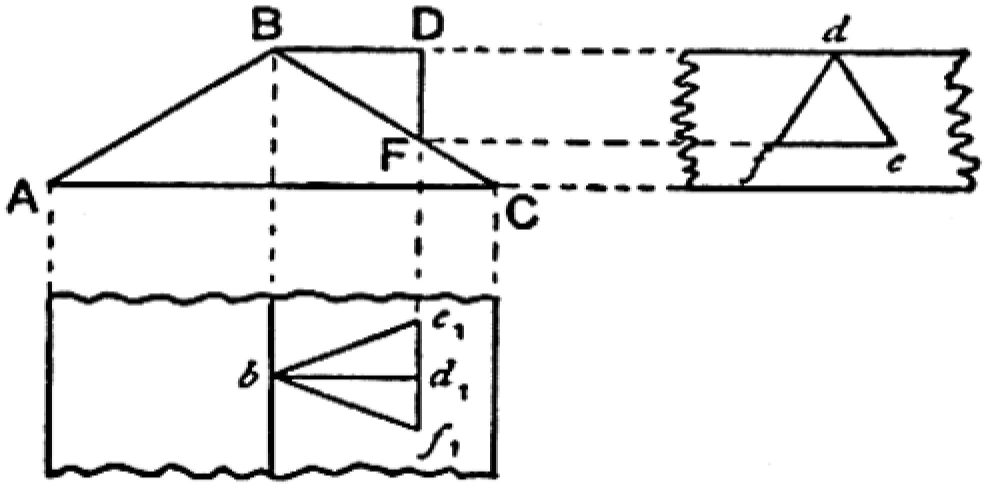


FIG. 71.

Fig. 71 shews the given elevation to scale with an end elevation dfe, and a plan $bd_1f_1e_1$ of the dormer window projected from it.

Since $AB = BC = 18$ ft., $AC = 30$ ft. and $BF = 2FC$,

$$\therefore \; BF = 12 \text{ ft.} \quad \text{and} \quad BD = 10 \text{ ft.}$$

Also
$$DF = \sqrt{12^2 - 10^2} = 2\sqrt{11},$$

and DF is the perpendicular height of the equilateral triangle def, the plan of which is $d_1e_1f_1$;

$$\therefore \; d_1e_1 = d_1f_1 = DF \tan 30^\circ$$
$$= \frac{2\sqrt{11}}{\sqrt{3}} = \frac{2\sqrt{33}}{3}.$$

From the plan,
$$bf_1 = \sqrt{(\overline{bd_1}|^2 + \overline{d_1f_1}|^2)}$$
$$= \sqrt{\left(100 + \frac{132}{9}\right)}$$
$$= \frac{\sqrt{1032}}{3};$$

$\therefore$ BF is a line inclined to the horizontal such that its plan is $\frac{\sqrt{1032}}{3}$ ft. long and F is $2\sqrt{11}$ ft. below B: that is, BF is inclined to the horizontal at an angle θ, such that

$$\tan\theta = \frac{2\sqrt{11}\times 3}{\sqrt{1032}} = 0{\cdot}6194\,;$$

$$\therefore\ \theta = 31^\circ\, 47'.$$

Since BD is horizontal and the plane DF is vertical, BD is perpendicular to the sloping side df.

The angle BF makes with df is a, where

$$\tan a = \frac{BD}{df} = 10 \bigg/ \frac{4\sqrt{33}}{3}$$

$$= \frac{7{\cdot}5}{\sqrt{33}}\,;$$

$$\therefore\ \log\tan a = 0{\cdot}8751 - 0{\cdot}7593$$

$$= 0{\cdot}1158\,;$$

$$\therefore\ a = 52^\circ\, 33'.$$

The area of the sloping face, whose plan is bd_1f_1, is

$$\tfrac{1}{2}bd_1 \times df\,;$$

$$\therefore\ \text{area} = 5\times 4\sqrt{33}/3 \text{ sq. ft.}$$

$$= 38{\cdot}3 \text{ sq. ft.}$$

EXERCISES XV.

1. Find the angle between a diagonal of a cube and any face which it meets.

2. A 60° set square stands with its shortest side in the horizontal plane. Its plane is vertical and inclined at 45° to another vertical plane P. Find the angle between the largest side of the set square and the plane P.

3. A hill side slopes at 29° to the horizontal, and there is a path inclined at 55° to a line of greatest slope. The hill is 1000 feet high. What is the length of the path if the hill side is plane ?

4. A plane area of 72·5 sq. ft. is inclined at 25° to the horizontal. Find its horizontal and vertical projections.

5. A seam of coal 6 ft. 6 in. thick is inclined at an angle of 15° to the horizontal. Find the volume of the coal under an acre of land.

6. A bar has a cross-section of 2·5 sq. in. (A cross-section is a section by a plane at right angles to the axis.) What is the area of a section making 45° with the cross-section?

The bar carries a load of 17,500 lb. What is the stress? What is the load per sq. in. on the inclined section? Resolve the total load normal (*i.e.* perpendicular) to and tangential (*i.e.* parallel) to the oblique section, and find how much per sq. in. it is each way.

7. A rectangle 4 ft. × 3 ft. has its plane vertical and inclined at 60° to a vertical plane P. One of its short sides stands in the horizontal plane. It is tilted about this side as a hinge until its plan is a square. What is its inclination to the horizontal plane in its new position? What are the angles between its diagonals and P in this position?

8. From a certain station A, a hilltop was observed to bear N.E. and its angle of elevation to be 24°. From a station S, 1200 yards due South of A and in the same horizontal plane, the bearing of the hill was N. 20° E. What is the height of the top of the hill?

9. From a point B, the angle of elevation to the top T of a tower TA was observed to be 24°. C is 400 feet from B, and the angles ABC, BCA are observed to be 95° and 54° respectively. Find the height of the tower.

10. There is a district in which the surface of the ground may be regarded as a sloping plane; its actual area is 3·246 square miles; it is shewn on the map as an area of 2·875 square miles; at what angle is it inclined to the horizontal? [B.E.]

11. The point D in space is equidistant from the three points A, B, C; prove that the foot of the perpendicular from D on the plane ABC is the circumcentre of the triangle ABC.

A tripod has three legs each 5 ft. long, and the feet are planted in the ground at points 3 ft. apart; find the height of the top of the tripod above the ground. [S.W.]

12. Prove that the height of a regular tetrahedron is equal to $\sqrt{(2/3)}$ of the length of each of its sides, and find the angle between two of its faces. [B.U.]

13. Opposite slant faces of a right square pyramid are at right angles; find the angle between two adjacent slant faces. [S.W.]

14. A, B, C, D are the corners of a rectangular picture. AB is the top edge, which is horizontal and 36 inches long. The edge AD is 24 inches long. The cord APB is attached at A and B and is 42 inches long. If the lower edge CD is supported against the wall on two nails in a horizontal line, so that the picture makes an angle of 10° with the wall, find the depth below the nail P of the edge CD. [I.C.E.]

15. The distance between two points AB, on opposite sides of a lake, was determined by observing the angles of elevation of a flagstaff of known height h situated at a point C such that the angle $CAB = 90°$. The angles of elevation of the flagstaff at A, B respectively were $\alpha°, \beta°$. Find an expression for AB in terms of h, α, β.

Shew also how to determine AB if h, α and the horizontal angle CBA are given, but not β. [I.M.E.]

16. The vertex V of a right pyramid is 10 inches vertically above the centre O of its base $ABCD$ which is a square of side 12 inches. Find, by calculation, to the nearest minute, (1) the angle between the edges VA, VB; (2) the angle between the edges VA, VC. [I.C.E.]

17. The base of a tetrahedron is an equilateral triangle ABC of 7 cm. side: the other edges VA, VB, VC are each 10 cm. long. Find (i) the inclination of VA to the plane ABC; (ii) the inclination of the face VAB to the plane ABC; (iii) the length of the perpendicular CD from C to the edge VB; and (iv) the angle between the faces VAB and VCB. Find also the volume of the tetrahedron.

Examples 18-25 *are taken from papers set at the Inter. B.Sc.* [L.U.]

18. A is the top, D is the foot of a vertical flagstaff; B and C are two points in the horizontal plane through D. If $BC = 214$ ft., $ABC = 73°$, $ACB = 66°$ and $ABD = 18°$, find the height of the flagstaff to the nearest inch.

19. From a point A in the horizontal plane through the foot of a tower, α is the angle subtended by the circumference and β is the elevation. Find the ratio of the height to the diameter of the tower. If the elevation be γ from a point B, where $AB = d$ and is towards the centre of the tower base, find the height and diameter.

20. A right pyramid, vertex V, stands on a regular pentagonal base $ABCDE$. If the length of a slant edge VA be 20 in., and if an edge AB of the base be 10 in., calculate the height and volume of the pyramid and the dihedral angle between a lateral face and the base.

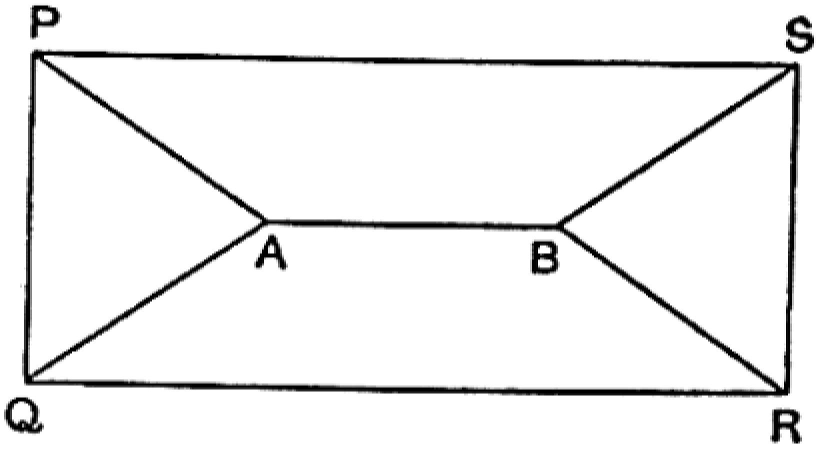
21. If N is the foot of the perpendicular from a point O to a plane, and if NX is drawn in this plane perpendicular to a line AB in the plane, prove that OX is perpendicular to AB.

AB is a horizontal line on the side of a hill and P is a point on the hill above AB such that the angle PAB is 30°. If the slope of AP is 1 in 20, calculate the angle of slope of the hill.

22. An observer notices that an aeroplane describing a horizontal circle of half a mile radius at a height 1,500 ft. is vertically overhead. Shew that when the angular elevation of the aeroplane becomes 23° it has described approximately either $\frac{7}{30}$ or $\frac{23}{30}$ of the circle.

23. How are angles between two planes and between a plane and a straight line measured?

$PQRS$ represents the plan of a symmetrical roof, AB being the ridge. If the height is 10 ft., $PS = 35$ ft., $PQ = 20$ ft., $AB = 17$ ft., find the inclinations to the horizontal of the straight line AP and the sloping parts of the roof. Determine the volume inside the roof.



24. By dividing a cube into 6 congruent pyramids, shew that the volume of any pyramid is $\frac{1}{3} \times$ base area $\times$ height.

The diagonal of the base of a square pyramid being 2 ft. and the height 5 ft., find the angle one of the lateral faces makes with the base. If the pyramid be cut by a plane through one of the base edges at an angle of 30° with the base, find the volume of the pyramid cut off and the angles the lateral edges make with the plane.

25. A triangular plate ABC rests with A on a horizontal plane so that AC and AB make 30° and 40° with the plane; it is kept in position by vertical props B_1B and C_1C resting on the plane. If $AB = 10$ ft., $BC = 8$ ft. and $CA = 9$ ft., find (i) the length of B_1C_1, (ii) the angles BC and ABC make with the plane, (iii) the volume

$$AB_1BCC_1A.$$

26. A swimming bath is 40 ft. wide and 150 ft. long. The shallow end is 3 ft. deep and slopes downwards at 3°, the other end is 9 ft. deep and slopes downwards at 2°. Determine the position of the line where the slopes intersect and the volume of water the bath would contain when full. What would be the depth of water at the deep end when the bath is $\frac{1}{4}$ full?

CHAPTER VI.

ANGLES IN GENERAL.

Angles in General. Let OX be any fixed straight line. Imagine a line OP pivoted at O and originally coinciding with OX.

If OP revolve in the plane of the paper and stop in any such position as OP_1, OP_2, OP_3 it will have swept out an angle (Fig. 72).

XOP_1 is an acute angle; XOP_2 is an obtuse angle and XOP_3 is a reflex angle. There is no reason why the rotation should not continue to any extent, and thus we arrive at the conception of angles of *any* magnitude.

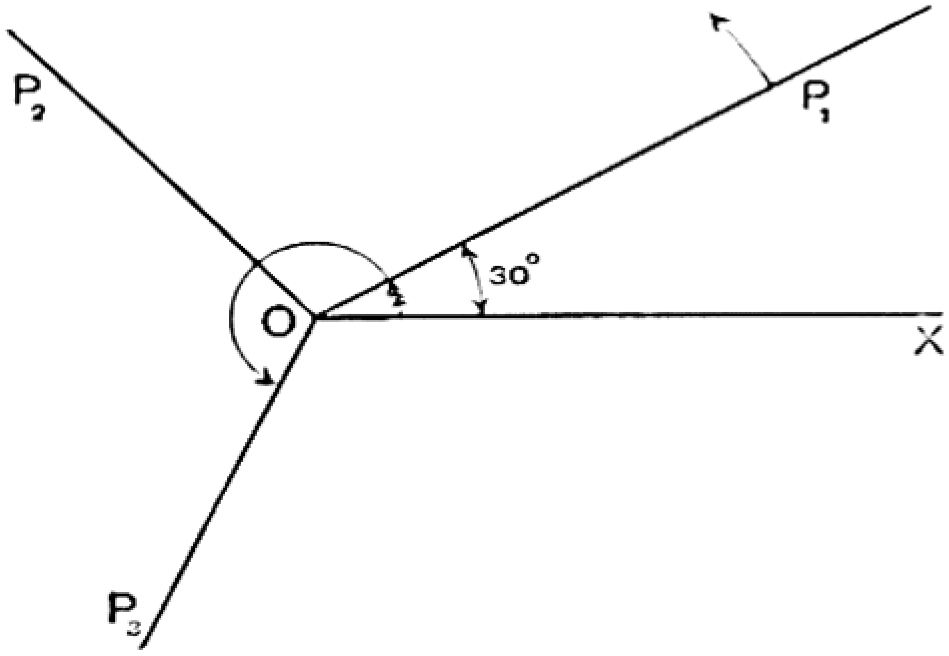


FIG. 72.

From the figure, it will be noticed that the rotation considered has been anti-clockwise. If OP had revolved in the clockwise direction, the angle XOP_3 would be an obtuse angle; the angles XOP_2 and XOP_1 would be reflex angles. Therefore, unless the direction of rotation is specified or implied, there is ambiguity. To avoid this, the following convention is adopted:

When OP revolves in an *anti-clockwise* direction, the angles swept out are taken as *positive*: when OP revolves in a *clockwise* direction, the angles swept out are taken as negative.

For example, with OX the line OP_1 bounds an angle of $30°$, of $360° + 30°$, of $-330°$, of $-330° - 360°$, or of $30° + n \cdot 360°$, where n is any positive or negative integer.

Circular Functions of Angles in General. Let OX be the *initial* or *standard* line. Draw OY, so that the angle XOY is $+90°$. Produce XO to X' and YO to Y', thus forming four quadrants, as shewn in Fig. 73.

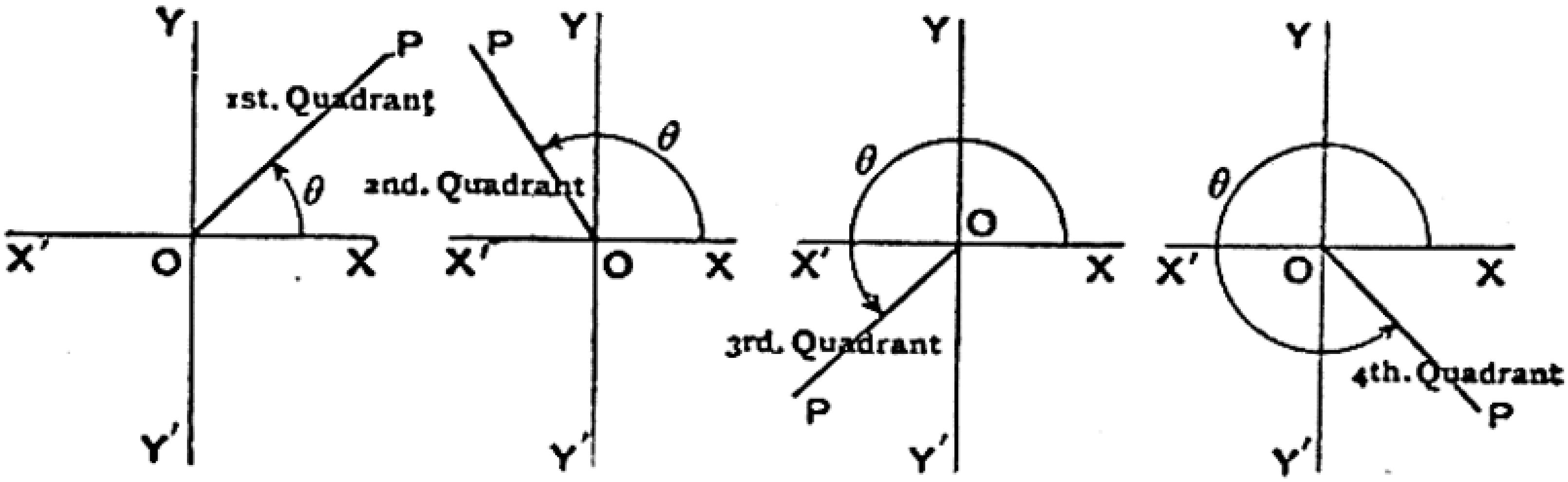


FIG. 73.

Let OP be the position of the revolving line after it has described any angle θ; let (x, y) be the coordinates of P referred to OX and OY as axes, and let $OP = r$.

Then

$$\left.\begin{array}{lll} \sin\theta = \dfrac{y}{r}, & \cos\theta = \dfrac{x}{r}, & \tan\theta = \dfrac{y}{x}, \\ \operatorname{cosec}\theta = \dfrac{r}{y}, & \sec\theta = \dfrac{r}{x}, & \cot\theta = \dfrac{x}{y}. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{I.}$$

In these definitions, r is always positive, while x and y obey the ordinary laws of signs for coordinates (see pp. 33 and 156).

Thus,

when OP lies in the	x is	y is	sin θ and cosec θ are	cos θ and sec θ are	tan θ and cot θ are
1st quadrant	+	+	+	+	+
2nd quadrant	−	+	+	−	−
3rd quadrant	−	−	−	−	+
4th quadrant	+	−	−	+	−

It is clear that these definitions of the circular function include those given on p. 124 as special cases.

Further, the smaller the angle XOP, the more nearly does the abscissa of P approach the value r and the ordinate of P the value zero; so if the line OP coincides with OX, θ is $0°$, $y = 0$ and $x = r$;

$$\therefore\ \sin 0° = 0, \quad \cos 0° = 1, \quad \tan 0° = 0.$$

Again, as y tends to zero, r/y increases indefinitely; *i.e.* as θ tends to 0, cosec θ increases indefinitely. This is generally expressed as $\operatorname{cosec} 0^\circ = \infty$. Similarly $\cot 0^\circ = \infty$. Thus we have

$$\operatorname{cosec} 0^\circ = \infty, \quad \sec 0^\circ = 1, \quad \cot 0^\circ = \infty.$$

In the same way, if OP coincides with OY, $\theta = 90^\circ$, $x = 0$ and $y = r$;

$$\therefore \sin 90^\circ = 1, \quad \cos 90^\circ = 0, \quad \tan 90^\circ = \infty;$$

$$\operatorname{cosec} 90^\circ = 1, \quad \sec 90^\circ = \infty, \quad \cot 90^\circ = 0.$$

Again, since the positions of the revolving line are the same when it has swept out the angle x° or $n \cdot 360^\circ + x^\circ$, we see that:

Any circular function of $\mathbf{n \cdot 360^\circ + x^\circ}$ = the same circular function of $\mathbf{x^\circ}$.

Ex. *Shew graphically the variation of sin θ as θ increases from 0° to 360°. What is learnt from an inspection of the graph?*

Draw a circle of unit radius on any convenient scale (Fig. 74), and draw diameters making angles of 30° with one another.

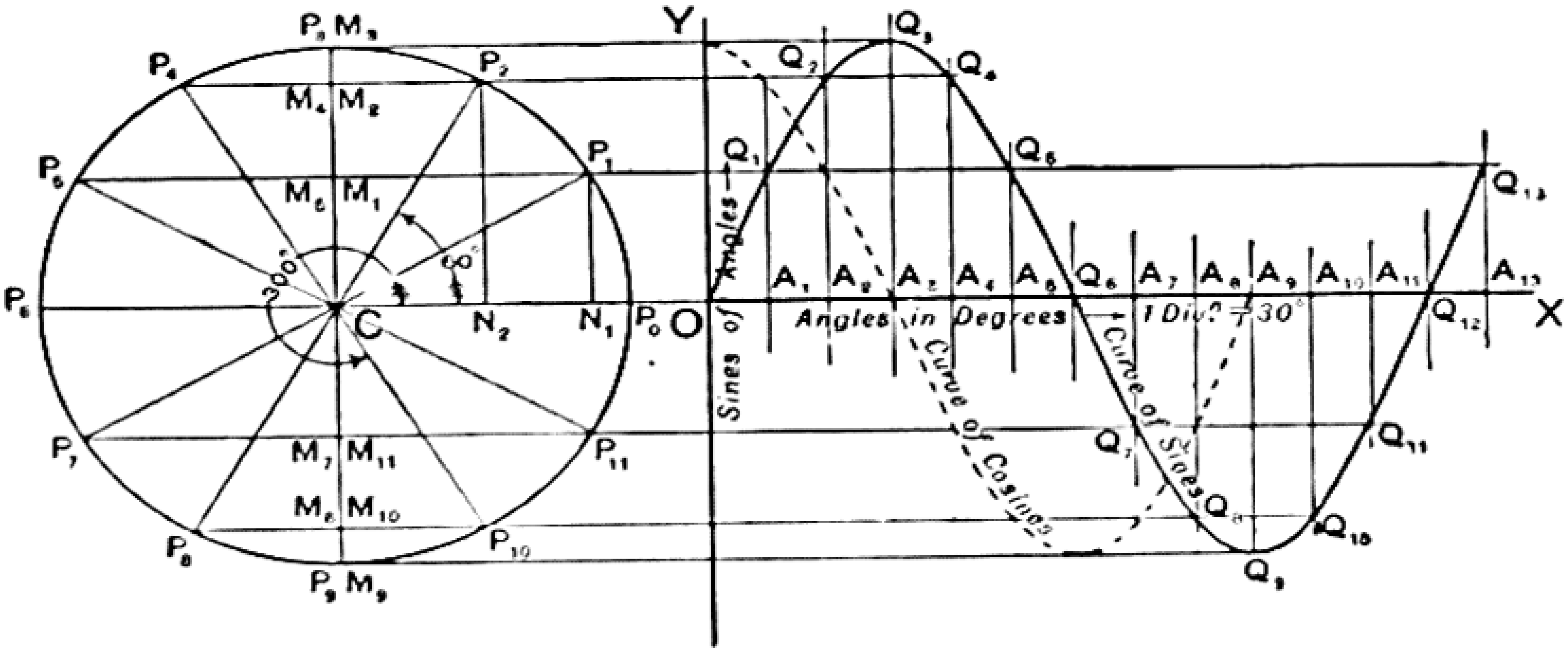


FIG. 74.

Take CP_0 as the standard line. Then CP_1, CP_2, ... CP_{11} make angles 30°, 60°, ... 330° with the standard line CP_0.

Draw the ordinates of the point P in different positions as shewn.

Since CP is of unit length, $\sin P_0CP$ equals the ordinate of P. This relation holds generally, since the ordinate of P changes sign as P crosses the standard line.

Tabulating, we have

Angle P_0CP -	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$
Sin P_0CP	0	P_1N_1	P_2N_2	P_3N_3	P_4N_4	P_5N_5	0
Angle P_0CP -	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	$390°$
Sin P_0CP -	P_7N_7	P_8N_8	P_9N_9	$P_{10}N_{10}$	$P_{11}N_{11}$	0	P_1N_1

To draw the graph, proceed as follows: Produce CP_0 to X, and on it take any point O as the origin. Along OX, on any convenient scale, mark off angles in degrees. Through the points thus obtained draw ordinates. In the figure every 30° is shewn.

The lengths of these ordinates are to be equal to the sines of the corresponding angles. The points Q are therefore obtained as follows: A_1Q_1 is to be $\sin 30°$, and therefore $A_1Q_1 = P_1N_1$. Then Q_1 is the point of intersection of M_1P_1 produced and the ordinate through A_1. Similarly Q_7 is the point of intersection of P_7M_7 produced and the ordinate through A_7. Q_{11} is the point of intersection of $M_{11}P_{11}$ and the ordinate through A_{11}.

A fair curve is drawn through the points Q, and thus the sine-curve is obtained. From the graph,

$$\sin 150° = A_5Q_5 = A_1Q_1 = \sin 30°,$$

$$\sin 120° = A_4Q_4 = A_2Q_2 = \sin 60°,$$

and, generally, inspection shews that points on the graph equidistant $x°$ from the line A_3Q_3 are at the same height above OX; in fact,

$$\sin(90° + x°) = \sin(90° - x°).$$

Similarly

$$\sin \theta° = \sin(180° - \theta°),$$

$$\sin \theta° = -\sin(180° + \theta°),$$

$$\sin \theta° = -\sin(360° - \theta),$$

For angles greater than 360° the curve repeats itself, *e.g.*

$$\sin 390° = A_{13}\,Q_{13} = A_1Q_1 = \sin 30°.$$

In general, $\sin x° = \sin(360° + x°)$.

The graph also shews that $\sin\theta$ increases most rapidly when θ is 0°, 360°, 720°, etc.; decreases most rapidly when θ is 180°, 540°, etc.; is a maximum when $\theta = 90°$, 450°, etc.; and a minimum when θ is 270°, 630°, etc.

By erecting ordinates at the points A equal to the abscissae of P, the **cosine-curve** can be obtained. A portion of the cosine-curve is shewn by the dotted line in Fig. 74. Inspection shews that the sine-curve is merely the cosine-curve shifted forward through 90°.

Functions like $\sin\theta$, $\cos\theta$, etc., the values of which recur after certain equal intervals in the value of θ, are called **periodic functions.** The interval is called the **period**, and in Fig. 74 is OA_{12}, A_1A_{13}, etc., *i.e.* 360°. These notes are particularly useful for electrical engineers.

Important Formulae. From equations I. (p. 152) above,

$$\frac{\sin\theta}{\cos\theta} = \frac{y}{x},$$

and, by definition,

$$\tan\theta = \frac{y}{x};$$

$$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta}$$

for angles of any magnitude.

Again, by Pythagoras's Theorem, in Fig. 73,

$$x^2 + y^2 = r^2.$$

Dividing in turn by r^2, x^2 and y^2, we have

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \text{ that is } (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

This is usually written

$$\cos^2\theta + \sin^2\theta = 1.$$

Again,
$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \text{ that is } 1 + \tan^2\theta = \sec^2\theta,$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2, \text{ that is } \cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

Thus the relations proved on p. 126 are true for angles in general.

Polar Coordinates. Let OX be the initial line and O the origin (Fig. 75).

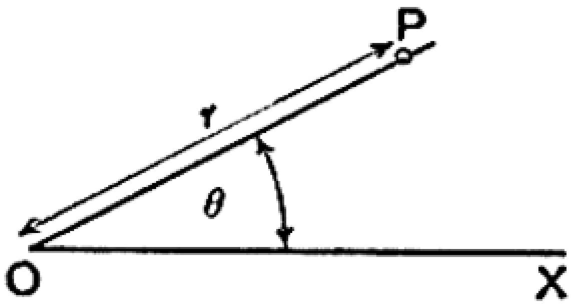


FIG. 75.

Draw any angle θ and along the line bounding the angle measure off any distance r, thus fixing a point P.

Then (r, θ) are the polar coordinates of P.

If (x, y) are the rectangular coordinates of P, the definitions of the circular functions give

$$\left.\begin{aligned} r\cos\theta &= x, \\ r\sin\theta &= y. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Ex. 1. *Plot the points* (2, 130°), (1·5, −45°), (2, 210°).

The points are shewn in Fig. 76.

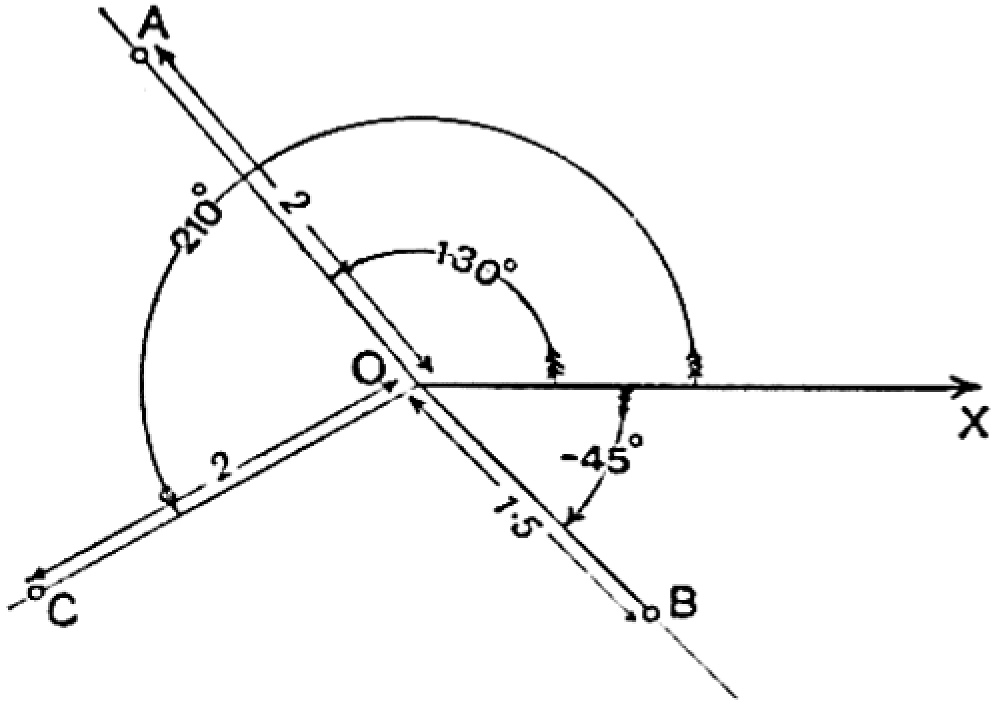


FIG. 76.

A is (2, 130°), B is (1·5, −45°) and C is (2, 210°).

Ex. 2. *Plot the points* (−2·5, 60°) *and* (−2, −45°), *and verify the relations* (1) *in each case.*

From the standard line OX (Fig. 77) measure off the angle $XOP' = 60°$ and take $OP' = 2{\cdot}5$.

Produce $P'O$ to P making $OP = OP'$. Then P is the point (−2·5, 60°).

From P, P' draw $PN, P'N'$ perpendicular to the initial line.

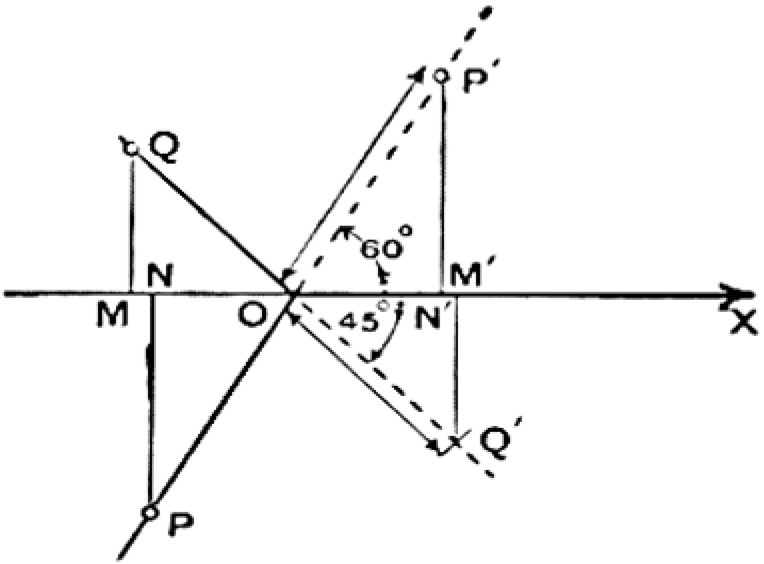


FIG. 77.

The x coordinate of P is ON,

$$i.e.\ -ON'.$$

But, by definition,

$$ON' = OP' \cos 60^\circ;$$

$\therefore$ the x coordinate of P is

$$-2{\cdot}5 \cos 60^\circ.$$

The y coordinate of P is NP,

$$i.e.\ -N'P'.$$

But, by definition, $N'P' = OP' \sin 60^\circ$;

$\therefore$ the y coordinate of P is $-2{\cdot}5 \sin 60^\circ$.

Again, from the standard line OX measure off the angle $XOQ' = -45^\circ$, and take $OQ' = 2$.

Produce $Q'O$ to Q making $OQ = OQ'$. Then Q is the point $(-2,\ -45^\circ)$.

The x coordinate of Q is OM,

$$i.e.\ -OM'.$$

But, by definition, $OM' = OQ' \cos(-45^\circ)$;

$\therefore$ the x coordinate of $Q = -2\cos(-45^\circ)$.

Similarly the y coordinate of $Q = -2\sin(-45^\circ)$.

Definition of π. Since all circles are similar figures, the length of the circumference of a circle is proportional to its diameter. The constant ratio—circumference divided by diameter—is represented by the symbol π. Hence, if C is the circumference of any circle, and r its radius measured in the same units,

$$\frac{C}{2r} = \pi \quad \text{or} \quad C = 2\pi r.$$

The area of the circle can be shewn to be πr^2.

The proof of this formula will be found on p. 409.

The area of a circle of radius 2 ft. is 4π sq. ft. From Ex. 7, p. 133, 4π lies between 11·8 and 13·0; *i.e.* π lies between 2·95 and 3·25. Closer approximations to π can be found by taking polygons with more sides (see Ex. 24, p. 136).

Circular Measure of an Angle. In drawing figures to scale, angles are given and measured in degrees. But in mathematical work in general, angles are measured in radians.

The **radian is the unit of circular measure.** It is the angle bounded by two radii of a circle which cut off an arc equal in length to the radius (Fig. 78).

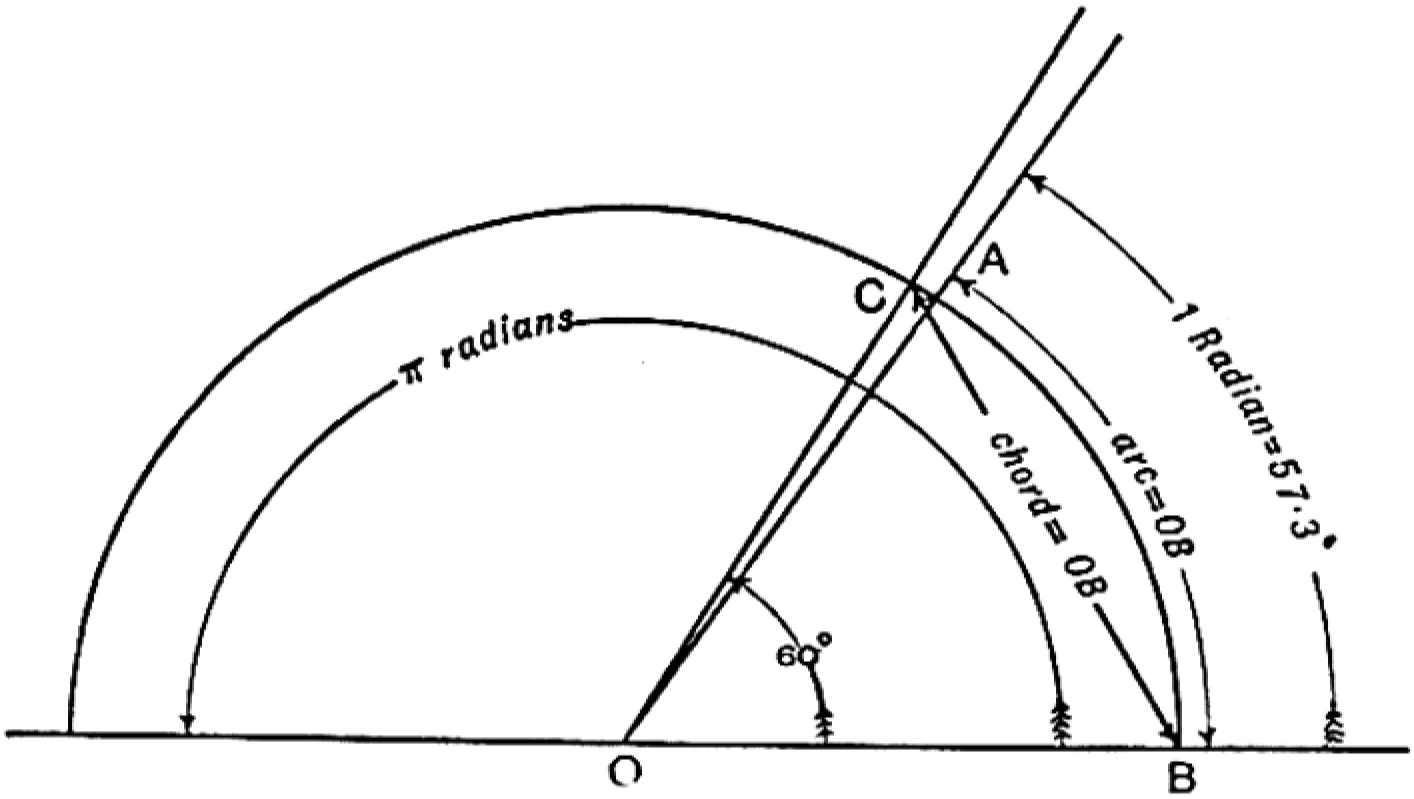


FIG. 78.

Note that it is the arc AB which is equal in length to the radius OB.

If the chord CB equals the radius OB, the angle $COB = 60°$.

Now since the circumference of a circle contains the radius 2π times, and each arc equal in length to one radius subtends one radian at the centre of the circle, the whole circumference of a circle subtends 2π radians at its centre;

$$\therefore\ 2\pi \text{ radians} = 360°, \quad \text{or} \quad 2\pi^c = 360°.$$

If an angle θ is given, it is understood, unless otherwise stated, to be measured in radians.

Ex. *How many radians are there in*

$$\text{(i) } 270°, \quad \text{(ii) } 50° ?$$

$$\text{Since} \qquad 180° = \pi^c,$$

$$\text{then} \qquad \text{(i) } 270° = \frac{270}{180}\pi^c = \frac{3\pi^c}{2},$$

$$\text{(ii) } \quad 50° = \frac{50}{180}\pi^c = \frac{5}{18}\pi^c.$$

Length of a Circular Arc. An angle of one radian is bounded by radii of a circle which cut off an arc equal in length to the radius; therefore an angle of two radians is bounded by radii which cut off an arc equal in length to twice the radius; and generally, an angle of θ radians is bounded by radii which cut off an arc equal in length to θ times the radius.

Hence, if an arc of a circle, radius r, *subtends* an angle of θ radians at the centre, its length is $r\theta$.

Area of Sector. Assuming the formula, πr^2, for the area of a circle, the area of a sector can be found as follows:

$$\frac{\text{area of sector } BOC}{\text{area of circle}} = \frac{\theta}{2\pi}. \quad \text{(Fig. 79.)}$$

$$\therefore \text{ area of sector } BOC = \frac{\theta}{2\pi} \times \pi r^2 = \tfrac{1}{2} r^2 \theta.$$

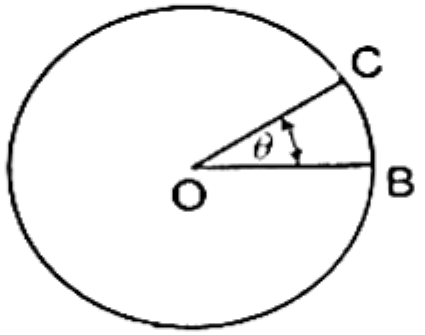


FIG. 79.

Thus, the area of a sector is $\frac{1}{2}$(radius) × (length of arc).

Ex. 1. *A sector containing an angle of* 135° *is cut from a circular sheet of metal, radius* 28 *in., and is formed into a circular cone. Find its volume.*

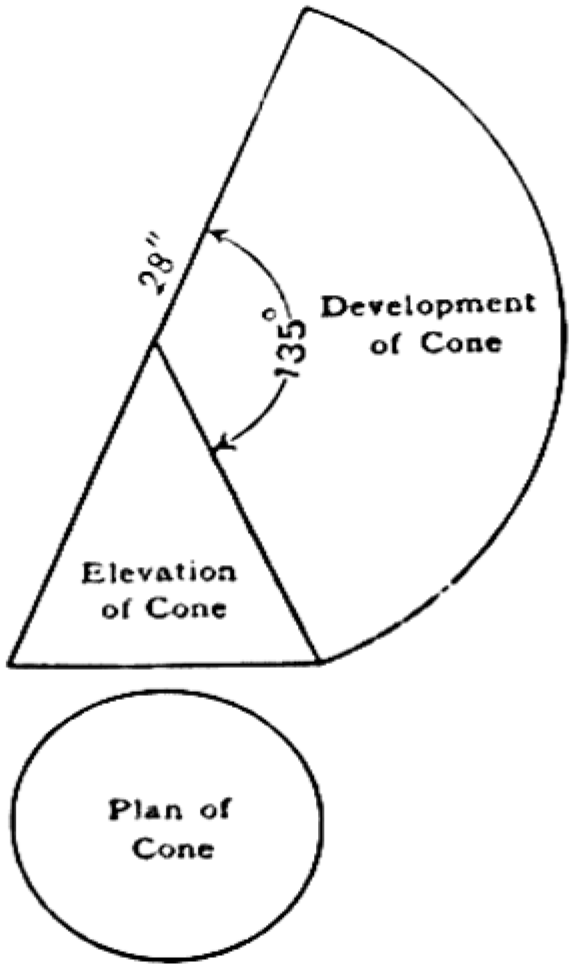


FIG. 80.

The slant side of the cone formed is 28 in. and the circumference of its base is $28 \times 135 \times \pi/180$ in. (Fig. 80), which is $2\pi r$ in. where r in. is the radius of the base.

Therefore $r = 14 \times 135/180 = 21/2$.

The vertical height is h in., where

$$h^2 = 28^2 - r^2;$$

$$\therefore \; h = \sqrt{(28 + \tfrac{21}{2})(28 - \tfrac{21}{2})}.$$

The volume of the cone is v cub. in., where

$$v = \tfrac{1}{3}\pi r^2 h$$

$$= \frac{\pi}{3}\left(\frac{21}{2}\right)^2 \sqrt{\frac{77 \cdot 35}{2 \cdot 2}}.$$

Using logs, $v = 2995$.

The volume of the cone is approximately 3000 cub. in.

Ex. 2. *Two points A, B are in latitude λ, and the difference of their longitudes is ϕ. Shew that the distance between them measured along the parallel of latitude is $R\phi \cos \lambda$, where R is the radius of the Earth.*

The Earth may be taken as a sphere, and the **Equator** is the circle of the intersection of the sphere by a plane through the centre perpendicular to the polar axis. In Fig. 81 NS represents the polar axis, and the Equator passes through the points A' and B'. The **meridian** of any place on the Earth's surface is the circle passing through that place and the poles.

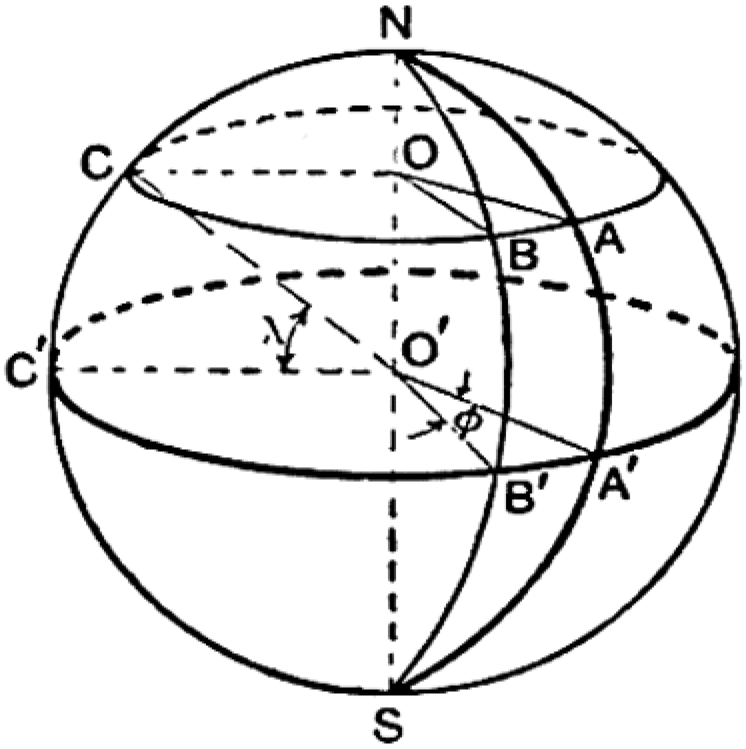


FIG. 81.

Longitudes are measured from the meridian of Greenwich.

The **longitude** of any place is measured by the angle subtended at the centre of the Earth by the arc of the Equator cut off by the meridians of Greenwich and that place. Thus the difference of the longitudes of A and B is the angle $A'O'B'$, *i.e.* ϕ.

The **latitude** of any place is measured by the angle subtended at the centre of the Earth by the arc of the meridian of the place intercepted between itself and the Equator. Thus the latitude of A is the angle $AO'A'$.

Clearly $AO'A' = BO'B' = CO'C' = \lambda$, the latitude.

Now, arc $AB = OA \times \phi$, for ϕ is the angle between the planes NAS and NBS.

Again, OA is the radius of the **parallel of latitude** through A;

$$\therefore \ OA = OC = R \cos \lambda;$$

$$\therefore \ \text{arc } AB = (R \cos \lambda)\phi.$$

Ex. 3. *If θ is an acute angle, prove that $\tan\theta$, θ and $\sin\theta$ are in descending order of magnitude.*

In Fig. 82 BOC is the angle θ, CN is perpendicular to OB and CT is the tangent at C.

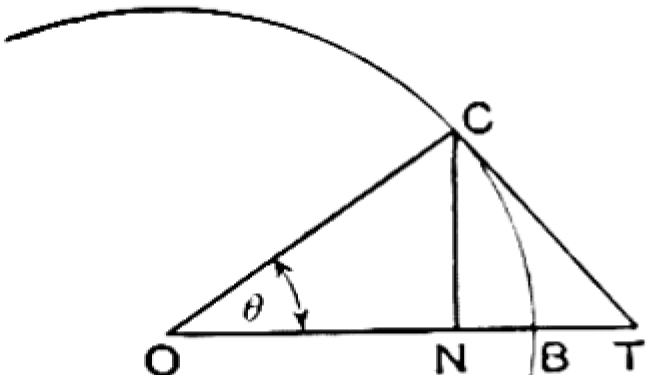


FIG. 82.

Area of $\triangle OCT = \frac{1}{2}OC.CT = \frac{1}{2}r.r\tan\theta$.

Area of sector $OCB = \frac{1}{2}r^2\theta$.

Area of $\triangle OCB = \frac{1}{2}OB.CN = \frac{1}{2}r \times r\sin\theta$.

Evidently

$$\triangle OCT > \text{sector } OCB > \triangle OCB;$$
$$\therefore\ \tfrac{1}{2}r^2\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta;$$
$$\therefore\ \tan\theta > \theta > \sin\theta.$$

Circular Functions of Small Angles. From the example just worked, $\tan\theta > \theta > \sin\theta$.

Dividing by $\sin\theta$ (which is positive as θ is an acute angle),

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1.$$

Now as θ tends to zero (usually written, $\theta \to 0$), $\cos\theta \to 1$;

$\therefore$ as $\theta \to 0$, $\frac{\theta}{\sin\theta}$ lies between unity and a number which tends to unity; $\therefore\ \frac{\theta}{\sin\theta} \to 1$ as $\theta \to 0$. ....(1)

This may be written (see also pp. 251-253)

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1. \qquad \text{.........................(2)}$$

N.B.—The student is warned against writing

$$\text{“}\frac{\theta}{\sin\theta} = 1, \text{ if } \theta = 0,\text{”}$$

since $\sin\theta = 0$, when $\theta = 0$ and $\frac{0}{0}$ is meaningless.

Either of the statements (1) and (2) shew that when θ is a small angle, $\sin\theta \doteqdot \theta$.

Again, dividing $\tan\theta > \theta > \sin\theta$ throughout by $\tan\theta$, which is positive, we have

$$1 > \frac{\theta}{\tan\theta} > \cos\theta;$$

whence, as before,

$$\frac{\theta}{\tan\theta} \to 1 \text{ as } \theta \to 0 \quad\text{or}\quad \lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1.$$

Further, if θ is a small angle, $\tan\theta \doteqdot \theta$.

Ex. 1. *If the diameter of the moon is* 2160 *miles, what angle does it subtend at the eye when it is* 240000 *miles away?*

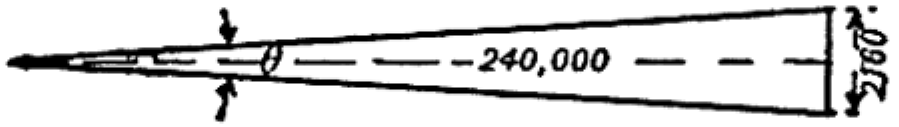


FIG. 83.

Let θ be the angle; from Fig. 83

$$\tan\frac{\theta}{2}=\frac{1080}{240000}=0{\cdot}0045;$$

$$\therefore\ \frac{\theta}{2}\doteqdot 0{\cdot}0045;$$

$$\therefore\ \theta\doteqdot 0{\cdot}009.$$

From the tables, $0{\cdot}009$ radian $\doteqdot 31'$.
The *angular diameter* of the moon is approximately $31'$.

Ex. 2. *If h be the height of an observer above the Earth's surface and R the radius of the Earth, find the dip of the horizon and the distance of the horizon.*

In Fig. 84, O represents the observer, C the centre of the Earth.

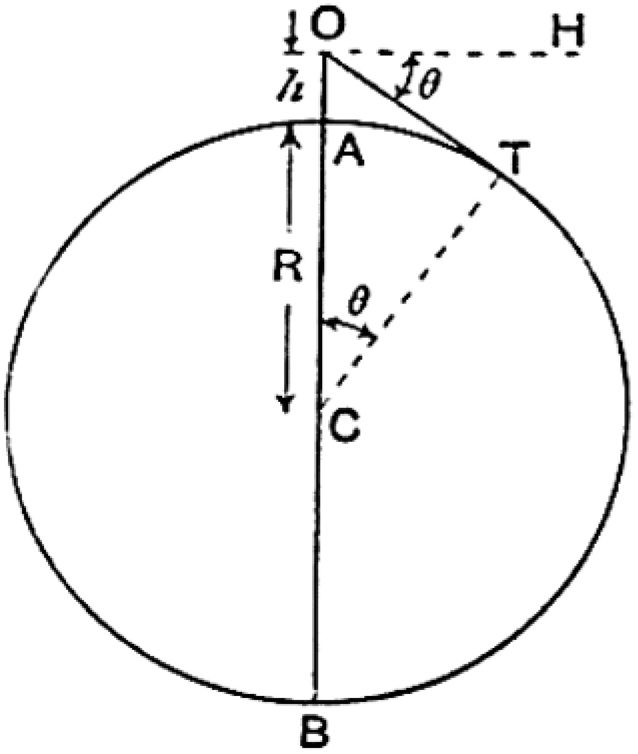


FIG. 84.

Then OC is a vertical line and OH is a horizontal line through the observer.

OT is the tangent to the Earth's surface from O.

The angle HOT is the **dip** and OT is the distance of the horizon.

From the figure,

$$\begin{aligned}OT^2&=OA\,.\,OB\\&=h\,.\,(h+2R)\\&\doteqdot 2hR,\end{aligned}$$

since h^2 is negligible in comparison with $2Rh$.

Again, $$\tan\theta=OT/CT\doteqdot\frac{\sqrt{2hR}}{R}\quad\text{or}\quad\sqrt{\frac{2h}{R}}.$$

Therefore the dip, θ, is approximately $\sqrt{2h/R}$ and the distance OT is approximately $\sqrt{2hR}$.

Obviously h and R must be measured in the same units.

EXERCISES XVI.

1. Draw the following angles:

$$220^\circ,\ 390^\circ,\ -235^\circ,\ \frac{5\pi^c}{6},\ -\frac{4\pi^c}{3},\ \frac{12\pi^c}{5}.$$

2. Express in radians 24°, 120°, 135°, 275°, $-13{\cdot}5^\circ$.

3. Express in degrees

$$\frac{2\pi^c}{3},\quad \frac{5\pi^c}{6},\quad \frac{7\pi^c}{8},\quad \frac{3\pi^c}{2},\quad 10^c.$$

4. Find the number of seconds in the angle subtended at the centre of a circle of radius 10 miles by an arc of length 4 ft.

5. Find the length of an arc of a circle of radius 10 ft. which subtends (i) an angle of 40° at the centre, (ii) an angle of 310° at the centre.

6. The centres of two pulleys of diameters 24 in. and 20 in. respectively are 10 ft. apart. What length of open belt is required (neglect lap)?

7. Find the length of one minute of arc of the Equator, given that the Earth's radius is 4000 miles.

8. What is the length of a railway curve which subtends an angle of $22\frac{1}{2}^\circ$ at the centre, the radius being $1\frac{1}{2}$ miles?

9. The three angles of a triangle are in arithmetic progression and the smallest is $\frac{1}{2}$ the greatest. Find the angles in radians.

10. What is the circular measure of the angle between the hour hand and the minute hand of a clock at half-past three?

11. A chord AB of a circle of radius r subtends an angle α at the centre of the circle. Find the area of the segment of the circle cut off by AB.

12. Two lines of rails make an angle of 75° with one another; they are to be connected by a circular arc of rails of central angle 105°. A straight path of length 800 yd. goes from one set of rails to the other and makes an angle of 30° with one set. Find the length of the circular rail wanted if the path is to be just avoided. [L.U.]

13. The development of a cone is a sector of radius R and angle θ. Find the sine of the semi-vertical angle of the cone. Find the vertical angle of the cone if $\theta = 4\pi/5$.

14. What is approximately the area of the Earth's surface visible to an airman at a height of 1 mile? (Radius of Earth = 3960 miles.)

15. If the distance of the horizon is 20 miles, what is the dip ?

16. From a lighthouse 210 ft. high the top of a second lighthouse 35 miles away is just visible. Find the height of the second lighthouse.

17. From the top of a rise on a level stretch of road two consecutive milestones are observed at depressions of 4° and 2° 50′ respectively. What is the height of the rise ?

18. Shew that the distance of an object h ft. high whose elevation is α° is roughly $20h/\alpha$ yards, if α is small.

19. A place is in latitude 52°. How many miles per minute does it travel in consequence of the Earth's rotation ? Assume the Earth's radius to be 3960 miles.

20. Referring to Ex. 2, p. 160, shew that

$$\text{the chord } AB = 2R\cos\lambda\sin\frac{\phi}{2}.$$

If $\lambda = 52^\circ$ and $\phi = 40^\circ$, find the angle subtended by AB at the centre of the Earth.

Examples 21-24 *are taken from papers set at the Inter. B.Sc.* [L.U.]

21. Shew that if θ is less than $\frac{1}{2}\pi$, then $\sin\theta$, θ and $\tan\theta$ are in ascending order of magnitude, and deduce that $\sin\theta$ and $\tan\theta$ tend to equality with θ as θ is indefinitely diminished.

Two landmarks A, B, when observed from a point C on the deck of a ship at anchor, are observed to be in line with C. On walking from C a distance b perpendicular to the line AC the landmarks subtend a small angle θ. Shew that, very approximately,

$$\frac{1}{BC} - \frac{1}{AC} = \frac{\theta}{b}.$$

The distance between A and B is known to be 2·4 miles and θ measured as 5′ 30″ when $b = 200$ ft. Find the distance of C from the nearer landmark B.

22. The Earth being a sphere of radius 3960 miles, find the length on the surface of a degree of longitude in latitude 40°.

What is meant by *the distance of the visible horizon ?*

Calculate it for a height of $\frac{1}{4}$ mile above the Earth's surface.

23. A point A is at a distance h from a sphere of radius r; shew that when h is small compared with r the length of a tangent drawn from the point to the sphere is given approximately by

$$d = \surd(2hr),$$

and that the angle subtended by a tangent at the centre of the sphere is approximately

$$\theta = \sqrt{\frac{2h}{r}}.$$

Let A be an aeroplane at a height of 2000 ft. and the sphere be the Earth. Taking the radius of the Earth as 4000 miles, find the distance and the dip of the horizon of the aeroplane.

24. Shew that if θ is less than $\pi/2$, then $\sin\theta$, θ and $\tan\theta$ are in ascending order of magnitude, and deduce that $\sin\theta$ and $\tan\theta$ tend to equality with θ as θ is indefinitely diminished.

The shadow of the top of a vertical spire is cast by the sun on horizontal ground, and is observed about noon to move 3·9 ft. in 2 minutes in a direction perpendicular to the centre line of the shadow. Assuming that the angular motion of the shadow round the top of the spire is at the rate of 360° in 24 hours, and that the elevation of the sun is 50°, calculate the height of the spire.

25. Shew that as $\theta \to 0$, $\dfrac{\sin m\theta}{\theta} \to m$, $\dfrac{\sin a\theta}{\sin b\theta} \to \dfrac{a}{b}$ and $\dfrac{\theta}{\tan p\theta} \to \dfrac{1}{p}$.

26. Shew that $\cos^2\theta \doteqdot 1 - \theta^2$ when θ is small.

27. Shew that as $n \to \infty$, $n \tan\dfrac{\alpha}{n}$ and $n \sin\dfrac{\alpha}{n}$ both $\to \alpha$.

Circular Functions of $-\theta$ and θ. Let OX be the standard line. Imagine two lines OP_1, OP_2, each of length r and originally coinciding with OX, to revolve at the same rate but in opposite directions. Then, if θ is the angle generated by OP_1 in any time, $-\theta$ is the angle generated by OP_2.

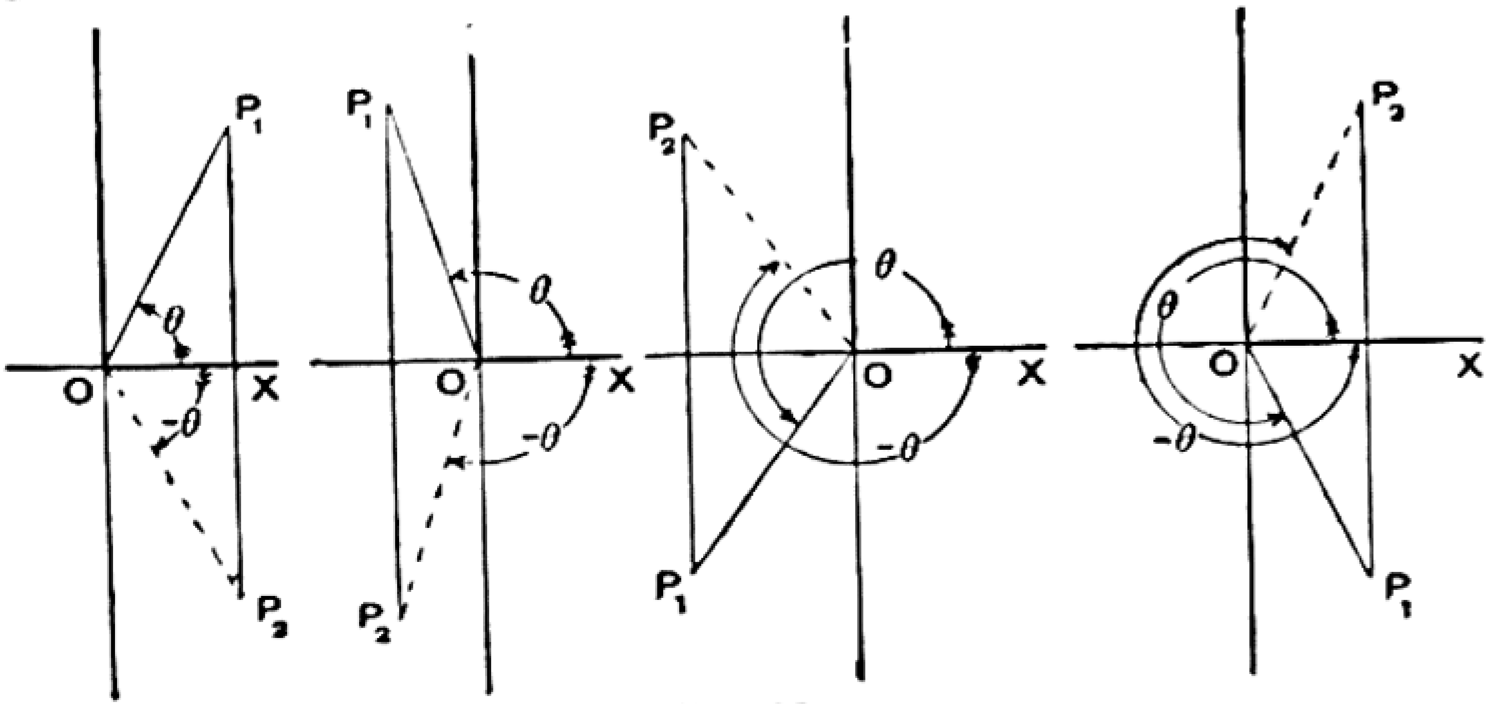


FIG. 85.

Fig. 85 shews that P_1 and P_2 are the images of one another in the line OX. Therefore the x coordinates of P_1 and P_2 are always the same, but the y coordinate of P_2 is always $-$(the y coordinate of P_1).

By definition,

$$\cos(-\theta)=\frac{\text{the } x \text{ coordinate of } P_2}{r}$$

$$=\frac{\text{the } x \text{ coordinate of } P_1}{r};$$

$$\therefore\ \cos(-\theta)=\cos(\theta),$$

$$\sin(-\theta)=\frac{\text{the } y \text{ coordinate of } P_2}{r}$$

$$=-\frac{\text{the } y \text{ coordinate of } P_1}{r};$$

$$\therefore\ \sin(-\theta)=-\sin\theta,$$

$$\tan(-\theta)=\frac{\text{the } y \text{ coordinate of } P_2}{\text{the } x \text{ coordinate of } P_2}$$

$$=\frac{-\text{the } y \text{ coordinate of } P_1}{\text{the } x \text{ coordinate of } P_1};$$

$$\therefore\ \tan(-\theta)=-\tan\theta.$$

Circular Functions of $\pi-\theta$ and θ. Let OX be the standard line. Imagine a line OP_1 of length r to sweep out the angle π, thus reaching the position OX'. Now let OP_1 revolve in the clockwise direction from OX' while a second line OP_2 of length r revolves at the same rate in the anti-clockwise direction from OX. Then if θ is the angle XOP_2 at any time, $\pi-\theta$ is the angle XOP_1 at the same time.

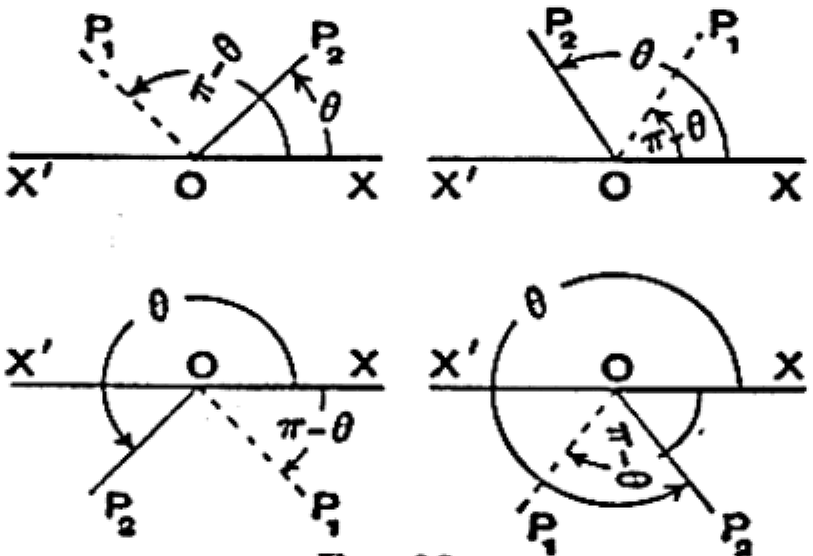


FIG. 86.

Fig. 86 shews that P_1 and P_2 are the images of one another in the line OY. Therefore the x coordinate of P_1 = − the x coordinate of P_2 and the y coordinates of P_1 and P_2 are always the same.

By definition,

$$\cos(\pi-\theta)=\frac{\text{the } x \text{ coordinate of } P_1}{r}$$

$$=\frac{-\text{the } x \text{ coordinate of } P_2}{r}:$$

$$\therefore\ \cos(\pi-\theta) = -\cos\theta,$$

$$\sin(\pi-\theta) = \frac{\text{the } y \text{ coordinate of } P_1}{r}$$

$$= \frac{\text{the } y \text{ coordinate of } P_2}{r};$$

$$\therefore\ \sin(\pi-\theta) = \sin\theta,$$

$$\tan(\pi-\theta) = \frac{\text{the } y \text{ coordinate of } P_1}{\text{the } x \text{ coordinate of } P_1}$$

$$= \frac{\text{the } y \text{ coordinate of } P_2}{-(\text{the } x \text{ coordinate of } P_2)};$$

$$\therefore\ \tan(\pi-\theta) = -\tan\theta.$$

Circular Functions of $\frac{1}{2}\pi-\theta$ and θ. Let OX be the standard line. Imagine a line OP_1 of length r to sweep out the angle $\pi/2$, thus reaching the position OY. Now let OP_1 revolve in the clockwise direction from OY while a second line OP_2 of length r revolves at the same rate in the anti-clockwise direction from OX.

Then, if θ is the angle XOP_2 at any time, $\frac{1}{2}\pi-\theta$ is the angle XOP_1 at the same time.

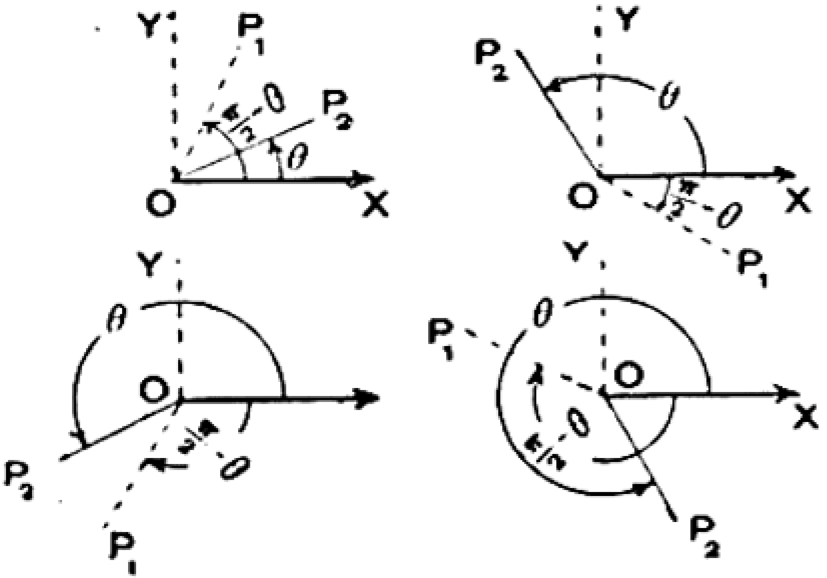


FIG. 87.

Consideration of Fig. 87 shews that the x coordinate of P_2 equals the y coordinate of P_1, and that the y coordinate of P_2 equals the x coordinate of P_1.

By definition,

$$\cos\left(\frac{\pi}{2}-\theta\right) = \frac{\text{the } x \text{ coordinate of } P_1}{r}$$

$$= \frac{\text{the } y \text{ coordinate of } P_2}{r};$$

$$\therefore\ \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta,$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \frac{\text{the } y \text{ coordinate of } P_1}{r}$$

$$= \frac{\text{the } x \text{ coordinate of } P_2}{r};$$

$$\therefore\ \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\text{the } y \text{ coordinate of } P_1}{\text{the } x \text{ coordinate of } P_1}$$

$$=\frac{\text{the } x \text{ coordinate of } P_2}{\text{the } y \text{ coordinate of } P_2};$$

$$\therefore\ \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta.$$

Circular Functions of $\pi+\theta$ and θ, and of $\frac{1}{2}\pi+\theta$ and θ. In a manner similar to that used in the foregoing proofs, it may be shewn that

$$\left.\begin{aligned}\sin(\pi+\theta)&=-\sin\theta,\\ \cos(\pi+\theta)&=-\cos\theta,\\ \tan(\pi+\theta)&=\ \ \tan\theta;\end{aligned}\right\}$$

$$\left.\begin{aligned}\sin\left(\frac{\pi}{2}+\theta\right)&=\ \ \cos\theta,\\ \cos\left(\frac{\pi}{2}+\theta\right)&=-\sin\theta,\\ \tan\left(\frac{\pi}{2}+\theta\right)&=-\cot\theta.\end{aligned}\right\}$$

Ex. 1. *Draw the graph of* $\tan x^\circ$ *as* x° *passes from* -360° *to* $+360^\circ$.

Using the tables, we have

x	0	30	60	90
$\tan x^\circ$	0	0·5774	1·732	∞

Since $\tan(180-x)^\circ=-\tan x^\circ$ (see p. 167), we have

x	120	150	180
$\tan x^\circ$	−1·732	−0·5774	0

Since $\tan (180+x)^\circ = \tan x^\circ$, we have

x	210	240	270	300	330	360
$\tan x^\circ$	0·5774	1·732	∞	$-1·732$	$-0·5774$	0

Finally, since $\tan(-x^\circ) = -\tan x^\circ$ (see p. 166), we have

x	-30	-60	-90	-120	-150	-180
$\tan x^\circ$	$-0·5774$	$-1·732$	$-\infty$	1·732	0·5774	0
x	-210	-240	-270	-300	-330	-360
$\tan x^\circ$	$-0·5774$	$-1·732$	$-\infty$	1·732	0·5774	0

There are, clearly, rapid increases in the value of $\tan x^\circ$ between 60° and 90°, between 90° and 120°, between 240° and 270° and similar ranges. To draw the graph accurately, it is advisable to consult the tables for additional points within these ranges.

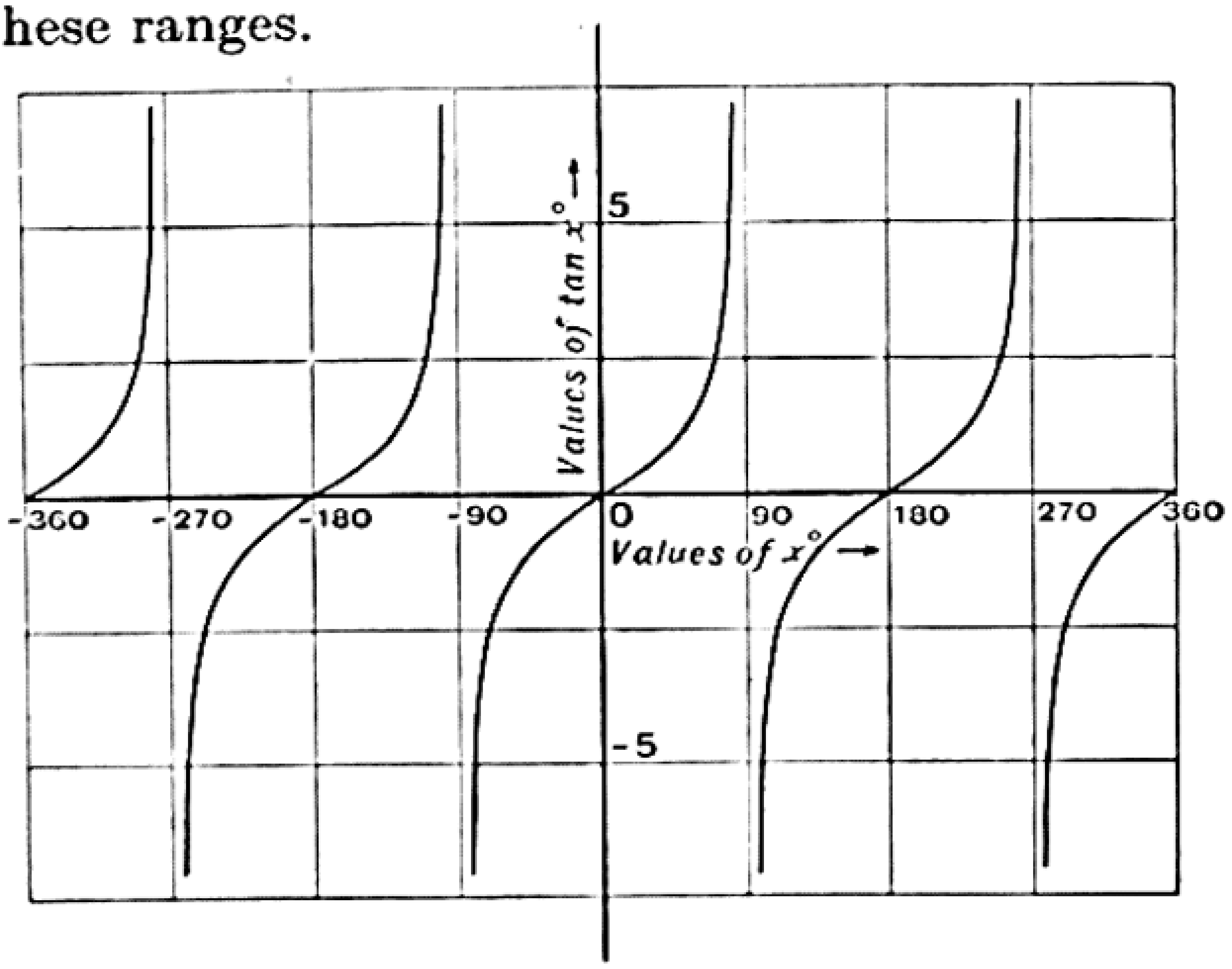


FIG. 88.—Graph of $\tan x^\circ$.

Fig. 88 shews the required graph. Note that as x° passes through 90° or any odd multiple of 90°, $\tan x^\circ$ changes from $+\infty$ to $-\infty$. This change in sign may be compared with the change in sign which occurs as the graph crosses the axis of x.

Ex. 2. *Draw the graph of* sec x° *as* x° *passes from* 0° *to* 360°.
Using the tables, we have

x°	0	20	40	60	80	90°
sec x°	1·0	1·064	1·305	2·0	5·759	∞

Since $\cos(180-\theta)^\circ = -\cos\theta^\circ$ (see p. 167),
$\therefore\ \sec(180-\theta)^\circ = -\sec\theta^\circ$, and we have

x	100	120	140	160	180
sec x°	– 5·759	– 2·0	– 1·305	– 1·064	– 1

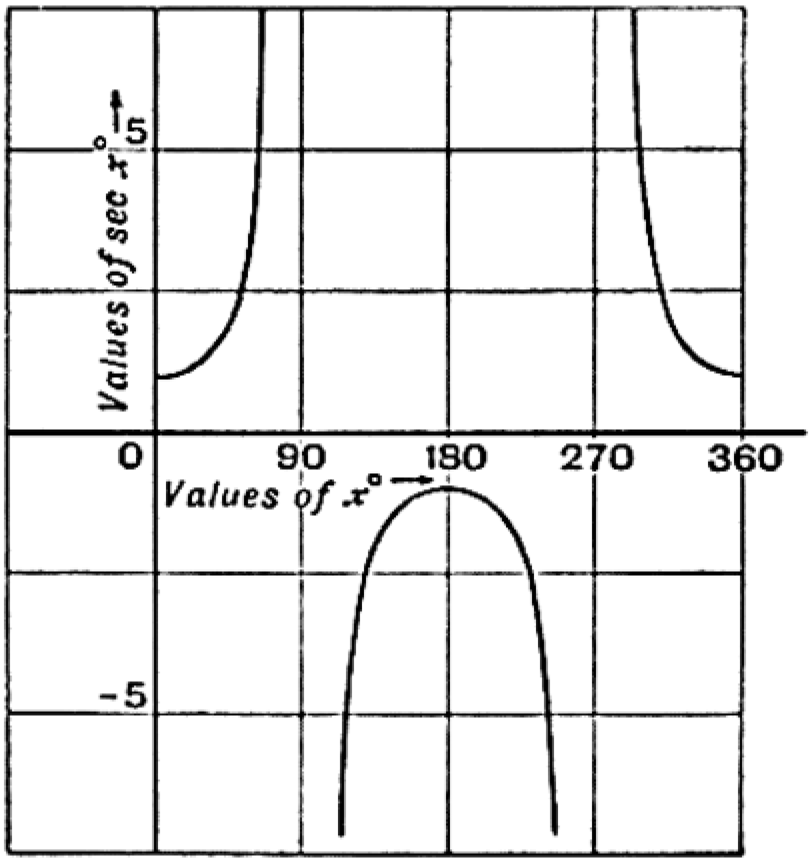


FIG. 89.—Graph of sec x°.

Since $\cos(-\theta)^\circ = \cos\theta^\circ$ (see p. 166),
$\therefore\ \cos(360-\theta)^\circ = \cos\theta^\circ$ (see p. 153),
$\therefore\ \sec(360-\theta)^\circ = \sec\theta^\circ$, and we have

x	200	220	240	260	270
sec x°	– 1·064	– 1·305	– 2·0	– 5·759	– ∞
x	280	300	320	340	360
sec x°	5·759	2·0	1·305	1·064	1·0

Fig. 89 shews the required graph.

General Solution of Trigonometric Equations. In order to solve a trigonometric equation we endeavour to reduce it to one or other of the following forms:

$$\cos x = \cos \alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

$$\sin x = \sin \alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

$$\tan x = \tan \alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III.)}$$

where x is the unknown angle and α is a known angle.

CASE I. In Fig. 90, if the angle P_1OX is α, the line which has swept out the angle x must coincide either with OP_1 or with OP_2, because $\cos(-\theta) = \cos\theta$ (see p. 166).

If it coincides with OP_1,

$$x = (\text{a multiple of } 2\pi) + \alpha.$$

If it coincides with OP_2,

$$x = (\text{a multiple of } 2\pi) - \alpha.$$

Consequently $x = 2n\pi \pm \alpha$, where n is any positive or negative integer.

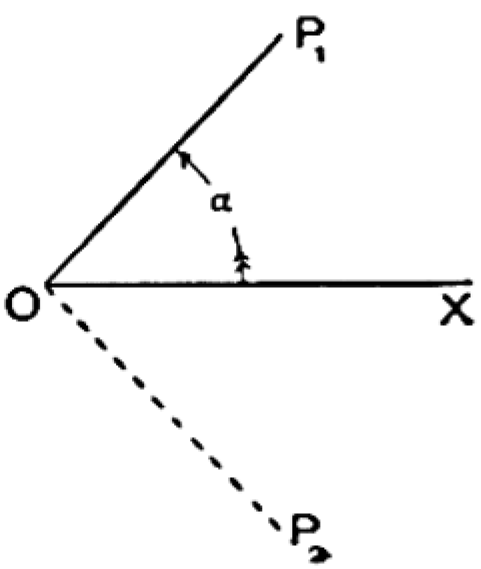


FIG. 90.

CASE II. In Fig. 91, if the angle P_1OX is α, the line which has swept out the angle x must coincide either with OP_1 or with OP_2, because $\sin(\pi - \theta) = \sin\theta$ (see p. 167).

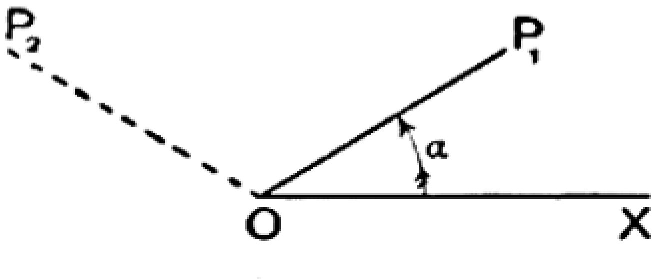


FIG. 91.

If it coincides with OP_1, $x = (\text{a multiple of } 2\pi) + \alpha$;

$$\therefore\ x = (\text{an } \textit{even} \text{ multiple of } \pi) + \alpha.$$

If it coincides with OP_2, $x = (\text{a multiple of } 2\pi) + \pi - \alpha$;

$$\therefore\ x = (\text{an } \textit{odd} \text{ multiple of } \pi) - \alpha.$$

Consequently $x = n\pi + (-1)^n\alpha$, where n is any integer.

Case III. In Fig. 92, if the angle P_1OX is α, the line which has swept out the angle x must coincide either with OP_1 or with OP_2, because $\tan(\pi+\theta)=\tan\theta$ (see p. 168).

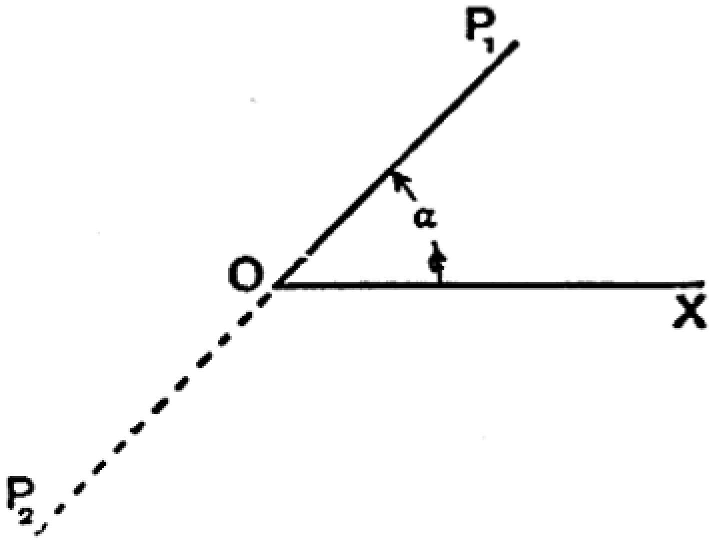


Fig. 92.

If it coincides with OP_1,

$$x = (\text{a multiple of } 2\pi) + \alpha;$$

$$\therefore\ x = (\text{an } \textit{even} \text{ multiple of } \pi) + \alpha.$$

If it coincides with OP_2,

$$x = (\text{a multiple of } 2\pi) + \pi + \alpha;$$

$$\therefore\ x = (\text{an } \textit{odd} \text{ multiple of } \pi) + \alpha.$$

Consequently $x = n\pi + \alpha$, where n is any positive or negative integer.

Ex. 1. *Solve the equation* $2\sec x - 3\cos x = 5$.

The equation is
$$\frac{2}{\cos x} - 3\cos x - 5 = 0;$$

$$\therefore\ 2 - 3\cos^2 x - 5\cos x = 0;$$

$$\therefore\ 3\cos^2 x + 5\cos x - 2 = 0;$$

$$\therefore\ (3\cos x - 1)(\cos x + 2) = 0;$$

$$\therefore\ \text{either } \cos x = \tfrac{1}{3} \text{ or } \cos x = -2.$$

Since $\cos x$ always lies between -1 and $+1$, $\cos x = -2$ gives no real solution.

From the tables, if $\cos x = 0{\cdot}3333$, $\cos x = \cos 70^\circ\,32'$ approximately;
$$\therefore\ x = n\,.\,360^\circ \pm 70^\circ\,32'.$$

Ex. 2. *Find all the values of* θ *between* 0 *and* 360° *which satisfy the equation*

$$7 = 2\sin(\theta-30)^\circ + 8\cos^2(\theta-30)^\circ.$$

Since $\cos^2(\theta-30)^\circ = 1 - \sin^2(\theta-30)^\circ$, the equation is

$$8\sin^2(\theta-30)^\circ - 2\sin(\theta-30)^\circ - 1 = 0;$$

$$\therefore\ [2\sin(\theta-30)^\circ - 1][4\sin(\theta-30)^\circ + 1] = 0;$$

$$\therefore\ \sin(\theta-30)^\circ = \tfrac{1}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{or}\quad \sin(\theta-30)^\circ = -\tfrac{1}{4}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(1) $\quad \operatorname{Sin}(\theta-30)^\circ = \tfrac{1}{2} = \sin 30^\circ;$

$$\therefore\ (\theta-30)^\circ = n\,.\,180^\circ + (-1)^n 30^\circ.$$

Giving n the values 0 and 1, we have

$$(\theta - 30)^\circ = 30^\circ \quad \text{or} \quad 150^\circ;$$

$$\therefore \ \theta = 60^\circ \quad \text{or} \quad 180^\circ.$$

If n is given any other integral value θ lies outside the given range.

(2) $\operatorname{Sin}(\theta - 30)^\circ = -0{\cdot}25 = -\sin 14^\circ 28' = \sin(-14^\circ 28')$;

$$\therefore \ (\theta - 30)^\circ = n \,.\, 180^\circ + (-1)^n(-14^\circ 28').$$

Giving n the values 0 and 1, we have

$$(\theta - 30)^\circ = -14^\circ 28' \quad \text{or} \quad 180^\circ + 14^\circ 28';$$

$$\therefore \ \theta = 15^\circ 32' \quad \text{or} \quad 224^\circ 28'.$$

Ex. 3. *Find by drawing the graphs of* $\sin x$ *and* $\frac{1}{3}(x+1)$ *the number of solutions of the equation* $3 \sin x = x + 1$. *Find the approximate value of the smallest positive root, and test your result by means of the tables.* (*x is the circular measure of the angle.*) [L.U.]

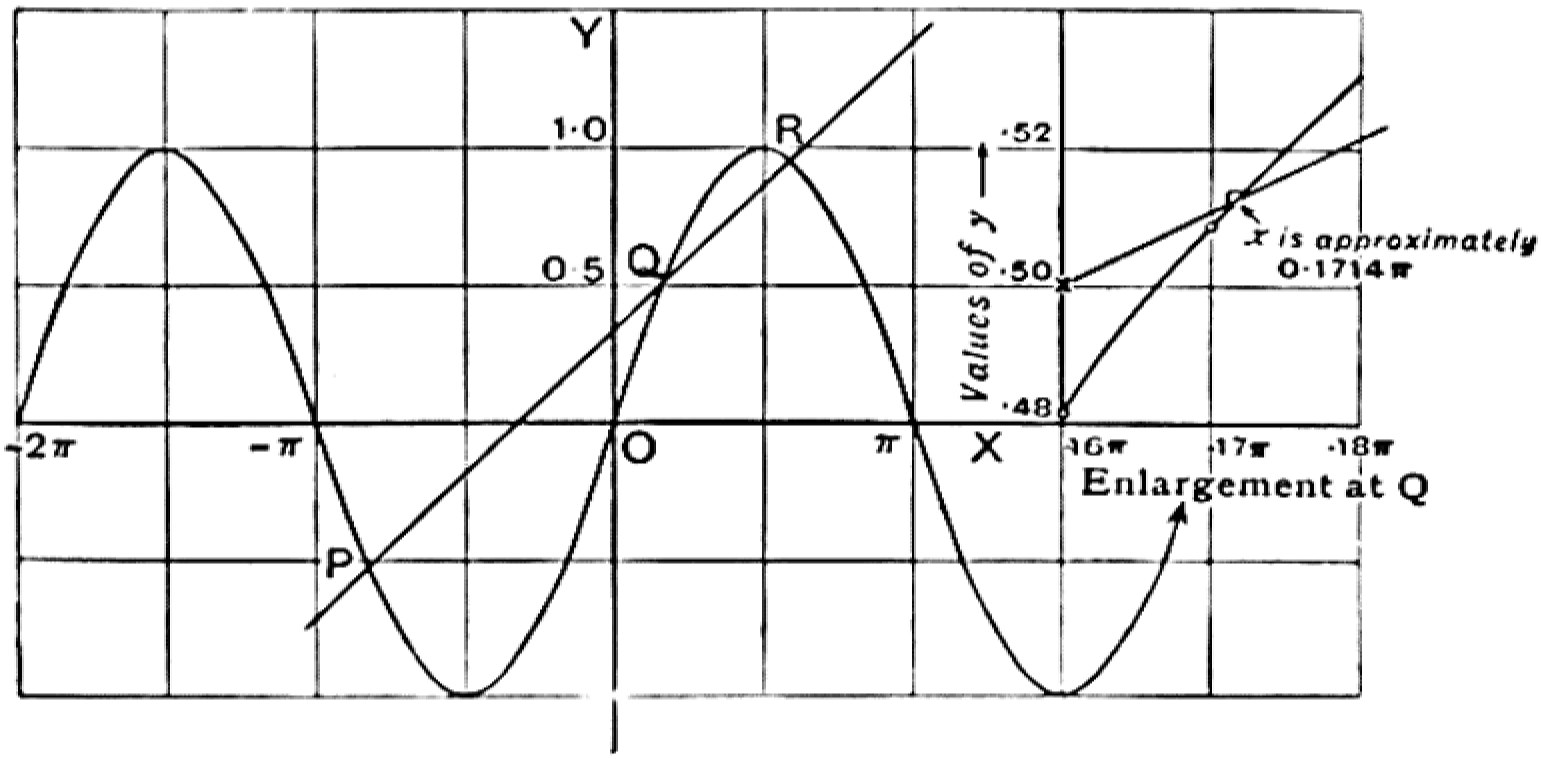


FIG. 93.

Fig. 93 shews the two graphs which intersect in the three points P, Q, R. There are therefore 3 roots, viz. the abscissae of the points P, Q, R, *i.e.* approximately $-0{\cdot}83\pi$, $0{\cdot}2\pi$ and $0{\cdot}6\pi$.

At Q, x is more accurately $0{\cdot}17\pi$.

Enlarging the graph in the neighbourhood of Q, a closer approximation is $x = 0{\cdot}1714\pi$.

EXERCISES XVII.

1. Complete the following table:

Angle.	Sine.	Cosine.	Tangent.	Angle.	Sine.	Cosine.	Tangent.
150°				-1000°			
135°				$\pi/3$			
480°				$5\pi/6$			
-330°				$3\pi/5$			
650°				$-\pi/4$			

2. Using the tables, draw the graphs of $\cos x$, $\cot x$ and $\operatorname{cosec} x$ from 0 to 2π.

3. Obtain the relations between the circular functions of $\pi+\theta$ and θ, and of $\frac{1}{2}\pi+\theta$ and θ given on p. 168.

4. Find equations connecting the circular functions of $3\pi/2+x$ with those of x.

5. Find the following in terms of x: (1) $\cos(270+x)^\circ$; (2) $\sin(360-x)^\circ$; (3) $\tan(720+x)^\circ$; (4) $\operatorname{cosec}(270-x)^\circ$.

6. What relations are obtained by writing $\frac{1}{2}\pi+x$ for θ in the identities connecting the circular functions of $\frac{1}{2}\pi-\theta$ and θ given on pp. 167-168.

7. Write down all the angles between -360° and $+360^\circ$ which have (i) the same sine as 40°, (ii) the same cosine as 50° and (iii) the same tangent as 60°.

8. The angle θ lies between 0 and $-\frac{1}{2}\pi$ and $\cos\theta=\frac{8}{17}$; find the value of $\cot\theta+3\operatorname{cosec}\theta$.

9. The angle θ lies between 0 and π and $\tan\theta=-\frac{5}{12}$; find $2\sin\theta+3\sec\theta$.

10. Solve the equations:

(i) $8\cos^2\theta+6\sin\theta-9=0$;

(ii) $17\cot\theta=4\operatorname{cosec}^2\theta+11$;

(iii) $3\sin\theta+2\cos^2\theta=0$;

(iv) $8\cos^4 x-6\cos^2 x+1=0$;

(v) $16\sin^4\theta-16\sin^2\theta+3=0$;

(vi) $\sqrt{3}\cot^2 x-(\sqrt{3}+1)\cot x+1=0$.

11. ABC is an equilateral triangle of side 14 inches. With each vertex as centre and radius 7 inches, arcs are drawn cutting the sides at their middle points P, Q, R. Find the area of the figure PQR bounded by these three circular arcs. [I.C.E.]

12. Solve the equations :

(i) $6\cos\theta + 5\sec\theta = 20$.

(ii) $6\sec\theta + 5\cos\theta = 20$. [I.M.E.]

13. Define the tangent of an angle of any magnitude, and find a general expression for all angles whose tangent is 2.

Solve the equations:

(i) $3(\tan^2\theta + \cos^2\theta) = 10$.

(ii) $\tan\theta \tan n\theta = 1$. [B.U.]

14. AB is a diameter of a circle and PQ is a chord perpendicular to it. If PQ subtend an angle 2θ at the centre of the circle, find an equation to determine θ so that the area of the triangle APQ is equal to the area between the chord PQ and the arc PBQ. Determine θ, to the nearest degree, with the help of tables. [B.U.]

15. Define the sine and cosine of an angle of any magnitude and sign, and prove that $\cos(90° + A) = -\sin A$.

Solve the equation $\tan^2 A = 3$, giving all the solutions. [B.U.]

16. AB is an arc of a circle of radius 32·6 cm., subtending an angle of 68° 18′ at the centre O. Calculate (i) the length of the arc AB, (ii) the area of the sector OAB, (iii) the area of the segment between the arc AB and the chord AB. [S.W.]

17. Plot the curve $y = 2\sin x + \sin 2x$ for the range $x = 45°$ to $x = 315°$, and read off the points where it meets the line

$$y = 3{\cdot}6 - 0{\cdot}02x,$$

where x is in degrees. [Leeds.]

18. Prove that if α is the smallest positive angle whose cosine is equal to k, where $0 \leq k \leq 1$, then every angle θ which satisfies the equation $\cos\theta = k$ is included in the expression $2n\pi \pm \alpha$, n being zero or any positive or negative integer.

Hence solve completely $\cos 2\theta = \cos 7\theta$. [B.U.]

19. Define a *radian*, and shew that it is a definite angle, stating the theorems used in the proof.

AB is the diameter of a semicircle. Find the position of a point P on the semicircle such that the length of the chord AP is equal to the length of the arc BP. [B.U.]

Examples 20-24 *are taken from papers set at the Inter. B.Sc.* [L.U.]

20. Draw graphs of the curves

$$\text{(i) } y=2x^2-1{\cdot}5x-0{\cdot}5, \qquad \text{(ii) } y=\sin\tfrac{1}{2}\pi x,$$

using the same scales and taking two inches as the unit of length for both x and y. From the graphs obtain approximate solutions of the equation

$$2\sin\tfrac{1}{2}\pi x=4x^2-3x-1.$$

21. Find by graphs all the solutions of the equation

$$3\cos\pi x+\sin\pi x=x,$$

x being the circular measure of an angle, and check your results from the tables.

22. Make graphs of the functions $\tan x$ and $\sin(x+\pi/8)$ on the same coordinate axes and with the same scales, and determine from their intersections the two positive values of x less than π which satisfy the equation $\sin(x+\pi/8)=\tan x$.

23. Construct points in the graph of $\tan 2\theta$ from $\theta=0$ to $\theta=\pi$. Find approximate values of two roots of the equation

$$3\tan 2\theta=2-\theta$$

by aid of the graph.

24. Define a *radian* and prove that it is a constant angle.

By drawing the graphs $y=\tan\theta$, $y=3(2-\theta)/4$, θ being in circular measure, find approximately the solutions of the equation

$$4\tan\theta+3\theta=6$$

which lie between 0 and π.

CHAPTER VII.

ELEMENTARY ALGEBRAIC OR COORDINATE GEOMETRY.

The Straight Line. Consider the equation $y = mx + c$.

Let A be a particular point whose coordinates (x_1, y_1) satisfy the equation.

Let P be *any other point* whose coordinates (x_2, y_2) satisfy the equation.

Using Fig. 94, $PN = y_2 - y_1$, $AN = x_2 - x_1$;

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{PN}{AN} = \tan\theta. \quad \ldots\ldots\text{(i)}$$

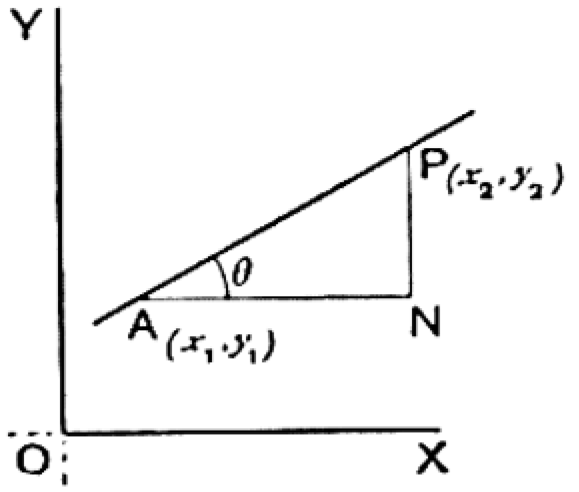


FIG. 94.

Now, since (x_1, y_1) and (x_2, y_2) satisfy the equation $y = mx + c$,

$$y_1 = mx_1 + c$$

and $$y_2 = mx_2 + c;$$

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} = m;$$

whence, from (i), $\tan\theta = m$, and therefore **$\tan\theta$** is fixed. That is, the direction from A to P is fixed. In other words, P lies on the fixed straight line which passes through A and has the slope m.

Since P was *any* point whose coordinates satisfied the equation $y = mx + c$, we see that *all* points whose coordinates satisfy this equation lie on the straight line which passes through A and has the slope m.

This has already been tested by numerous examples (pp. 43-62).

Parallel Lines. Straight lines which have the same slope are parallel; therefore $y = mx + c$ and $y = m'x + c'$ **represent parallel straight lines if $\mathbf{m = m'}$.**

Perpendicular Lines. Fig. 95 represents two perpendicular lines $y = mx + c$ and $y = m'x + c'$.

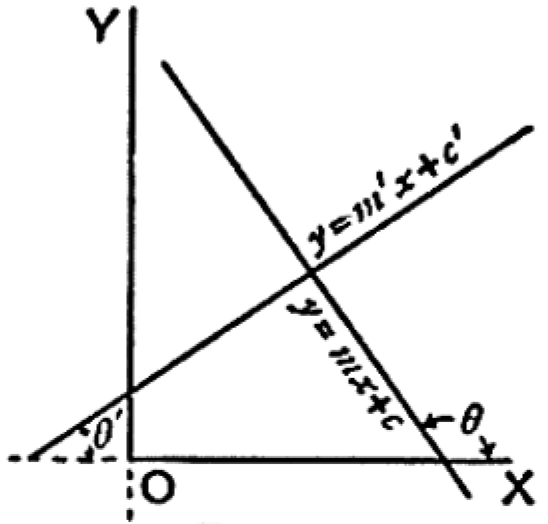


FIG. 95.

We have $\theta = \frac{1}{2}\pi + \theta'$, $\tan\theta = m$ and $\tan\theta' = m'$.

$$\text{Hence} \quad m = \tan\theta$$
$$= \tan(\tfrac{1}{2}\pi + \theta')$$
$$= -\cot\theta'$$
$$= -\frac{1}{m'}.$$

$\therefore$ the lines $y = mx + c$ and $y = m'x + c'$ **are perpendicular if**

$$\mathbf{mm' = -1.}$$

Ex. 1. *Find the equation of the straight line through* (x', y') *parallel to the line* $y = 3x + 1$.

The slope of the required line must be 3.

Let its equation be $\quad y = 3x + c.$ ....(1)

Since it passes through (x', y'),

$$y' = 3x' + c;$$

$$\therefore\ c = y' - 3x'.$$

Substituting in (1), we have

$$y = 3x + y' - 3x'$$

as the required equation.

Ex. 2. *Find the equation to the straight line through* (3, −1) *perpendicular to the line* $4x - 5y - 2 = 0$.

The given straight line is

$$5y = 4x - 2$$

$$\text{or} \quad y = \tfrac{4}{5}x - \tfrac{2}{5},$$

and therefore its slope is $\frac{4}{5}$.

Consequently the slope of every perpendicular straight line is $-\frac{5}{4}$.

Let the equation of the required line be

$$y = -\tfrac{5}{4}x + c. \quad \dots\dots\dots\dots\dots\dots(1)$$

Since it passes through $(3, -1)$,

$$-1 = -\tfrac{15}{4} + c;$$

$$\therefore \; c = \tfrac{11}{4}.$$

Substituting in (1), we have

$$y = -\tfrac{5}{4}x + \tfrac{11}{4}$$

or $\quad 4y + 5x = 11$

as the required equation.

Equation of the Straight Line through a given point $(\mathbf{x}', \mathbf{y}')$ having a given slope $\mathbf{m}$.

Let the equation be $\quad y = mx + c.$
Since (x', y') lies on the line, we have

$$y' = mx' + c.$$

The equation of the line is therefore

$$y = mx + (y' - mx'),$$

i.e. $\quad \mathbf{y - y' = m(x - x')}.$

The form of this equation should be noted. The equation represents a line having slope m, and it satisfies the condition that $y = y'$ when $x = x'$.

Ex. 1. *Write down the equation of the line joining the points* (x_1, y_1) *and* (x_2, y_2).

The slope of the line is $(y_2 - y_1)/(x_2 - x_1)$, and it passes through (x_1, y_1).

Its equation is, therefore,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \quad \dots\dots\dots\dots\dots(1)$$

Alternatively, since it passes through (x_2, y_2), its equation is

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2). \quad \dots\dots\dots\dots\dots(2)$$

It should be verified that equations (1) and (2) are identical.

Ex. 2. *If A, B are the points* (x_1, y_1), (x_2, y_2), *find the coordinates of the point P in AB which is such that* $pAP = qPB$.

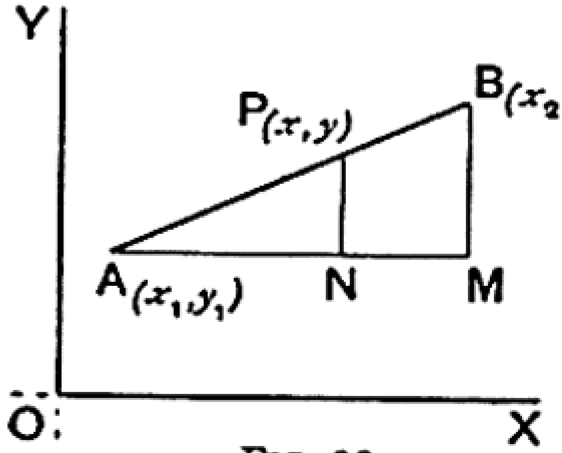


FIG. 96.

Let the coordinates of P be (x, y). Using Fig. 96,

$$\frac{PN}{BM} = \frac{AP}{AB}$$

$$= \frac{q}{p+q}, \text{ since } \frac{AP}{PB} = \frac{q}{p},$$

i.e. $$\frac{y - y_1}{y_2 - y_1} = \frac{q}{p+q};$$

$$\therefore (p+q)y = (p+q)y_1 + q(y_2 - y_1)$$

$$= py_1 + qy_2;$$

$$\therefore y = \frac{py_1 + qy_2}{p+q}.$$

Similarly, $$x = \frac{px_1 + qx_2}{p+q}.$$

COR. The coordinates of the mid point of AB are $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$.

Ex. 3. *If A, B* (Fig. 97) *are the points* (1, 2), (4, 5) *and P divides AB in the ratio* 3 : 1, *find the equation of the straight line through P perpendicular to AB.*

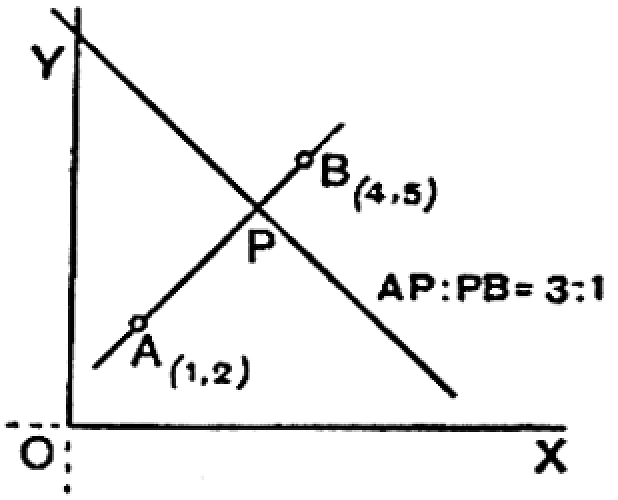


FIG. 97.

Since $AP = 3PB$, the coordinates of P are

$$\frac{1 \times 1 + 3 \times 4}{1+3}, \quad \frac{1 \times 2 + 3 \times 5}{1+3},$$

i.e. $$\frac{13}{4}, \quad \frac{17}{4}.$$

The slope of AB is $\frac{5-2}{4-1}$, i.e. 1.

Therefore the slope of the required line is -1.
Its equation is therefore

$$y - \frac{17}{4} = -1\left(x - \frac{13}{4}\right),$$

i.e. $$2y + 2x = 15.$$

Ex. 4. *A triangle is determined by the lines* $y+x-6=0$, $3y-x+2=0$ *and* $3y=5x+2$. *Find its orthocentre and centroid.*

The first two lines intersect in the point whose coordinates satisfy

$$\left.\begin{aligned} y+x-6&=0,\\ 3y-x+2&=0. \end{aligned}\right\}$$

That is, in the point A, given by $x=5$, $y=1$.

N.B.—The solution of the equations $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ can be written in the form

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}, \text{ see p. 235.}$$

It is well to remember this, since the coordinates of the intersection of two lines are often required in examples in Algebraic Geometry.

The line through A perpendicular to $3y=5x+2$ has for its equation

$$y-1=-\tfrac{3}{5}(x-5), \quad \text{.......................(1)}$$

since its slope is $-\frac{3}{5}$.

Again, the point of intersection of the lines $3y-x+2=0$ and $3y=5x+2$ is the point B, given by $x=-1$, $y=-1$.

The line through B, perpendicular to $y+x-6=0$,

is $$y+1=x+1. \quad \text{.......................(2)}$$

The orthocentre is the intersection of the lines (1) and (2), that is, of

$$\left.\begin{aligned} 5y+3x&=20\\ \text{and}\qquad y-x&=0. \end{aligned}\right\}$$

That is, the orthocentre is the point $2\frac{1}{2}$, $2\frac{1}{2}$.

This may be checked by verifying that the line joining $(2\frac{1}{2}, 2\frac{1}{2})$ to the vertex C, viz. the intersection of $3y=5x+2$ and $y+x-6=0$, is perpendicular to $3y-x+2=0$.

The vertex C is the point $(2, 4)$.

The coordinates of the mid point of BC are

$$(-1+2)/2,\ (-1+4)/2; \quad \textit{i.e.}\ (\tfrac{1}{2}, 1\tfrac{1}{2}).$$

Therefore the median through A, being the join of $(5, 1)$ and $(\frac{1}{2}, 1\frac{1}{2})$, is given by

$$9y+x=14. \quad \text{.......................(3)}$$

Again, the coordinates of the mid-point of AB are $(2, 0)$. Therefore the median through C is given by

$$x=2. \quad \text{.............................(4)}$$

Now the centroid is the intersection of the medians, and from equations (3) and (4) its coordinates are $(2, 1\frac{1}{3})$.

Ex. 5. *ABC is a triangle* (Fig. 98). *A is the point* $(4, -2)$. *The slope of AB is* $4/3$; *AC passes through the point* $(2\frac{1}{2}, 0)$; *and the equation of BC is* $9y+4x=46$. *Find the coordinates of B, C and the circumcentre.*

The equation of AB is $y+2=\frac{4}{3}(x-4)$,

$$i.e. \qquad 3y-4x+22=0. \qquad\qquad (1)$$

The slope of AC is $\dfrac{-2-0}{4-2\frac{1}{2}}$, *i.e.* $-\dfrac{4}{3}$.

The equation of AC is $y=-\dfrac{4}{3}(x-2\frac{1}{2})$,

$$i.e. \qquad 3y+4x-10=0. \qquad\qquad (2)$$

The equation of BC is

$$9y+4x-46=0. \qquad\qquad (3)$$

The coordinates of B satisfy equations (1) and (3). Solving these equations, B is $(7, 2)$. Similarly, solving equations (2) and (3), C is $(-2, 6)$.

The mid point M of AB is

$$(4+7)/2, \ (-2+2)/2,$$

$$i.e. \qquad \frac{11}{2}, \quad 0.$$

The line through M perpendicular to AB has slope $-3/4$, and therefore its equation is

$$y=-\frac{3}{4}\left(x-\frac{11}{2}\right),$$

$$i.e. \qquad 8y+6x-33=0. \qquad\qquad (4)$$

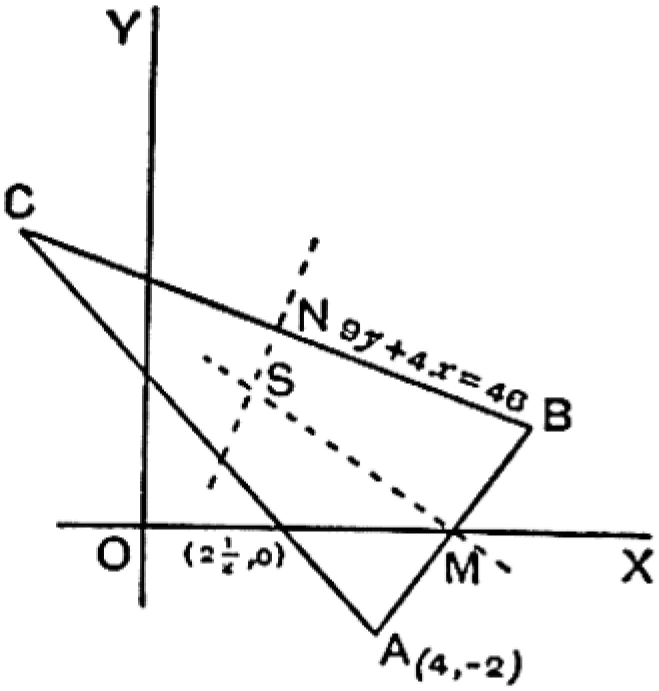


FIG. 98.

The mid point N of BC is $(7-2)/2$, $(2+6)/2$,

$$i.e. \quad 5/2, \quad 4.$$

The line through N perpendicular to BC has slope $9/4$, and therefore its equation is

$$y-4=\frac{9}{4}\left(x-\frac{5}{2}\right),$$

$$i.e. \qquad 8y-18x+13=0. \qquad\qquad (5)$$

The coordinates of the circumcentre S satisfy equations (4) and (5).

Solving these equations, S is $23/12$, $43/16$.

EXERCISES XVIII.

1. Find the equations of the straight lines through

(*a*) $(1, 2)$ of slope 3; (*b*) $(-2, 5)$ of slope $\frac{1}{2}$;

(*c*) $(-3, -2)$ of slope $-\frac{1}{3}$; (*d*) $(6, 0)$ of slope $-\frac{2}{3}$.

2. Find the equation of the straight lines joining

(*a*) $(1, 2)$ and $(-5, -1)$; (*b*) $(3, 4)$ and $(0, 1)$;

(*c*) $(-2, -1)$ and $(-7, 4)$.

3. Find the equations of the straight lines through the point $(1, 2)$ parallel and perpendicular to

(*a*) $y=3x+1$; (*b*) $2y+x=0$; (*c*) $2x-3y+4=0$.

4. Shew that the middle point of the line joining $(5, 3)$ and $(5, 7)$ is the same as the middle point of the line joining $(-1, 1)$ and $(11, 9)$.

5. Shew that $(-5, 6)$, $(2, 8)$, $(11, -7)$ and $(4, -9)$ are the vertices of a parallelogram.

6. The vertices of a triangle are $(4, 1)$, $(1, -6)$ and $(-2, 2)$. Find the coordinates of the point of intersection of the medians. (This point divides any median in the ratio $2 : 1$.)

7. Find the equation of the straight line through the intersection of $2x+3y=13$ and $5x-2y=4$ which is parallel to

$$2x-3y+4=0.$$

8. Find the equation of the straight line through the intersection of $x+2y=0$ and $y+4x+7=0$ which is perpendicular to $3x-y=0$.

9. Find the coordinates of the orthocentre of the triangle whose vertices are $(-1, 0)$, $(3, 0)$ and $(0, 3)$.

10. Find the equation of the straight line through (x', y') perpendicular to the line $ax+by+c=0$.

Obtain the coordinates of the circumcentre of the triangle formed by the lines $x-2y=1$, $3x-y=3$, $2x+y=17$. [Durham.]

11. Find the coordinates of the point dividing the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m : n$.

Two sides of a parallelogram are $y=2x$ and $2y=x$. If the coordinates of its centre are $(2, 3)$, find the coordinates of the four vertices. [Man.]

12. A rectangle whose sides are equally inclined to the axes of reference has a vertex at the point $(9, -3)$, and one of its diagonals is the line $3x+y=12$. Find the coordinates of the other three vertices and the equation of the other diagonal. [B.U.]

13. Find the coordinates of the point which divides in a given ratio the line joining two points with given coordinates.

Four points A, B, C, D have coordinates $(2, 1)$, $(-7, 6)$, $(8, -3)$, $(1, 2)$ respectively. Find the ratios in which AC and BD divide each other. [B.U.]

14. Find the condition that the straight lines $y = mx$, $y = m'x$ should be perpendicular, the axes being rectangular.

Two vertices of a triangle are at the points $(3, -1)$, $(-2, 3)$, and the orthocentre is at the origin; find the coordinates of the remaining vertex. Give a figure. [S.W.]

15. The coordinates of three points A, B, C are $A(1, 2)$, $B(-3, 4)$, $C(5, -2)$. Find the equations of (1) the sides BC, CA, AB of the triangle ABC; (2) the perpendiculars to the sides at their middle points respectively; (3) the perpendiculars from the angular points on the opposite sides respectively.

Deduce from (3) the coordinates of the orthocentre of the triangle. [S.W.]

Other Forms of the Equation of a Straight Line.

I. In terms of the intercepts made on the axes.

In Fig. 99 the line cuts OX and OY in A and B respectively, where $OA = a$, $OB = b$.

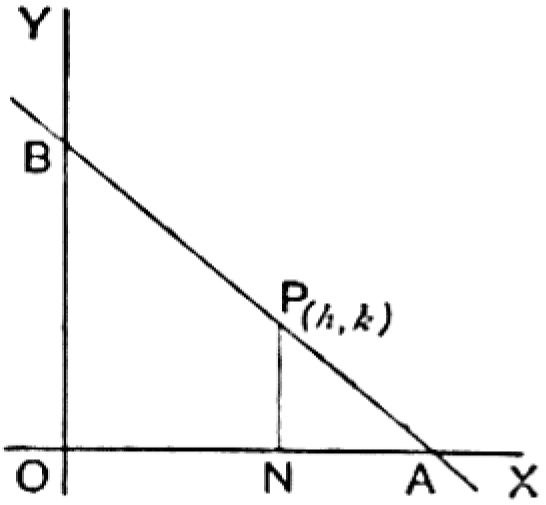


FIG. 99.

Let $P\ (h, k)$ be any point on the line and PN the ordinate at P.

By similar triangles, $\dfrac{NP}{OB} = \dfrac{NA}{OA}$,

i.e. $$\frac{k}{b} = \frac{a-h}{a},$$

i.e. $$\frac{h}{a} + \frac{k}{b} = 1.$$

And this is the condition that the point (h, k) shall lie on the line

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This is the standard form of the equation of a straight line which makes intercepts a and b on the axes of X and Y respectively.

II. In terms of the perpendicular from the origin and the angle the perpendicular makes with the axis of X.

In Fig. 100, p is the length of the perpendicular OD from the origin to the given line and a is the angle DOX.

Let $P\ (h, k)$ be any point on the line and PN the ordinate at P.

Let Q be the foot of the perpendicular from N to OD and let NP meet OD in R.

Then $$OQ = ON\cos a = h\cos a,$$
$$QD = NP\cos QRN$$
$$= k\cos(90^\circ - a) = k\sin a\,;$$
$$\therefore\ h\cos a + k\sin a = OD = p.$$
And this is the condition that the point (h, k) shall lie on the line
$$x\cos a + y\sin a = p.$$

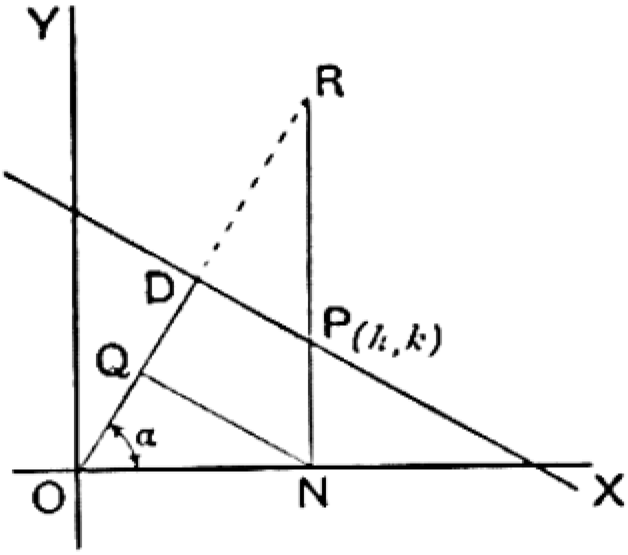


FIG. 100.

This is the standard form of the equation of a straight line such that the perpendicular from the origin is of length p and makes an angle a with the axis of X.

The sign of $\mathbf{p - x'\cos a - y'\sin a}$, where $\mathbf{(x', y')}$ does not lie on the straight line $\mathbf{p - x\cos a - y\sin a = 0}$.

Fig. 101 shews PQ, the line $p - x\cos a - y\sin a = 0$, OP being p, the perpendicular from the origin.

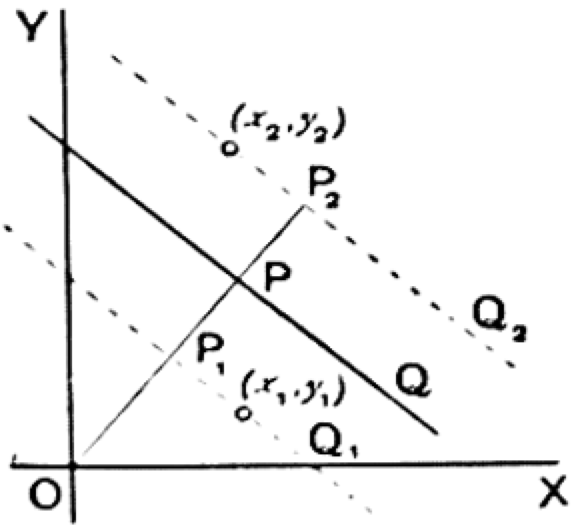


FIG. 101.

(x_1, y_1) is a point on the same side of the line PQ as the origin; (x_2, y_2) is a point on the opposite side.

The lines P_1Q_1, P_2Q_2, through (x_1, y_1) and (x_2, y_2) parallel to PQ, meet OP in P_1 and P_2 respectively.

If $OP_1 = p_1$, the equation of P_1Q_1 is
$$p_1 - x\cos a - y\sin a = 0.$$
Since (x_1, y_1) lies on this line,
$$p_1 = x_1\cos a + y_1\sin a.$$
From the figure,
$$P_1P = p - p_1 = p - x_1\cos a - y_1\sin a.$$
Hence $p - x_1\cos a - y_1\sin a$ is the actual length of the perpendicular from (x_1, y_1) to PQ and is **positive**.

Similarly, if $OP_2 = p_2$, the equation of P_2Q_2 is

$$p_2 - x\cos\alpha - y\sin\alpha = 0.$$

Since (x_2, y_2) lies on this line,

$$p_2 = x_2\cos\alpha + y_2\sin\alpha.$$

From the figure,

$$PP_2 = p_2 - p = x_2\cos\alpha + y_2\sin\alpha - p.$$

Hence $x_2\cos\alpha + y_2\sin\alpha - p$ is the actual length of the perpendicular from (x_2, y_2) to PQ and is positive.

Therefore, $p - x_2\cos\alpha - y_2\sin\alpha$ is negative.

Hence, generally, $p - x'\cos\alpha - y'\sin\alpha$ is positive or negative according as (x', y') and the origin lie on the same or on opposite sides of the line

$$p - x\cos\alpha - y\sin\alpha = 0.$$

Comparing Coefficients. If the equations $ax + by + c = 0$ and $a'x + b'y + c' = 0$ represent the *same* straight line, then each coefficient in either equation must be the same multiple of the corresponding coefficient in the other equation, *i.e.*

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

This is called "comparing coefficients."

The sign of $\mathbf{ax' + by' + c}$, where $\mathbf{(x', y')}$ does not lie on the straight line $\mathbf{ax + by + c = 0}$. Assume c is *positive*. (The equation of a line can always be written so that the constant term is positive.)

Let $p - x\cos\alpha - y\sin\alpha = 0$ be the same straight line as $ax + by + c = 0$, so that p is the actual length of the perpendicular from the origin to the straight line.

Comparing coefficients, $\dfrac{a}{-\cos\alpha} = \dfrac{b}{-\sin\alpha} = \dfrac{c}{p}$;

$$\therefore\ ax' + by' + c = \frac{c}{p}(-x'\cos\alpha - y'\sin\alpha + p).$$

Now c and p are both positive. Therefore $ax' + by' + c$ and $p - x'\cos\alpha - y'\sin\alpha$ have the same sign.

Consequently, from the above, if the equation of the line is given in the form

$$ax + by + c = 0,$$

where c is *positive*, then

$$ax' + by' + c$$

is positive or negative according as (x', y') and the origin lie on the same or on opposite sides of the line $ax + by + c = 0$.

Ex. 1. *B, C, are the feet of the perpendiculars from a point A(h, k) on the axes OX, OY. Find the equation of the straight line through A perpendicular to BC.*

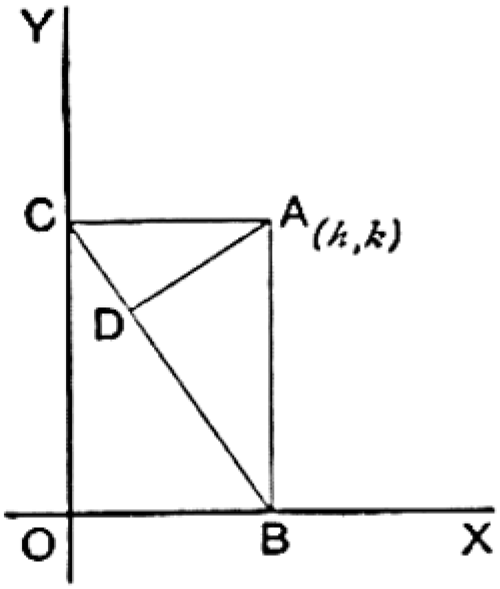


FIG. 102.

Using Fig. 102, $OB=h$, $OC=k$.

The equation to BC is therefore $x/h+y/k=1$.

The slope of BC is $-k/h$.

Therefore the equation of the perpendicular from A to BC is

$$y-k=\frac{h}{k}(x-h),$$

$$\text{i.e.}\qquad ky-hx=k^2-h^2.$$

Ex. 2. *If in the last example A moves along the line $y=2x$, find the locus of D, the foot of the perpendicular from A on BC.*

In Fig. 102, let x', y' be the coordinates of D.

$$\text{Then } x', y' \text{ satisfy}\quad \left.\begin{aligned} ky-hx&=k^2-h^2\\ \text{and}\quad x/h+y/k&=1\ ;\end{aligned}\right\}$$

$$\therefore\quad \left.\begin{aligned} ky'-hx'&=k^2-h^2\\ \text{and}\quad hy'+kx'&=hk.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\text{I.}$$

But since (h, k) satisfy $y=2x$,

$$k=2h.$$

Substituting for k in I., we have

$$\left.\begin{aligned} 2y'-x'&=3h,\\ y'+2x'&=2h.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\text{II.}$$

Eliminating h from II., we have

$$\frac{2y'-x'}{y'+2x'}=\frac{3}{2},$$

$$\text{i.e.}\qquad y'=8x'.$$

This is the condition that $D\,(x', y')$ lies on the line

$$y=8x,$$

which is the equation of the required locus.

Ex. 3. *Write the equation* $16x-30y+13=0$ *in the form* $x\cos\alpha+y\sin\alpha=p$; *find the values of* α *and* p.

Comparing $\qquad 16x-30y+13=0$

with $\qquad x\cos\alpha+y\sin\alpha-p=0,$

we have
$$\frac{\cos\alpha}{16}=\frac{\sin\alpha}{-30}=\frac{-p}{13};$$

and, since $\cos^2\alpha+\sin^2\alpha=1$,
$$\frac{p^2}{13^2}(16^2+30^2)=1;$$
$$\therefore\ p^2=\frac{13^2}{1156}=\left(\frac{13}{34}\right)^2; \qquad \therefore\ p=\frac{13}{34};$$
$$\therefore\ \cos\alpha=-\frac{16}{34}\text{ and }\sin\alpha=+\frac{30}{34};$$
$$\therefore\ \alpha\text{ lies between }90^\circ\text{ and }180^\circ,$$
$$\text{and}\qquad \tan\alpha=-1{\cdot}875;$$
$$\therefore\ \alpha=180^\circ-61^\circ 56'=118^\circ 4'.$$

Ex. 4. *Find the locus of a point which moves so that the square of its distance from a given point varies as its perpendicular distance from a fixed straight line.*

Take the fixed point O as origin and a line through O perpendicular to the fixed straight line as the axis of X.

The equation to the fixed straight line may be written
$$x=a.$$

Using Fig. 103, let the coordinates of the moving point P be (h, k).

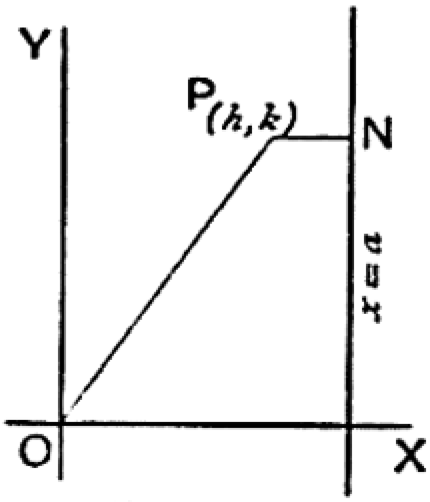


FIG. 103.

We are given $OP^2=c\,.\,PN$, where c is a constant and PN is perpendicular to the line $x=a$.

Hence $\qquad h^2+k^2=c\,.\,(a-h),$

which is the condition that h, k, lies on
$$x^2+y^2=c(a-x).$$

Since this equation may be written
$$\left(x+\frac{c}{2}\right)^2+y^2=ca+\frac{c^2}{4},$$
the locus of P is a circle whose centre is the point $(-\tfrac{1}{2}c, 0)$ and whose radius is $\sqrt{ca+c^2/4}$.

EXERCISES XIX

1. Find the equations of the straight lines whose intercepts on the axes of X and Y respectively are

(i) 2, 3 ; (ii) -2, 3 ; (iii) 5, 7 ; (iv) -6, -4.

2. Find the equations of the lines joining the points

(i) (3, 0) and (0, 4) ; (ii) $(-2, 0)$ and $(0, -5)$.

3. Find the equation of the lines which pass through $(-1, 4)$ and make equal intercepts on the axes.

4. What are the intercepts made by the line $ax+by+c=0$ on the axes ?

5. Write the following lines in the form $x\cos\alpha+y\sin\alpha=p$, and determine in each case the values of α and p.

(i) $4x+3y=6$; (ii) $3x-4y=6$;
(iii) $5y-12x=13$; (iv) $5x+12y+13=0$.

6. A and B are points on the axes of X and Y respectively. If (h, k) is the middle point of AB, what is the equation of AB ?

7. Prove that the equation of a straight line may be put into the form

$$x\cos\alpha+y\sin\alpha=p.$$

Find the equation of the line through the point (1, 2) whose distance from the point (3, 1) has the greatest possible value. [B.U.]

8. Obtain geometrically the equation of any straight line in the form

$$x\cos\alpha+y\sin\alpha-p=0.$$

Express the line $8x-15y+17=0$ in this form, and construct the angle α without actually drawing the line. [S.W.]

9. P is a point which lies on the locus whose equation is $y^2=4ax$. PN is drawn perpendicular to OX. Find the locus of the middle point of PN.

10. A point P moves so that its distance from the point $(a, 0)$ is equal to its distance from the y-axis. Find the equation of its locus.

11. A straight line AB of constant length c slides so that A and B move along the axes of X and Y respectively. Shew that the locus of its middle point is a circle. Find also the equation of each of the points of trisection of AB.

12. APQ is a straight line passing through a fixed point $A(a, 0)$ and meeting the straight lines $x=b$ and $y=b$ in P and Q respectively. If O is the origin and B the point $(b, 0)$, shew that the locus of the intersection of OP and BQ is a straight line.

13. A is the point $(2, 0)$. P and Q move on OY and are such that PQ always equals 2. Find the locus of the foot of the perpendicular from Q to AP.

14. If A is the point $(a, 0)$ and B the point $(0, b)$, find the standard equation to AB, viz., $x/a+y/b=1$, (i) by using the method of Ex. 1, p. 179, (ii) by using the fact that, if P is any point on AB, then $\Delta OAP+\Delta OBP=\Delta OAB$.

15. Deduce the equation of a line in the form

$$p-x\cos\alpha-y\sin\alpha=0$$

from the form $x/a+y/b=1$.

The length of the perpendicular from the point (x', y') to the straight line $ax+by+c=0$ is $\dfrac{ax'+by'+c}{\sqrt{a^2+b^2}}$.

In Fig. 101, let $ax+by+c=0$ be the equation of PQ and let $p-x\cos a-y\sin a=0$ be the same straight line, so that p is the perpendicular OP. Comparing coefficients,

$$-\frac{\cos a}{a}=-\frac{\sin a}{b}=\frac{p}{c}. \quad \text{........................(1)}$$

As on page 185, the perpendicular from (x', y') to PQ, neglecting sign, is

$$p-x'\cos a-y'\sin a,$$

or, using the values of $\cos a$ and $\sin a$ from (1),

$$\frac{p}{c}(c+ax'+by'). \quad \text{........................(2)}$$

Now, since $\cos^2 a+\sin^2 a=1$, we have from (1)

$$\frac{p^2}{c^2}(a^2+b^2)=1,$$

$$i.e. \qquad p=\pm\frac{c}{\sqrt{a^2+b^2}}.$$

Hence the expression (2) already found for the perpendicular, becomes

$$\pm\frac{ax'+by'+c}{\sqrt{a^2+b^2}}.$$

Alternatively, the length of the perpendicular may be found as follows :

Let P (Fig. 104) be the point (x', y') and let Q, the foot of the perpendicular from P on the line $ax+by+c=0$, be (h, k).

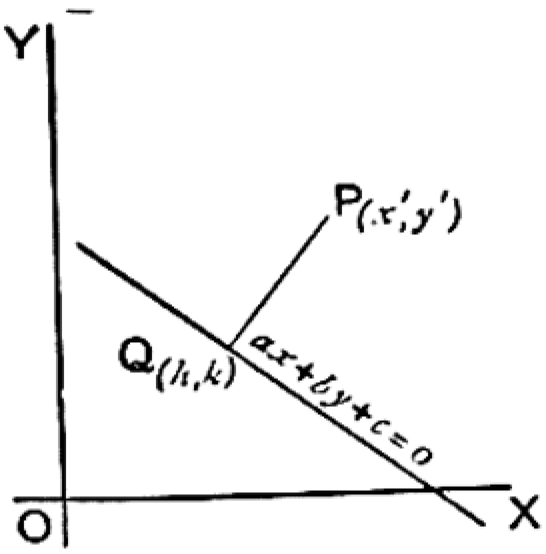


FIG. 104.

The slope of the given line is $-a/b$, and therefore the slope of PQ is b/a.

But the slope of PQ is $(y'-k)/(x'-h)$;

$$\therefore\ \frac{y'-k}{x'-h}=\frac{b}{a},$$

$$\text{i.e.}\qquad a(y'-k)-b(x'-h)=0. \qquad\text{.....................(1)}$$

Again, since (h, k) lies on the given line

$$ah+bk+c=0,$$

$$\therefore\ ax'+by'+c-(ah+bk+c)=ax'+by'+c,$$

$$\text{i.e.}\qquad a(x'-h)+b(y'-k)=ax'+by'+c. \qquad\text{..............(2)}$$

Squaring and adding (1) and (2), we have

$$a^2\{(x'-h)^2+(y'-k)^2\}+b^2\{(y'-k)^2+(x'-h)^2\}=(ax'+by'+c)^2,$$

$$\text{i.e.}\qquad (a^2+b^2)\{(x'-h)^2+(y'-k)^2\}=(ax'+by'+c)^2.$$

$$\text{But}\qquad PQ^2=(x'-h)^2+(y'-k)^2.$$

$$\text{Therefore}\qquad PQ^2=\frac{(ax'+by'+c)^2}{a^2+b^2},$$

$$\text{i.e.}\qquad PQ=\pm\frac{ax'+by'+c}{\sqrt{a^2+b^2}}. \qquad\text{........................(3)}$$

The numerical value of the length of the perpendicular is therefore $(ax'+by'+c)/\sqrt{a^2+b^2}$.

Ex. 1. *Find the equations of the bisectors of the angle between the lines* $3x+4y=12$ *and* $y=4$.

Let $P\,(h,\,k)$ be any point on a bisector (Fig. 105).

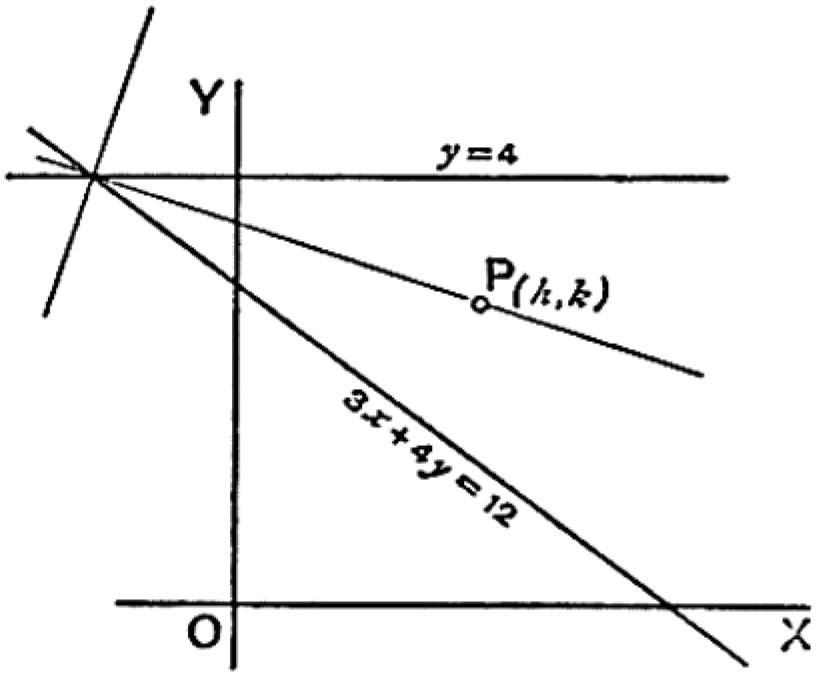


FIG. 105.

The perpendicular from P on $y=4$ is $4-k$. ....(1)

The perpendicular from P on $12-3x-4y=0$ (note c is positive) is $\pm(12-3h-4k)/5$, where the sign is determined as follows:

In the figure, P and the origin are on opposite sides of $12-3x-4y=0$. Therefore $12-3h-4k$ is negative, and consequently the perpendicular is

$$-(12-3h-4k)/5. \quad \text{......................(2)}$$

As P is on a bisector, the perpendiculars (1) and (2) are equal;

$$\therefore\ 4-k=-(12-3h-4k)/5,$$

which is the condition that $(h,\,k)$ shall lie on the line

$$3x+9y-32=0, \quad \text{............................(3)}$$

and this is therefore the equation of a bisector.

If P were taken on the other bisector, and below $y=4$, we should obtain

$$4-k=+(12-3h-4k)/5,$$

and the equation of the other bisector is consequently

$$3x-y+8=0. \quad \text{.........................(4)}$$

Inspection shews that (3) and (4) are at right angles, and Fig. 105 shews which is the bisector of the acute and which the bisector of the obtuse angle between the given lines.

Ex. 2. *Prove that the straight line whose equation is*

$$y+3x=16$$

touches the circle whose equation is $x^2+y^2-4x=6$.

We have to shew that the perpendicular from the centre of the circle to the given line is equal to the radius of the circle.

The equation of the circle can be written

$$(x-2)^2+y^2=10.$$

The centre of this circle is $(2, 0)$ and its radius is $\sqrt{10}$ (see p. 64).

The perpendicular from $(2, 0)$ to the given line is

$$(0+3\times 2-16)/\sqrt{10}, \quad i.e. \ -\sqrt{10}.$$

Hence, numerically, this perpendicular equals the radius of the circle.

Therefore the given line touches the circle.

Ex. 3. *The equations of the sides AB, BC, CA of a triangle are* $4x+3y-12=0$, $3x-4y+16=0$ *and* $4x-3y-12=0$. *Find the equations of the internal bisectors of the angles B and C.*

In Fig. 106 let (h, k) be the coordinates of *any* point I on the bisector of the angle B.

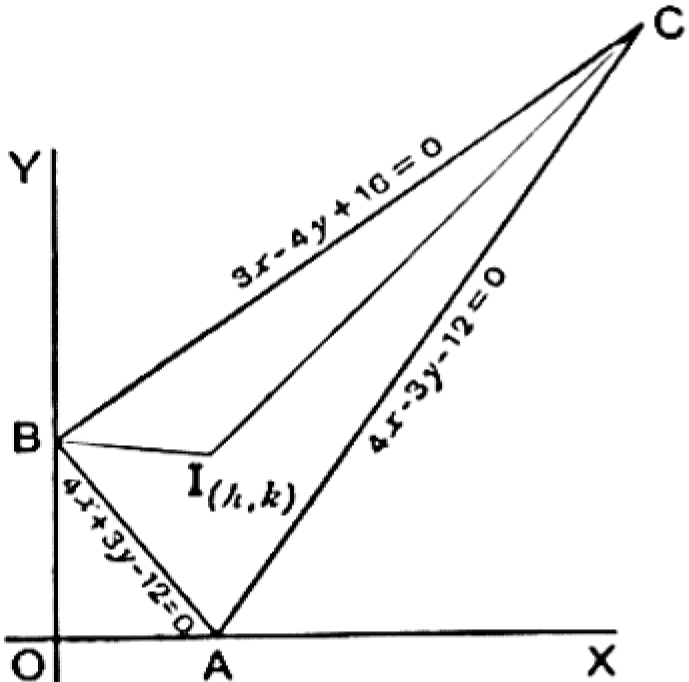


FIG. 106

The equation of AB is

$$12-4x-3y=0,$$

making the 'c' positive.

$\therefore$ the actual length of the perpendicular from I on AB is

$$-(12-4h-3k)/\sqrt{4^2+3^2}, \ldots(1)$$

since I and the origin are on opposite sides of AB.

Again, the actual length of the perpendicular from I on BC is

$$(16+3h-4k)/\sqrt{3^2+4^2}, \ldots..(2)$$

since I and the origin are on the same side of BC.

Equating (1) and (2), we have

$$-12+4h+3k=16+3h-4k,$$

$$i.e. \quad h+7k=28. \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The equation of IB is, therefore,

$$x+7y=28.$$

Similarly, equating the actual lengths of the perpendiculars on BC and CA, from any point (h', k') on the bisector of the angle C, we have

$$\frac{16+3h'-4k'}{\sqrt{3^2+4^2}}=\frac{12-4h'+3k'}{\sqrt{3^2+4^2}},$$

$$i.e. \qquad 7h'-7k'=-4. \quad \text{..........................}(4)$$

The equation of the internal bisector of the angle C is, therefore,

$$7x-7y+4=0.$$

From (3) and (4) the incentre is the point $(3, \frac{25}{7})$.

Ex. 4. *A straight line AB of constant length c slides so that A and B move along OX and OY respectively. If P is a point on AB such that $AP:PB=1:r$, find the locus of P.*

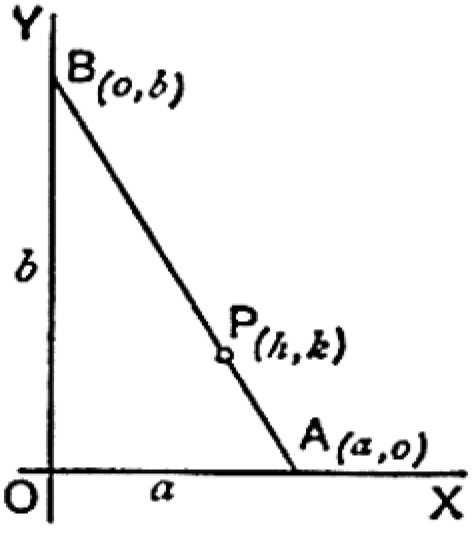


FIG. 107.

At any moment let the intercepts OA, OB be a, b respectively (Fig. 107), so that

$$a^2+b^2=c^2. \quad \text{..............................}(1)$$

If P is the point (h, k), then, since $r\,.\,AP=PB$,

$$h=\frac{ra}{r+1}, \quad k=\frac{b}{r+1} \text{ (see p. 180)};$$

$$\therefore\ a=\frac{(r+1)h}{r}, \quad b=(r+1)k.$$

Using (1),
$$(r+1)^2\frac{h^2}{r^2}+(r+1)^2k^2=c^2,$$

and this is the condition that h, k should lie on the curve whose equation is

$$\frac{x^2}{r^2}+y^2=\frac{c^2}{(r+1)^2}.$$

Ex. 5. *A and B are the fixed points* (1, 0), (0, 2) ; *P is any point on AB ; PN, PM are perpendicular to OX, OY respectively; BN and AM meet in Q. Find the locus of Q as P moves along AB.*

Fig. 108 shews one position of P and the corresponding position of Q.

If P is the point (x', y'), N is the point $(x', 0)$ and M the point $(0, y')$.

Let the coordinates of Q be h, k.

The equation of AM is $x + y/y' = 1$ and Q lies on AM ;

$$\therefore \; h + \frac{k}{y'} = 1. \; \ldots\ldots\ldots\ldots(1)$$

The equation of BN is $x/x' + y/2 = 1$ and Q lies on BN ;

$$\therefore \; \frac{h}{x'} + \frac{k}{2} = 1. \; \ldots\ldots\ldots\ldots(2)$$

The equation of AB is $x + y/2 = 1$ and P lies on AB ;

$$\therefore \; x' + \frac{y'}{2} = 1. \; \ldots\ldots\ldots\ldots(3)$$

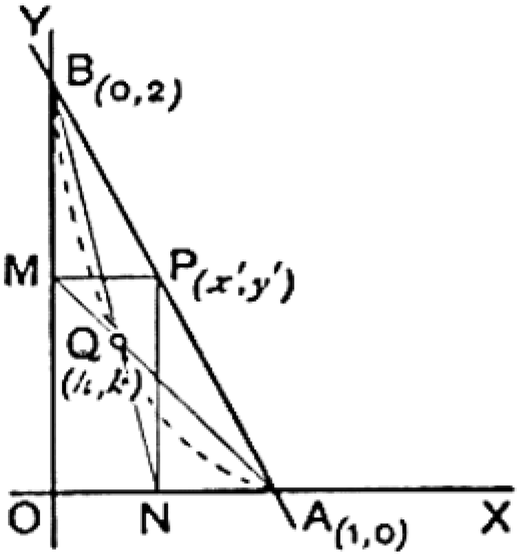


FIG. 108.

The object is to obtain a relation between h and k. Evidently the best way of doing this is to find x' and y' from (1) and (2) and substitute in (3).

From (1), $$y' = \frac{k}{1-h}.$$

From (2), $$x' = \frac{2h}{2-k}.$$

Using (3), $$\frac{2h}{2-k} + \frac{k}{2(1-h)} = 1,$$

i.e. $$4h - 4h^2 + 2k - k^2 = 4 - 2k - 4h + 2hk,$$

i.e. $$4h^2 + k^2 + 2hk - 8h - 4k + 4 = 0.$$

And this is the condition that the point Q lies on the locus whose equation is

$$4x^2 + y^2 + 2xy - 8x - 4y + 4 = 0.$$

EXERCISES XX.

1. Find the lengths of the following perpendiculars:

(a) from (1, 1) to $3x-4y+2=0$;
(b) from $(-2, -1)$ to $4x-3y=0$;
(c) from (7, 1) to $13y-5x=1$.

2. Find the perpendiculars from the origin to each of the lines in Ex. 1.

3. Prove that the points $(2, -5)$ and $(-1, 4)$ are equidistant from the line $3x+y+5=0$.

4. Find the distance between the lines $3x-4y+13=0$ and $3x-4y-3=0$.

5. Find the equation to the circle whose centre is the point (1, 2) and which touches the straight line $2x-y=4$.

6. Shew that the origin is the incentre of the triangle whose sides are $y=1$, $3x-4y=5$ and $5x+12y+13=0$.

Shew also that $(-2, 16)$ is the centre of the escribed circle touching $y=1$.

7. Shew that the product of the perpendiculars from the points $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$ on the straight line $2x\cos\theta-3y\sin\theta=6$ is independent of θ.

8. Establish a formula for the length of the perpendicular from a point (x', y') to a line $ax+by+c=0$.

Find the equations of the lines parallel to the line $3x+4y=7$, and passing at distance 7·5 from the point $(1, -2)$. Verify by checking the intercepts of the lines in a figure. [S.W.]

9. Find the equation of the line through the intersection of $3x+2y-1=0$ and $2x-5y+3=0$ perpendicular to the line $y=2x+1$.

10. Find the equation of the line through the intersection of $3x+4y-1=0$ and $5x-y+4=0$ which is parallel to the line $x+9y=0$.

11. The equations representing one pair of opposite sides of a parallelogram are $2x-3y-1=0$ and $2x-3y+5=0$.

The equations representing the other pair of opposite sides are $5x+6y=0$ and $5x+6y+2=0$.

Find the equations of the diagonals.

12. Give three forms of the equation of a straight line, explaining the meaning of the constants in each case.

Find the equation of the line which passes through the point (1, 2) and is perpendicular to the line $2x - 3y = 10$. [Durham.]

13. Find the distance of the point (1, 2) from the line

$$5x + 12y = 16.$$

Shew that the equation of the locus of a point $P(x, y)$ which moves so as to be at equal distances from the point (1, 1) and the line $x + 2y - 7 = 0$ is

$$4x^2 + y^2 - 4xy + 4x + 18y - 39 = 0.$$ [Durham.]

14. Prove that the area of a triangle having one vertex at the origin and its other vertices at the points

$$(x_1, y_1), (x_2, y_2) \quad \text{is} \quad \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

If L and M are the feet of the perpendiculars from the point (1, 5) to the lines $2x - 3y = 0$ and $3x - y = 0$, find the area of the triangle OLM, O being the origin. [Durham.]

15. The vertices of a triangle ABC are the points $(0, b)$, $(-a, 0)$, $(a, 0)$. Find the equations to its sides, and determine the locus of a point which moves inside the triangle so that the rectangle contained by the perpendiculars from P to AB, AC is equal to the square on the perpendicular to BC.

Prove that the locus is an arc of a circle, and find its centre. [S.W.]

16. Find the coordinates of the point dividing in the ratio $k : l$ the line joining the points (x_1, y_1), (x_2, y_2).

Through the point $A(-4, 6)$ a variable line AMN is drawn cutting OX, OY in M, N. A point Q is taken on MN such that $MQ = \frac{1}{4}MN$. Find the locus of Q for all positions of AMN. [S.W.]

17. Q is any point on the line $x - a = 0$. If A is the point $(a, 0)$ and QR, the bisector of the angle OQA, meets OX in R, find the locus of the foot of the perpendicular from R to OQ.

The Circle. It has already been shewn (p. 64) that the equation

$$x^2 + y^2 + 2ax + 2by + c = 0$$

represents a circle.

This equation can be written

$$(x + a)^2 + (y + b)^2 = a^2 + b^2 - c,$$

shewing that the centre is the point $(-a, -b)$ and that the square of its radius is $a^2 + b^2 - c$.

Equation of the Tangent at (x', y') to the Circle

$$x^2 + y^2 = a^2.$$

Let P be the point (x', y') (Fig. 109).

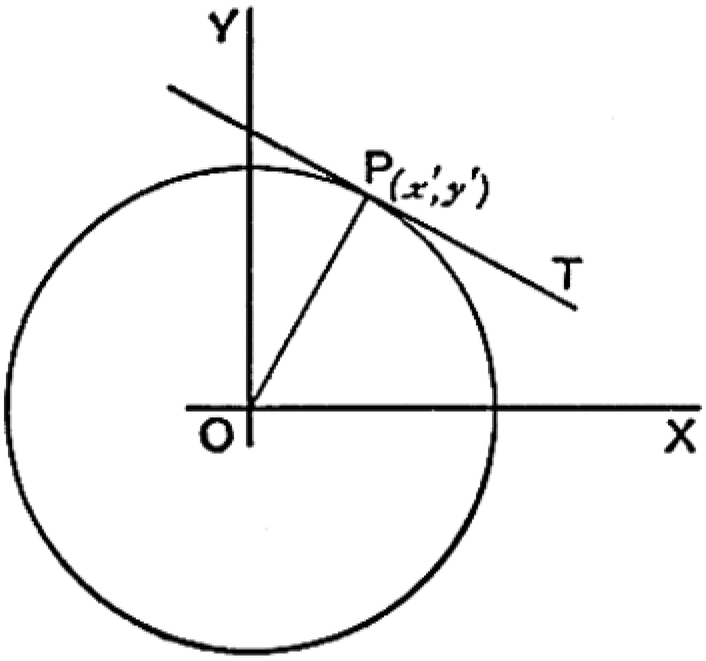


FIG. 109.

The centre, O, of the circle is the origin.

The slope of OP is y'/x'.

The tangent PT is perpendicular to the radius OP, and therefore its slope is $-x'/y'$.

The equation of PT is, therefore,

$$y - y' = -\frac{x'}{y'}(x - x'),$$

i.e. $\quad xx' + yy' = x'^2 + y'^2,$

i.e. $\quad xx' + yy' = a^2.$

Equation of the Tangent at (x', y') to the Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Let P be the point (x', y') (Fig. 110).

The centre, C, of the circle is the point $(-g, -f)$.

The slope of CP is $(y' + f)/(x' + g)$.

Therefore the slope of PT, the tangent at P, is

$$-(x' + g)/(y' + f),$$

and its equation is

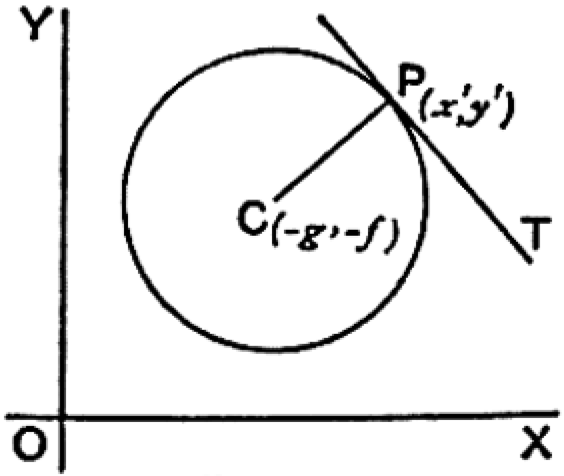


FIG. 110.

$$y - y' = -\frac{x' + g}{y' + f}(x - x'),$$

i.e. $\quad (x - x')(x' + g) + (y - y')(y' + f) = 0,$

i.e. $\quad x(x' + g) + y(y' + f) = x'^2 + y'^2 + gx' + fy'. \quad \text{.........(1)}$

But since P lies on the circle, $x'^2 + y'^2 + 2gx' + 2fy' + c = 0$ and therefore the right-hand side of (1) is $-gx' - fy' - c$;

$\therefore$ the equation of the tangent may be written

$$x(x' + g) + y(y' + f) + gx' + fy' + c = 0.$$

Length of Tangent. Let T be any point (h, k) lying outside the circle $x^2+y^2+2gx+2fy+c=0$, and let TP be the tangent to the circle (Fig. 110).

Then
$$\begin{aligned} TP^2 &= TC^2 - CP^2 \\ &= (h+g)^2+(k+f)^2-\{g^2+f^2-c\} \\ &= h^2+k^2+2gh+2fk+c. \end{aligned}$$

Consequently if $S=0$ is the equation of the circle so that S means
$$x^2+y^2+2gx+2fy+c,$$
then the square of the tangent from the point (x', y') to the circle is S', where S' is the value of S when x' and y' are substituted for x and y.

Ex. 1. *Find the equation of the circle which touches the axis of X at the point* (3, 0) *and cuts off a chord of length* 8 *on OY.*

In Fig. 111 A is the point (3, 0) and C is the centre.

$\therefore$ CA is a radius and the abscissa of C is 3.

Let the ordinate at C be k. Then k is the radius of the circle.

If Q is the foot of the perpendicular from C on OY, then
$$PQ=4$$
and
$$CP^2=PQ^2+QC^2,$$
i.e.
$$k^2=4^2+3^2=25;$$
$$\therefore\ k=\pm 5.$$

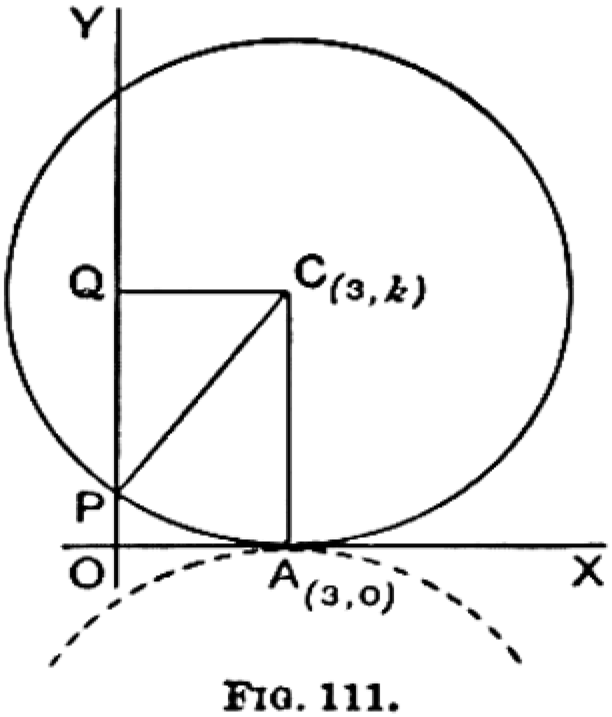


FIG. 111.

But in the figure k is positive, therefore the equation of the circle is
$$(x-3)^2+(y-5)^2=5^2,$$
i.e.
$$x^2+y^2-6x-10y+9=0.$$

Taking $k=-5$, the equation $x^2+y^2-6x+10y+9=0$ is obtained.

This represents a circle which is the reflection of the other in the axis of X.

Ex. 2. *A point P moves so that the ratio of its tangents to the circles* $x^2+y^2+4x+3=0$ *and* $x^2+y^2-6x+5=0$ *is* $1:2$. *Find the locus of P, and shew that it divides the line of centres internally and externally in the ratio of the radii.*

Let P be the point (h, k) (Fig. 112).

Then
$$PT^2=h^2+k^2+4h+3,$$
and
$$PT'^2=h^2+k^2-6h+5.$$

Since $\dfrac{PT}{PT'}=\frac{1}{2}$, $4(h^2+k^2+4h+3)=h^2+k^2-6h+5$,

i.e.
$$3h^2+3k^2+22h+7=0.$$

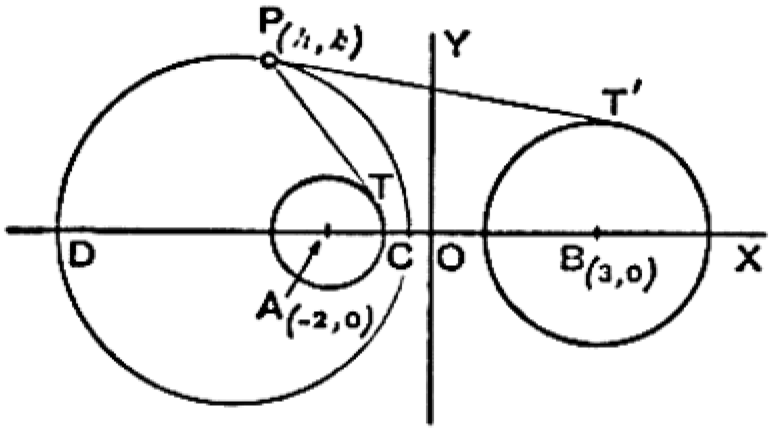


FIG. 112.

Therefore P lies on the *circle* whose equation is
$$3x^2+3y^2+22x+7=0.$$

The points C, D, where this locus cuts the axis of X, are the points at which $y=0$.

Hence the abscissae of these points satisfy the equation
$$3x^2+22x+7=0,$$
i.e.
$$(3x+1)(x+7)=0.$$

C is the point $(-\frac{1}{3}, 0)$ and D the point $(-7, 0)$.

The centres A, B of the given circles are $(-2, 0)$ and $(3, 0)$, and their radii are 1 and 2 respectively.

$$AC=-\tfrac{1}{3}+2=5/3 \quad \text{and} \quad CB=3+\tfrac{1}{3}=10/3;$$
$$\therefore \ AC/CB=1/2.$$

Again, $AD=-2+7=5$ and $BD=3+7=10$;
$$\therefore \ AD/BD=1/2.$$

Ex. 3. *Find the equations of the tangents to the circle*
$$x^2+y^2=a^2$$
which are parallel to the line $y=mx$.

Let (x', y') be a point on the circle, so that
$$x'^2+y'^2=a^2. \quad \text{..........................(1)}$$

Let the tangent $xx'+yy'=a^2$ be parallel to $y=mx$:
$$\therefore \ -\frac{x'}{y'}=m. \quad \text{..........................(2)}$$

From (1) and (2), $y'^2(m^2+1)=a^2$;

$$\therefore\ y' = \pm\frac{a}{\sqrt{1+m^2}}.$$

Using (2), $$x' = \mp\frac{am}{\sqrt{1+m^2}}.$$

Hence the equations of the required tangents are

$$-\frac{amx}{\sqrt{1+m^2}}+\frac{ay}{\sqrt{1+m^2}}=a^2$$

and $$\frac{amx}{\sqrt{1+m^2}}-\frac{ay}{\sqrt{1+m^2}}=a^2,$$

i.e. $$y-mx=a\sqrt{1+m^2}$$

and $$mx-y=a\sqrt{1+m^2}.$$

The Intersections of the Line $y=mx+c$ and the Circle $x^2+y^2=a^2$.

In Fig. 113 PQ are the intersections of the line and circle. The coordinates of P and Q satisfy *both* the given equations. Therefore the abscissae of P and Q are given by

$$x^2+(mx+c)^2=a^2 \quad \text{(see p. 68, Ch. III.)},$$

i.e. $$x^2(1+m^2)+2mcx+c^2-a^2=0. \quad \ldots(1)$$

If the line actually cuts the circle as in the figure, the roots of (1) must be real;

$$\therefore\ 4m^2c^2-4(1+m^2)(c^2-a^2)>0,$$

i.e. $$m^2c^2-(1+m^2)c^2+(1+m^2)a^2>0,$$

i.e. $$a^2(1+m^2)>c^2.$$

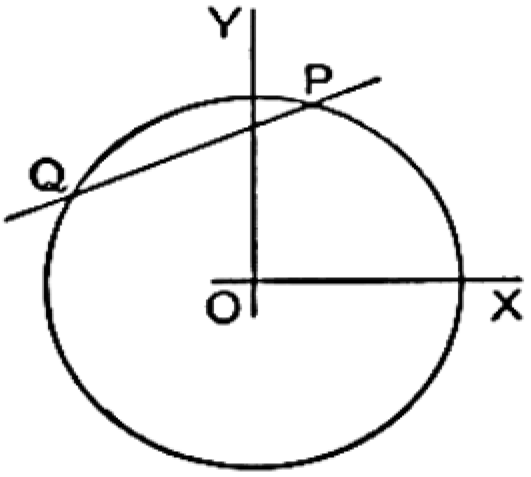


FIG. 113.

If the line touches the circle, P and Q coincide and the roots of (1) must be equal;

$$\therefore\ a^2(1+m^2)=c^2.$$

If the line does not cut the circle, the roots of (1) are imaginary;

$$\therefore\ a^2(1+m^2)<c^2.$$

Equation of Circle whose Diameter is the Line joining (x_1, y_1) and (x_2, y_2).

Let A, B (Fig. 114) be the points (x_1, y_1), (x_2, y_2).

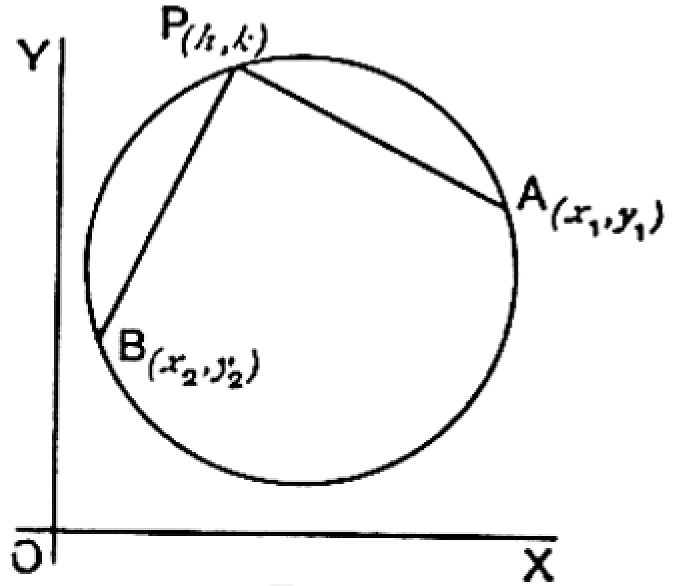


FIG. 114.

Let $P(h, k)$ be any point on the circle.

The slope of AP is $(k-y_1)/(h-x_1)$.

The slope of PB is $(k-y_2)/(h-x_2)$.

Since $A\hat{P}B$ is a right angle,

$$\frac{(k-y_1)(k-y_2)}{(h-x_1)(h-x_2)} = -1.$$

Therefore the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

Ex. 1. *Find the equation of the circle whose diameter is the chord of intersection of the straight line* $3y+x=6$ *and the curve* $4x^2+9y^2=36$.

Let (x_1, y_1) and (x_2, y_2) be the points of intersection of the straight line and the curve.

Then y_1, y_2, the ordinates of the points of intersection, are the roots of the equation obtained by eliminating x between the equations

$$3y+x=6$$

$$\text{and} \qquad 4x^2+9y^2=36,$$

i.e. y_1, y_2 are the roots of the equation

$$4(6-3y)^2+9y^2=36 \qquad \text{.....................(1)}$$

On reduction, (1) becomes $5y^2-16y+12=0$;

$$\therefore \; y_1=6/5, \quad y_2=2.$$

Since $3y_1+x_1=6$, $\qquad x_1=12/5$.

Similarly $\qquad x_2=0$.

The equation to the circle whose diameter is the line joining $(12/5, 6/5)$ and $(0, 2)$ is

$$\left(x-\frac{12}{5}\right)(x-0)+\left(y-\frac{6}{5}\right)(y-2)=0,$$

$$\textit{i.e.} \qquad x^2+y^2-\frac{12}{5}x-\frac{16y}{5}+\frac{12}{5}=0$$

$$\text{or} \qquad 5(x^2+y^2)-12x-16y+12=0.$$

N.B.—It is not necessary to find the values of x_1, x_2, y_1 and y_2 in order to find the equation of this circle, for the equation could have been obtained as follows:

The equation of the circle having (x_1, y_1) and (x_2, y_2) as the ends of a diameter can be written

$$x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0. \quad \ldots\ldots(1)$$

Since y_1, y_2 are the roots of $5y^2-16y+12=0$,

$$y_1+y_2=\frac{16}{5} \quad \text{and} \quad y_1y_2=\frac{12}{5}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly x_1, x_2 are the roots of the equation obtained by eliminating y between $3y+x=6$ and $4x^2+9y^2=36$, *i.e.* x_1, x_2 are the roots of the equation $5x^2-12x=0$, and therefore

$$x_1+x_2=\frac{12}{5} \quad \text{and} \quad x_1x_2=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The equation of the required circle is obtained by substituting in (1) from (2) and (3).

Ex. 2. *Find the equations of the tangents from the origin to the circle* $x^2+y^2-6x-2y+9=0$.

The line $lx+my=0$ passes through the origin and meets the given circle in points the abscissae of which are given by the equation

$$x^2+\left(-\frac{l}{m}x\right)^2-6x-2\left(-\frac{l}{m}x\right)+9=0,$$

$$\text{i.e.} \qquad (l^2+m^2)x^2+mx(2l-6m)+9m^2=0. \quad \ldots\ldots\ldots\ldots(1)$$

If $lx+my=0$ touches the circle, the roots of (1) must be equal;

$$\therefore \; m^2(2l-6m)^2=36m^2(l^2+m^2),$$

$$\text{i.e.} \qquad m^2l\{4l+3m\}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$l=0$ gives $y=0$ as one of the tangents required.

$l=-3m/4$ gives $3x-4y=0$ as another tangent from the origin.

$m^2=0$ gives $x=0$. A figure shews that $x=0$ is *not* a tangent from the origin. The factor m^2 appearing in (2) was introduced by the step by which (1) was obtained.

EXERCISES XXI.

1. Find the equation of the tangent

(i) at (3, 4) to $x^2+y^2=25$;

(ii) at (2, -3) to $x^2+y^2=13$;

(iii) at (5, 3) to $x^2+y^2-4x-2y=8$;

(iv) at (3, -2) to $x^2+y^2-4x=1$;

(v) at ($-\frac{1}{2}$, $\frac{1}{2}$) to $2x^2+2y^2-4x-12y+3=0$.

2. Find the equations of the tangents to the circle $x^2+y^2=4$ which make angles of 60° with the axis of x.

3. For what values of c does the line $y=2x+c$ touch the circle $x^2+y^2-3x+2y+1=0$?

4. Shew that the circle $x^2+y^2-6x-6y+9=0$ touches the axes of X and Y.

5. Shew that $x^2+y^2-4x-6y+4=0$ touches the axis of X, and find the length of the chord intercepted by the axis of Y.

6. Prove that the lengths of the tangents to the circles $x^2+y^2+2x-3=0$ and $x^2+y^2+3x-3=0$ from (0, 6) are equal.

7. Find the equation of the circle circumscribing the triangle whose vertices are (2, 0), (-3, 0) and (0, 4), and find the equations of the tangents at the vertices.

8. Use the fact that the perpendicular from the centre of a circle to any tangent equals the radius, in order to find the condition that $y=mx+k$ touches

(i) $x^2+y^2=a^2$; (ii) $x^2+y^2+2gx+2fy+c=0$.

9. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which are parallel to the line $y=mx$.

10. Find the locus of the middle points of chords of the circle $x^2+y^2+2gx+2fy+c=0$ which are parallel to the line $y=mx$.

11. Find the locus of the middle points of chords, which pass through the origin, of the circle

(i) $x^2+y^2-4x=0$; (ii) $x^2+y^2-4x-2y=20$.

12. Find the equation of the circle through the points (5, 5), (4, 8), (-7, -3). Give a figure, shew that the point (-4, -4) is also on the circle, and mark the centre. [S.W.]

13. Write down the condition that the two straight lines $lx+my+n=0$ and $l'x+m'y+n'=0$ shall be (i) parallel, (ii) perpendicular.

Prove that the equation to the circle on the line joining $(-2, 3)$ and $(4, -5)$ as diameter is

$$(x+2)(x-4)+(y-3)(y+5)=0. \qquad \text{[S.W.]}$$

14. Shew that the equation

$$x^2+y^2+2gx+2fy+c=0$$

represents a circle, and find its centre and radius.

Find also an expression for the length of the tangent from (x', y') to the circle.

P, Q are the points $(8, 12)$, $(-9, 9)$, and PT, QT' are tangents to the circle $x^2+y^2=36$. Prove $PQ^2=PT^2+QT'^2$. [S.W.]

15. Find the coordinates of the foot, X, of the perpendicular from A $(2, 3)$ to that line through the intersection, P, of the lines $2x+5y-2=0$ and $3x+2y+1=0$ which is parallel to $x+2y-1=0$.

Verify that X lies on the circle whose diameter is AP. [B.U.]

16. Shew that the square of the tangent from (x', y') to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$x'^2+y'^2+2gx'+2fy'+c.$$

The sum of the squares of the tangents from a point P to two given circles is constant. Prove that the locus of P is a circle whose centre lies midway between the centres of the given circles. [B.U.]

17. Find the coordinates of the mid-point of the chord cut off from the circle $x^2+y^2+2fy+2gx+c=0$ by the line $y=mx$.

Apply your result to the circle $x^2+y^2-2x+3y+2=0$ and the line $y+2x=0$. [S.W.]

18. Find the condition that the line $y=mx$ shall touch the circle $x^2+y^2+2gx+2fy+c=0$.

Find the equation of the circle which touches the axis of x at $(2, 0)$ and passes through the point $(3, -1)$. Obtain the equation of the other tangent through the origin. [B.U.]

19. Obtain the equation of a circle in the form

$$x^2+y^2+2gx+2fy+c=0.$$

Find the equation of the circle which touches the axis of x and passes through the points $(1, 2)$ and $(3, 2)$. [B.U.]

20. Prove that the equation of a circle can be expressed in the form $x^2+y^2+Ax+By+C=0$. Express the radius in terms of A, B and C.

A circle is described on the line joining the points $(3, 7)$ and $(9, 1)$ as diameter; shew that $x+y=4$ is a tangent to this circle, and find the point of contact. [Man.]

21. Define the tangent to a circle at a given point.

If $lx+my=0$ touches the circle $x^2+y^2+ax+by+c=0$, prove

$$l^2(b^2-4c)-2lmab+m^2(a^2-4c)=0.$$

Find the tangents from the origin to the circle

$$x^2+y^2+10x+2y+13=0.$$ [S.W.]

22. Find the equations of the tangents drawn from the origin of coordinates to the circle

$$2x^2+2y^2-12x-16y+25=0.$$ [S.W.]

23. Shew how to determine the points in which the straight line $ax+by=c$ cuts the circle $x^2+y^2=1$; and examine the cases $c^2>a^2+b^2$, $c^2=a^2+b^2$, $c^2<a^2+b^2$.

Find the equations of the two tangents which can be drawn from the origin to the circle

$$x^2+y^2+4x-6y+4=0.$$ [S.W.]

24. Find the locus of the point which moves so that the lengths of tangents from it to two given concentric circles are inversely as their radii. [B.U.]

25. Prove that the chord through the points (x_1, y_1), (x_2, y_2) on the circle $x^2+y^2=a^2$ has for equation

$$x(x_1+x_2)+y(y_1+y_2)=a^2+x_1x_2+y_1y_2;$$

and deduce the equation of the tangent at (x_1, y_1). [S.W.]

26. Find the equations of the two tangents at the points where the line $2x-y=15$ cuts the circle $x^2+y^2=65$; find also the coordinates of their intersection, and sketch the figure. [S.W.]

27. Find the points where the line $y-mx=7-6m$ cuts the circle $x^2+y^2=85$, and hence or otherwise find the equation of the tangent at the point (6, 7). [S.W.]

28. Find the equation of the circle whose centre is (1, 2) and for which the tangent from the origin is of length 2.

Find the points on the line $x-y+1=0$ the tangents from which to the circle $x^2+y^2-3x=0$ are of length 2. [Durham.]

29. Shew that the equation $x^2+y^2+2gx+2fy+c=0$ represents a circle, and state its radius and the coordinates of its centre.

Find the equation of the circle with centre at the point (1, 2) and touching the line $2x+y=9$. Shew that this line also touches the circle $4x^2+4y^2-4x-24y+17=0$. [Durham.]

30. Shew that the circle described on the line joining the points (0, 1), (p, q), as diameter cuts the axis of X in points whose abscissae are the roots of the equation $x^2-px+q=0$.

Concurrent Lines. Any straight line through the intersection of

$$ax+by+c=0 \quad \text{and} \quad a'x+b'y+c'=0 \quad \text{..........(1)}$$

can be written in the form

$$ax+by+c+\lambda(a'x+b'y+c')=0, \quad \text{..............(2)}$$

where λ may have any value.

For let (h, k) be the point of intersection of the straight lines (1).

Then

$$ah+bk+c=0$$

and

$$a'h+b'k+c'=0;$$

$$\therefore \; ah+bk+c+\lambda(a'h+b'k+c')=0.$$

And this is the condition that (h, k) lies on the straight line whose equation is (2).

Ex. 1. *Find the equation of the line joining the origin to the point of intersection of* $x-y+2=0$ *and* $2x+y+1=0$.

Any line through the point of intersection of the given lines is given by

$$(x-y+2)+\lambda(2x+y+1)=0. \quad \text{................(1)}$$

If (1) passes through the origin,

$$2+\lambda=0;$$

$$\therefore \; \lambda=-2.$$

Substituting in (1), the equation of the required line is

$$(x-y+2)-2(2x+y+1)=0, \qquad i.e. \quad x+y=0.$$

Ex. 2. *Find the equation of the line parallel to* $3x-2y=0$ *and concurrent with* $2x+y+1=0$ *and* $3x-y+5=0$.

Any line concurrent with the given lines is given by

$$(2x+y+1)+k(3x-y+5)=0. \quad \text{................(1)}$$

The slope of (1) is $\quad -\dfrac{2+3k}{1-k}.$

But since the required line is to be parallel to $3x-2y=0$,

$$-\frac{2+3k}{1-k}=\frac{3}{2};$$

$$\therefore \; -4-6k=3-3k;$$

$$\therefore \; k=-\frac{7}{3}.$$

Substituting in (1), the equation of the required line is

$$3(2x+y+1)-7(3x-y+5)=0,$$

$$i.e. \qquad -15x+10y-32=0.$$

Ex. 3. *Find the equation of the line joining the intersection of* $2y-x-3=0$ *and* $3y-x-5=0$ *to the intersection of*

$$2y-3x+13=0 \quad \textit{and} \quad y-2x+8=0.$$

Any line through the intersection of the first pair of lines is

$$(2y-x-3)+\lambda(3y-x-5)=0. \qquad \text{..............(1)}$$

Any line through the intersection of the second pair of lines is

$$(2y-3x+13)+\mu(y-2x+8)=0. \qquad \text{..............(2)}$$

If (1) and (2) represent the required join, they must be identical.

Comparing coefficients, we have

$$\frac{2+3\lambda}{2+\mu}=\frac{-1-\lambda}{-3-2\mu}=\frac{-3-5\lambda}{13+8\mu}. \qquad \text{.................(3)}$$

Now, if $\frac{a}{b}=\frac{c}{d}$, we know that

$$\frac{a}{b}=\frac{c}{d}=\frac{ma+nc}{mb+nd}.$$

Hence each ratio in (3) equals

$$\frac{2(2+3\lambda)+(-1-\lambda)}{2(2+\mu)+(-3-2\mu)}, \quad i.e.\ 3+5\lambda.$$

Also each ratio in (3) equals

$$\frac{4(-1-\lambda)+(-3-5\lambda)}{4(-3-2\mu)+(13+8\mu)}, \quad i.e. -7-9\lambda;$$

$$\therefore\ 3+5\lambda=-7-9\lambda;$$

$$\therefore\ 14\lambda=-10 \quad \text{or} \quad \lambda=-5/7.$$

Using (1), the required line is

$$7(2y-x-3)-5(3y-x-5)=0,$$

$$i.e. \qquad y+2x-4=0.$$

Equations of Circles passing through the Points of Intersection of two given Circles.

Let the given circles be

$$\left.\begin{aligned} x^2+y^2+2gx+2fy+c&=0, \\ x^2+y^2+2g'x+2f'y+c'&=0. \end{aligned}\right\} \quad \text{..............(1)}$$

Consider the equation

$$x^2+y^2+2gx+2fy+c+\lambda(x^2+y^2+2g'x+2f'y+c')=0. \quad \text{...(2)}$$

It represents a circle, since the coefficients of x^2 and y^2 are the same, viz. $(1+\lambda)$, and there is no term involving xy.

Let (p, q) be *one* of the points of intersection of the circles (1).

Then $$p^2+q^2+2gp+2fq+c=0$$

and $$p^2+q^2+2g'p+2f'q+c'=0;$$

$$\therefore\ p^2+q^2+2gp+2fq+c+\lambda(p^2+q^2+2g'p+2f'q+c')=0,$$

which is the condition that the point (p, q) lies on the circle (2).

In the same way, it can be shewn that the other point of intersection of the circles (1) lies on the circle (2).

Hence (2) represents a circle passing through the points of intersection of the two given circles.

N.B.—If in (2) λ be put equal to -1, (2) becomes

$$2(g-g')x+2(f-f')y+c-c'=0, \quad \text{..............(3)}$$

which represents a straight line, since it is of the first degree.

And (3) is therefore the equation of the common chord of the circles (1).

The coordinates of the centre of the circle, given by (2) above, are

$$-\frac{g+\lambda g'}{1+\lambda}, \quad -\frac{f+\lambda f'}{1+\lambda}.$$

These are infinitely great if $\lambda=-1$. Hence the common chord given by (3) may be regarded as one of the circles (2): viz. that which has its centre at infinity.

If S be written for $x^2+y^2+2gx+2fy+c$

and S' for $x^2+y^2+2g'x+2f'y+c'$,

then $S+\lambda S'=0$ is the equation of any circle passing through the intersections of the circles $S=0$ and $S'=0$.

And $S - S' = 0$ is the equation of their common chord, being that one of the family of circles $S + \lambda S' = 0$ which has its centre at infinity.

Ex. 1. *Find the equation of the circle passing through the point* (1, −2) *and through the intersections of the circle*

$$x^2 + y^2 - 4y + 3 = 0$$

and the line $$x - y + 1 = 0.$$

Any circle through the intersections of the given line and circle is $$x^2 + y^2 - 4y + 3 + k(x - y + 1) = 0.$$

If this passes through the point (1, −2), we have

$$1 + 4 + 8 + 3 + k(1 + 2 + 1) = 0;$$

$$\therefore \; k = -4.$$

The required circle is therefore

$$x^2 + y^2 - 4x - 1 = 0.$$

Ex. 2. *Find the equation of the circle whose diameter is the common chord of the circles*

$$x^2 + y^2 + 3x + 2y + 1 = 0 \quad \textit{and} \quad x^2 + y^2 + 3x + 4y + 2 = 0.$$

Any circle through the intersection of the given circles is

$$x^2 + y^2 + 3x + 2y + 1 + k(x^2 + y^2 + 3x + 4y + 2) = 0,$$

i.e. $$(x^2 + y^2)(1 + k) + 3(1 + k)x + 2(1 + 2k)y + 1 + 2k = 0. \quad \text{...(1)}$$

The centre of this circle is $-3/2, \; -(1 + 2k)/(1 + k)$.

Since this circle is to have the common chord of the given circles as diameter, its centre must lie on the common chord, *i.e.* on the line

$$2y + 1 = 0;$$

$$\therefore \; 2\left(-\frac{1 + 2k}{1 + k}\right) + 1 = 0$$

or $$k = -\frac{1}{3}.$$

Substituting in (1), the equation of the required circle is

$$\frac{2}{3}(x^2 + y^2) + 2x + \frac{2}{3}y + \frac{1}{3} = 0,$$

i.e. $$2(x^2 + y^2) + 6x + 2y + 1 = 0.$$

Ex. 3. *Find the equations of the circles passing through the intersections of the circles*

$$x^2+y^2-2x+1=0 \quad and \quad x^2+y^2-5x-6y+4=0$$

and touching the line $2x-y+3=0$.

Any circle through the intersections of the given circles is

$$x^2+y^2-2x+1+\lambda(x^2+y^2-5x-6y+4)=0,$$

$$i.e.\quad (1+\lambda)(x^2+y^2)-(2+5\lambda)x-6\lambda y+1+4\lambda=0, \quad\ldots\ldots\ldots(1)$$

Substituting $y=2x+3$ in (1), the quadratic giving the abscissae of the points of intersection of the straight line and the circle (1) is

$$(1+\lambda)\{x^2+(2x+3)^2\}-(2+5\lambda)x-6\lambda(2x+3)+1+4\lambda=0$$

$$\text{or}\quad 5(1+\lambda)x^2+(10-5\lambda)x+(10-5\lambda)=0. \quad\ldots\ldots\ldots\ldots(2)$$

If the straight line touches circle (1), the roots of (2) are equal,

$$i.e.\quad (10-5\lambda)^2=20(1+\lambda)(10-5\lambda),$$

$$i.e.\quad (10-5\lambda)\{-10-25\lambda\}=0,$$

$$i.e.\quad \lambda=2 \quad \text{or} \quad -2/5.$$

Taking $\lambda=2$, (1) becomes

$$3(x^2+y^2)-12x-12y+9=0$$

$$\text{or}\quad x^2+y^2-4x-4y+3=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Taking $\lambda=-2/5$, (1) becomes

$$\frac{3}{5}(x^2+y^2)+\frac{12}{5}y-\frac{3}{5}=0$$

$$\text{or}\quad x^2+y^2+4y-1=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The required circles are given by the equations (3) and (4).

EXERCISES XXII.

1. Find the equation of the common chord of the circles :

(i) $x^2+y^2-3x-y+2=0$ and $x^2+y^2-2x+y+3=0$;

(ii) $x^2+y^2+4x-y=0$ and $2x^2+2y^2+x-y+1=0$;

(iii) $3x^2+3y^2-4=0$ and $2x^2+2y^2+3x-2y-4=0$.

2. Shew that common chords of the following circles taken in pairs are concurrent:

$$x^2+y^2+4x+y=0, \qquad 2x^2+2y^2+3x+5y+9=0$$
$$\text{and} \qquad x^2+y^2+y=0.$$

Give the coordinates of this point—the *radical centre.*

3. Find the equation of the circle which passes through

(i) the point $(-2, 3)$ and the intersections of the circle $x^2+y^2-7x+12=0$ and the line $y=0$;

(ii) the point $(-1, -2)$ and the intersections of the circle $x^2+y^2+8x-6y=0$ and the line $10x-8y+1=0$;

(iii) the origin and the intersections of the circles

$$x^2+y^2+4x-y+3=0 \quad \text{and} \quad x^2+y^2+2x-y-3=0.$$

4. Find the equation of the circle which touches the line $x+2y=5$ and passes through the intersections of the circles

$$x^2+y^2-4=0 \quad \text{and} \quad x^2+y^2-2x-4y+4=0.$$

5. Find the equation of the circle through the origin and the points of intersection of $x^2+y^2-2x=4$ and $5x-2y=6$.

Write down its radius and the coordinates of its centre, and find the equation of the tangent to it at the origin. [Durham.]

6. Find the equations of the circles which pass through the ends of the common chord of the two circles

$$x^2+y^2+4x+2y-3\tfrac{1}{2}=0 \quad \text{and} \quad x^2+y^2-8x-4y-5=0,$$

and touch the line $x=7$. [B.U.]

7. If $S=0$ and $S'=0$ are the equations of any two curves, shew that $S+\lambda S'=0$ is the equation of a curve passing through the intersections of $S=0$ and $S'=0$.

8. Shew that $x-y=0$ passes through the points of intersection of $y^2=4a(x+a)$ and $x^2=4a(y+a)$. Illustrate by rough graphs.

9. Find the common chords of the circle $x^2+y^2=25$ and the ellipse $2x^2+y^2=34$, and illustrate by rough graph.

****Equation of the Second Degree.** The equation of the second degree in x and y represents in general a curve (called a **conic**). For example, consider the equation

$$x^2+4xy+5y^2-2x-5y-1=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It can be written $\quad x^2+2x(2y-1)=1+5y-5y^2,$

which becomes, on adding $(2y-1)^2$ to both sides,

$$(x+2y-1)^2=2+y-y^2;$$

$$\therefore\ x+2y-1=\pm\sqrt{(2-y)(1+y)}. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Note that, if $y > 2$ or if $y < -1$, the product $(2-y)(1+y)$ is negative; so that, for such values of y, x would be unreal. Choosing values of y ranging from -1 to $+2$, the corresponding values of x can easily be calculated from equation (2). In the table of values given, the upper value for x corresponds to the $+$, the lower to the $-$, in equation (2).

y	-1	$-0·8$	$-0·5$	$-0·2$	0	$0·2$	$0·5$	$0·8$	1	$1·2$	$1·5$	$1·8$	2
x	3	$3·35$	$3·12$	$2·73$	$2·41$	$2·07$	$1·5$	$0·87$	$0·41$	$-0·07$	$-0·88$	$-1·85$	-3
	3	$1·85$	$0·88$	$0·07$	$-0·41$	$-0·87$	$-1·5$	$-2·07$	$-2·41$	$-2·73$	$-3·12$	$-3·35$	-3

The graph is shewn in Fig. 115.

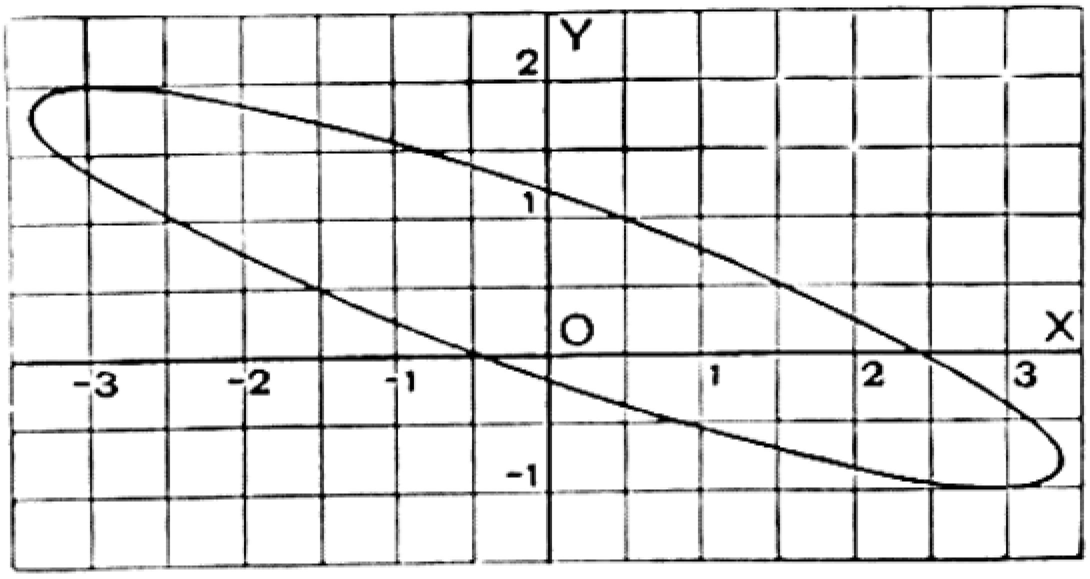


FIG. 115.

Consider now the equation

$$x^2 + 4xy + 3y^2 + 4x + 14y - 5 = 0. \qquad \text{..............(3)}$$

It can be written

$$x^2 + 2x(2y+2) = 5 - 14y - 3y^2,$$

or, adding $(2y+2)^2$ to both sides,

$$(x + 2y + 2)^2 = 9 - 6y + y^2;$$

$$\therefore\ x + 2y + 2 = \pm(3 - y). \qquad \text{....................(4)}$$

Points can be plotted as before, and they will be found to lie on one or other of two straight lines. This, however, is clear from (4), which shews that the coordinates of the points which lie on the locus whose equation is (3) must satisfy *either* $x+2y+2=3-y$ *or* $x+2y+2=-(3-y)$; *i.e.* the points must lie on one or other of the straight lines $x+3y-1=0$ and $x+y+5=0$.

Ex. *Shew that* $2x^2+xy-y^2+x+4y-3=0$ *represents a pair of straight lines.*

The terms of the second degree are $2x^2+xy-y^2$,

$$i.e. \quad (2x-y)(x+y).$$

Consider the expression $(2x-y+a)(x+y+b)$. On expanding, it becomes

$$2x^2+xy-y^2+(a+2b)x+(a-b)y+ab. \quad \text{............}(1)$$

Now let $\quad a+2b=1 \quad$ and $\quad a-b=4$,

so that the coefficients of x and y in (1) are the same as in the given equation.

$$\therefore\ a=3 \quad \text{and} \quad b=-1; \qquad \therefore\ ab=-3,$$

which agrees with the constant term in the given equation.

Hence the given equation can be written in the form

$$(2x-y+3)(x+y-1)=0,$$

and is consequently equivalent to the pair of straight lines

$$\left.\begin{aligned} 2x-y+3&=0,\\ x+y-1&=0. \end{aligned}\right\}$$

** EXERCISES XXIII.

1. Find the equations of the straight lines represented by

(*a*) $3x^2+5xy-2y^2-2x+3y-1=0$,
(*b*) $3x^2+10xy+3y^2=0$,
(*c*) $4x^2-y^2+2x+y=0$,
(*d*) $10x^2+13xy-3y^2+22x+16y+12=0$,
(*e*) $abx^2-(a^2+b^2)\,xy+aby^2=0$.

2. Trace the conics

(*a*) $x^2-2xy+y^2+2x-6y+9=0$,
(*b*) $4x^2-4xy+5y^2+8x-20y-44=0$.

3. Shew that $21x^2+4xy-y^2-3x-11y-24$ can be expressed as the product of two factors of the form $(7x-y+m)(3x+y+n)$.

Hence shew that $21x^2+4xy-y^2-3x-11y-24=0$ represents a pair of straight lines; find m and n, and plot the lines. [Man.]

** **The homogeneous equation of the second degree**

$$ax^2+2hxy+by^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

represents always two straight lines.

If $a \neq 0$, (1) can be multiplied through by a and becomes

$$a^2x^2+2ahxy=-aby^2,$$

$$i.e. \qquad (ax+hy)^2=(h^2-ab)y^2.$$

The two straight lines are, therefore, $ax+hy=\pm(\sqrt{h^2-ab})y$, and are real, coincident, or imaginary according as h^2-ab is greater than, equal to, or less than zero.

If $a=0$, (1) becomes $y(2hx+by)=0$, which represents the two real lines $y=0$ and $2hx+by=0$.

The condition that $ax^2+2hxy+by^2=0$ should represent a pair of **perpendicular** straight lines is $\mathbf{a}+\mathbf{b}=0$.

Let the two lines be $y-mx=0$ and $y+\frac{1}{m}x=0$.

The equation of the two lines together is

$$(y-mx)\left(y+\frac{1}{m}x\right)=0,$$

$$i.e. \quad y^2+\left(\frac{1}{m}-m\right)xy-x^2=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This equation is, therefore, the same as

$$ax^2+2hxy+by^2=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing coefficients of x^2 and y^2 in (1) and (2), $\frac{a}{-1}=\frac{b}{1}$, or $a+b=0$.

Ex. 1. *Find the condition that the slope of one of the lines in $ax^2+2hxy+by^2=0$ is three times that of the other.*

Let the lines be $y-mx=0$ and $y-3mx=0$.

Multiplying out, $y^2-4mxy+3m^2x^2=0$.

Comparing with $by^2+2hxy+ax^2=0$, we have

$$\frac{b}{1}=\frac{2h}{-4m}=\frac{a}{3m^2};$$

$$\therefore\ m^2=\frac{a}{3b} \quad\text{and}\quad m=-\frac{h}{2b};$$

$$\therefore\ \left(\frac{a}{3b}\right)=\left(-\frac{h}{2b}\right)^2,$$

$$i.e. \qquad 4ab=3h^2.$$

Ex. 2. *Find the condition that one of the lines*

$$ax^2+2hxy+by^2=0$$

should coincide with one of the lines

$$a'x^2+2h'xy+b'y^2=0.$$

Let the equation of the line common to each pair be

$$y-mx=0.$$

Then $y-mx$ is a factor of $ax^2+2hxy+by^2$;

$$\therefore \quad ax^2+2hmx^2+bm^2x^2=0,$$

$$\text{i.e.} \qquad a+2hm+bm^2=0. \qquad \text{(1)}$$

Similarly,

$$a'+2h'm+b'm^2=0. \qquad \text{(2)}$$

Treating (1) and (2) as simultaneous equations in m and m^2, we obtain

$$m=-\frac{ab'-a'b}{2(hb'-h'b)}. \qquad \text{(3)}$$

and

$$m^2=-\frac{ah'-a'h}{bh'-b'h}. \qquad \text{(4)}$$

From (3) and (4),

$$\left\{\frac{ab'-a'b}{2(hb'-h'b)}\right\}^2=-\frac{ah'-a'h}{bh'-b'h},$$

$$\text{i.e.} \qquad (ab'-a'b)^2=4(ah'-a'h)(b'h-bh').$$

** EXERCISES XXIV.

1. Find the condition that the lines $ax^2+2hxy+by^2=0$
   (i) are equally inclined to the axis of X;
   (ii) make complementary angles with the axis of X.

2. Find the condition that the slope of one of the lines $ax^2+2hxy+by^2=0$ is double that of the other.

3. Find the condition that the sum of the slopes of the lines $ax^2+2hxy+by^2=0$ shall be 6.

4. Prove that the equation $3x^2+5xy-2y^2=0$ represents a pair of lines through the origin parallel to the lines

$$3x^2+5xy-2y^2-2x+3y-1=0.$$

5. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two lines, what is the equation of the parallel lines through the origin?

6. Prove that the equation

$$bx^2 - 2hxy + ay^2 = 0$$

represents a pair of straight lines through the origin which are perpendicular to the lines given by

$$ax^2 + 2hxy + by^2 = 0.$$

7. What is the equation of the pair of lines through the origin perpendicular to the lines given by the equation in Ex. 5 ?

** **The equation of the bisectors of the angles between the lines**

$$ax^2 + 2hxy + by^2 = 0$$

is $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

Let the lines be $y - mx = 0$ and $y - m'x = 0$.

Then $$y^2 - (m + m')xy + mm'x^2 = 0$$

and $$by^2 + 2hxy + ax^2 = 0$$

represent the same pair of lines. Comparing coefficients,

$$\frac{1}{b} = \frac{-(m + m')}{2h} = \frac{mm'}{a},$$

i.e. $$m + m' = -\frac{2h}{b} \quad \text{and} \quad mm' = \frac{a}{b}. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

If (α, β) lies on the bisector of either the acute or obtuse angle between the lines $y - mx = 0$ and $y - m'x = 0$, the perpendiculars from (α, β) to these lines are equal,

i.e. $$\frac{\beta - m\alpha}{\sqrt{1 + m^2}} = \pm \frac{\beta - m'\alpha}{\sqrt{1 + m'^2}} \quad \text{(see p. 190)};$$

$$\therefore \ (1 + m'^2)(\beta - m\alpha)^2 = (1 + m^2)(\beta - m'\alpha)^2;$$

$$\therefore \ \beta^2(m'^2 - m^2) - 2\alpha\beta(m - m' + mm'^2 - m^2m') + \alpha^2(m^2 - m'^2) = 0.$$

Dividing through by $(m - m')$, this becomes

$$(\alpha^2 - \beta^2)(m + m') - 2\alpha\beta(1 - mm') = 0,$$

i.e., using (1), $$(\alpha^2 - \beta^2)\left(-\frac{2h}{b}\right) - 2\alpha\beta\left(1 - \frac{a}{b}\right) = 0,$$

i.e. $$-(\alpha^2 - \beta^2)h + \alpha\beta(a - b) = 0;$$

and this is the condition that (α, β) should lie on

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

**** The equation of the pair of lines joining the origin to the intersections of the circle $x^2+y^2+2gx+2fy+c=0$ and the line**

$$lx+my=n.$$

Consider the equation

$$x^2+y^2+(2gx+2fy)\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2=0. \quad\ldots\ldots(1)$$

It is homogeneous and of the second degree, and therefore represents a pair of lines through the origin.

Let $P(h, k)$ be one of the points of intersection of the given circle and line.

Then $$h^2+k^2+2gh+2fk+c=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$lh+mk=n,$$

i.e. $$\frac{lh+mk}{n}=1. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The left-hand side of (1), when h, k are substituted for x, y respectively, becomes

$$h^2+k^2+(2gh+2fk)\left(\frac{lh+mk}{n}\right)+c\left(\frac{lh+mk}{n}\right)^2,$$

i.e. $h^2+k^2+2gh+2fk+c$, using (3),

i.e. 0, using (2).

Therefore P lies on the locus whose equation is (1).

Similarly Q, the second point of intersection of the circle and the line, also lies on (1).

But (1) represents a pair of lines which pass through the origin.

Therefore OP, OQ are the lines represented by (1).

Ex. 1. *Find the equation of the lines joining the origin to the points of intersection of*

$$2x^2+2y^2+5x+3y-2=0 \quad \textit{and} \quad 2x+3y+2=0.$$

The equation of the given line can be written

$$\frac{2x+3y}{-2}=1.$$

The pair of lines required are therefore

$$2x^2+2y^2+(5x+3y)\left(\frac{2x+3y}{-2}\right)-2\left(\frac{2x+3y}{-2}\right)^2=0;$$

i.e. $4x^2+4y^2-(10x^2+21xy+9y^2)-(4x^2+12xy+9y^2)=0,$

i.e. $10x^2+33xy+14y^2=0,$

i.e. $(2x+y)(5x+14y)=0.$

Ex. 2. *Prove that the straight lines joining the origin to the points of intersection of* $\frac{x}{a}+\frac{y}{b}=2$ *and* $(x-a)^2+(y-b)^2=r^2$ *are perpendicular if* $a^2+b^2=r^2$.

The equation of the given line is

$$\frac{bx+ay}{2ab}=1.$$

The lines joining the points of intersection to the origin are

$$x^2+y^2-(2ax+2by)\left(\frac{bx+ay}{2ab}\right)+(a^2+b^2-r^2)\left(\frac{bx+ay}{2ab}\right)^2=0. \quad (1)$$

In this, the coefficient of x^2 is

$$1-\frac{2ab}{2ab}+(a^2+b^2-r^2)\frac{b^2}{4a^2b^2},$$

$$i.e. \quad \left(\frac{a^2+b^2-r^2}{4a^2}\right).$$

Similarly, the coefficient of y^2 is $\frac{a^2+b^2-r^2}{4b^2}$.

The lines given by (1) are perpendicular if the sum of these coefficients is zero, *i.e.* if

$$(a^2+b^2-r^2)\left(\frac{1}{4a^2}+\frac{1}{4b^2}\right)=0,$$

i.e. if $a^2+b^2=r^2.$

**** The Discriminant.** If

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of straight lines, then

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

The left hand side is called the **discriminant** of the equation, and is usually written in the form of a determinant (see p. 231).

If $a \neq 0$, the equation may be written

$$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac.$$

Adding $(hy + g)^2$ to each side, we have

$$(ax + hy + g)^2 = (h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac.$$

This represents two straight lines if the expression on the right is a perfect square in y, *i.e.* if

$$(gh - af)^2 = (h^2 - ab)(g^2 - ac),$$

which reduces to $a^2bc + 2afgh - a^2f^2 - abg^2 - ach^2 = 0$; or, since $a \neq 0$,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

If $a = 0$, it can be shewn that the same condition is required.

Ex. 1. *For what value of* λ *does*

$$\lambda x^2 + 13xy - 3y^2 + 22x + 16y + 12 = 0$$

represent a pair of straight lines?

The discriminant is

$$\lambda(-3)(12) + 2 . 8 . 11(\tfrac{13}{2}) - \lambda . 8^2 - (-3)11^2 - 12(\tfrac{13}{2})^2 .$$

$$\therefore \quad \lambda(-36 - 64) + 13(88 - 39) + 363 = 0,$$

$$i.e. \qquad -100\lambda + 637 + 363 = 0,$$

$$i.e. \qquad 100\lambda = 1000;$$

$$\therefore \quad \lambda = 10.$$

Ex. 2. *For what values of* λ *does*

$$xy + 2y^2 - \lambda x - 5y - 3 = 0$$

represent a pair of straight lines?

The discriminant is

$$2\left(-\frac{5}{2}\right)\left(-\frac{\lambda}{2}\right)\frac{1}{2} - 2\left(-\frac{\lambda}{2}\right)^2 - (-3)\left(\frac{1}{2}\right)^2;$$

$$\therefore \quad \frac{5\lambda}{4} - \frac{\lambda^2}{2} + \frac{3}{4} = 0,$$

$$i.e. \qquad 2\lambda^2 - 5\lambda - 3 = 0,$$

$$i.e. \qquad (\lambda - 3)(2\lambda + 1) = 0.$$

If $\lambda = 3$, the lines are given by the equations

$$y - 3 = 0 \quad \text{and} \quad x + 2y + 1 = 0.$$

If $\lambda = -\frac{1}{2}$, the lines are given by the equations

$$2y + 1 = 0 \quad \text{and} \quad x + 2y - 6 = 0.$$

** EXERCISES XXV.

1. Write down the equation of the bisectors of the angles between

(a) $x^2 + 2xy - 3y^2 = 0$, (b) $4x^2 - 7xy + 3y^2 = 0$,
(c) $6x^2 - xy - y^2 = 0$.

2. Shew that

$$ax^2 + 2hxy + by^2 = 0 \quad \text{and} \quad (a+k)x^2 + 2hxy + (b+k)y^2 = 0$$

have the same bisectors.

3. Shew that the homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents two straight lines through the origin.

Hence shew that the equation

$$x^2 + y^2 = (lx + my)^2$$

represents the two straight lines through the origin and through the intersections of the circle $x^2 + y^2 = a^2$ and the line $lx + my = a$. [S.W.]

4. The chord of intersection of $x \cos \alpha + y \sin \alpha = p$ and $x^2 + y^2 = a^2$ subtends a right angle at the origin. Shew that $a^2 = 2p^2$, and interpret this result geometrically.

5. Find the equation of the straight lines joining the origin to the points of intersection of

$$x^2 + y^2 + 2px + 2qy = 0 \quad \text{and} \quad ax + by + c = 0.$$

Find the condition that these lines are at right angles.

6. Find the equation of the lines joining the origin to the points of intersection of $x^2 + y^2 = 25$ and $3x + 4y = 25$. Shew that the lines are coincident, and interpret the result.

7. Find the condition that one of the lines given by

$$ax^2 + 2hxy + by^2 = 0$$

may be perpendicular to one of the lines given by

$$a'x^2 + 2h'xy + b'y^2 = 0.$$

8. Find the product of the perpendiculars from the point (p, q) on the lines $ax^2 + 2hxy + by^2 = 0$.

9. Find the equation of the lines joining the origin to the points of intersection of $x - y = 1$ and $x^2 + y^2 - 8x - 2y + 15 = 0$, and shew that $x - y = 1$ touches the circle.

10. Find the values of λ for which the following equations represent pairs of straight lines :

(a) $\lambda x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$,
(b) $2x^2 + \lambda xy - y^2 - 3x + 6y - 9 = 0$,
(c) $12x^2 + 36xy + \lambda y^2 + 7x + 11y + 1 = 0$,
(d) $6x^2 + \lambda xy - 10y^2 + x + 31y - 15 = 0$.

11. Shew that the pair of lines given by $(a-b)(x^2-y^2)+4hxy=0$, and the pair of lines given by $(x^2-y^2)/(a-b)=xy/h$, are such that each pair bisects the angles between the other pair.

12. Shew that the point (h, k) is equidistant from the pair of lines given by $a^2(x^2+y^2)-(hy-kx)^2=0$.

13. Any straight line through the fixed point C cuts the straight lines given by $ax^2+by^2=0$ in the points P, Q. If C is the point $(0, c)$, shew that the equation of the locus of the middle point of PQ is $ax^2+by(y-c)=0$.

14. Find the condition that

$$xy+2gx+2fy+c\equiv(x+l)(y+m),$$

and hence shew that $xy+2gx+2fy+c=0$ represents two straight lines if $4fg=c$.

15. Shew that the curves of which the equations are

$$x^2+4xy+5y^2-2x-5y-1=0 \quad \text{and} \quad y^2+x+y-3=0$$

intersect in points which lie on a pair of parallel straight lines.

CHAPTER VIII.

THE REMAINDER THEOREM. FACTORS. PARTIAL FRACTIONS. DETERMINANTS.

Ex. *Find the remainder when* ax^3+bx^2+cx+d *is divided by* $x-h$.

$$\begin{array}{lll}
x-h\,] & ax^3+bx^2+cx+d\,[& ax^2+(ah+b)x+(ah^2+bh+c) \\
 & \underline{ax^3-ahx^2} & \\
 & (ah+b)x^2+cx+d & \\
 & \underline{(ah+b)x^2-(ah^2+bh)x} & \\
 & \qquad (ah^2+bh+c)x+d & \\
 & \qquad \underline{(ah^2+bh+c)x-(ah^3+bh^2+ch)} & \\
 & \qquad\qquad\qquad \underline{\underline{ah^3+bh^2+ch+d.}} &
\end{array}$$

The remainder is, therefore, ah^3+bh^2+ch+d.

It is clear that

$$ax^3+bx^2+cx+d \equiv (x-h)\{ax^2+(ah+b)x+(ah^2+bh+c)\}$$
$$+ah^3+bh^2+ch+d, \;\ldots\ldots\ldots\ldots\ldots(1)$$

where the symbol $\equiv$ is read " is identically equal to " ; that is to say, the expression on the left is exactly the same as that on the right, no matter what the value of x may be.

The **identity** (1) is a particular case of the general statement that

$$\text{Dividend} \equiv \text{Divisor} \times \text{Quotient} + \text{Remainder}. \quad \ldots\ldots(2)$$

If the divisor is a linear expression (p. 43) in x, the remainder does not contain x. Consideration of the above example should make this clear.

Any expression involving x is called a **function of** x and may be written $f(x)$, $F(x)$, $\phi(x)$, etc.

In particular, if $f(x)$ stands for $3x^3 - x + 7$,

then $f(2)$ means $3 \,.\, 2^3 - 2 + 7$, *i.e.* 29;

$f(-x)$ means $3(-x)^3 - (-x) + 7$,

i.e. $-3x^3 + x + 7$;

$f(0)$ means $3(0)^3 - 0 + 7$, *i.e.* 7;

and $f(y)$ means $3y^3 - y + 7$.

Identity (1) can therefore be written

$$f(x) \equiv (x - h)\,\phi(x) + f(h),$$

where $f(x)$ means the Dividend and $\phi(x)$ the Quotient.

The Remainder Theorem.

Let $f(x)$ denote the expression $ax^n + bx^{n-1} + cx^{n-2} + \dots + k$.

If $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

Let $\phi(x)$ be the quotient and R the remainder.

Since $(x - a)$ is linear, R does not contain x, and

$$f(x) \equiv (x - a) \times \phi(x) + R, \text{ see (2) above.}$$

Since this is an identity, it is therefore true for all values of x. If we substitute a for x, the coefficient of $\phi(x)$ vanishes, and we have

$$f(a) \equiv 0 \times \phi(a) + R.$$

That is $$R = f(a). \quad \dots\dots\dots\dots (3)$$

Ex. 1. *Find the remainder when $x^6 - 5x + 10$ is divided by $x + 2$.*

By (3) above, the remainder is $(-2)^6 - 5(-2) + 10$,

i.e. $64 + 10 + 10$ or 84.

Ex. 2. *Find the remainder when*

$$(a + 2b)x^3 - (2a - b)x^2 + 3ax - (2a + 3b)$$

is divided by $x - 1$.

The remainder is

$$(a + 2b) \,.\, 1^3 - (2a - b) \,.\, 1^2 + 3a - (2a + 3b)$$

or $$a + 2b - 2a + b + 3a - 2a - 3b,$$

which is 0.

Evidently the given expression is exactly divisible by $\boldsymbol{x - 1}$; that is, $x - 1$ is a factor of

$$(a + 2b)x^3 - (2a - b)x^2 + 3ax - (2a + 3b).$$

Factor Theorem. If $(x-a)$ is a factor of $f(x)$, then $f(a)=0$. Conversely, if $f(a)=0$, then $(x-a)$ is a factor of $f(x)$.

Ex. 1. *Find the factors of* $2x^3+3x^2-1$.

Let $f(x)$ stand for $2x^3+3x^2-1$.

By trial, $f(-1)=0$; therefore $(x+1)$ is a factor of

$$2x^3+3x^2-1;$$

$$\therefore\ 2x^3+3x^2-1=(x+1)\,.\,\phi(x),$$

where $\phi(x)$ by inspection, or actual division, is $2x^2+x-1$, and

$$2x^2+x-1=(2x-1)(x+1);$$

$$\begin{aligned}\therefore\ 2x^3+3x^2-1&=(x+1)(2x-1)(x+1)\\&=(x+1)^2(2x-1).\end{aligned}$$

Ex. 2. *Find the factors of*

$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

Let E stand for the expression

$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

If $a=b$,

$$\begin{aligned}E&=b^2(b-c)+b^2(c-b)+0\\&=0;\end{aligned}$$

$$\therefore\ a-b \text{ is a factor of } E.$$

The expression E is symmetrical in a, b and c, *i.e.* remains unchanged when a is changed into b, b into c and c into a.

Hence, since $a-b$ is a factor, so also are $b-c$ and $c-a$.

Again, E is of the third degree in a, b and c, and we know already three linear factors $(a-b)$, $(b-c)$ and $(c-a)$. If there is another factor, it must be a mere number.

Thus $E\equiv N\,.\,(a-b)(b-c)(c-a)$, where N is a number.

Put $a=0$, $b=1$, $c=2$, and we have

$$E=0+1^2(2-0)+2^2(0-1)=2-4=-2$$

and $N(a-b)(b-c)(c-a)=N(0-1)(1-2)(2-0)=2N$;

$$\therefore\ N=-1$$

and $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$.

Ex. 3. *Find the factors of*

$$a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4).$$

Let E stand for the expression.

If $a=b$, $\quad E=b(b^4-c^4)+b(c^4-b^4)+0=0$;

$$\therefore\ a-b \text{ is a factor of } E.$$

Similarly, $b-c$ and $c-a$ are factors.

Since E is symmetrical in a, b and c and of the fifth degree, and since $(a-b)(b-c)(c-a)$ is symmetrical and of the third degree, we may write $E\equiv(a-b)(b-c)(c-a)$ multiplied by a symmetrical expression of the second degree in a, b and c;

$$\therefore\ E\equiv(a-b)(b-c)(c-a)\{l(a^2+b^2+c^2)+m(ab+bc+ca)\},$$

where l and m are mere numbers.

Put $a=0$, $b=1$, $c=2$, and we have

$$1(2^4)+2(-1^4)=(-1)(-1)(2)\{l(5)+m(2)\};$$

$$\therefore\ 14=10l+4m. \quad\text{..............................(1)}$$

Again, put $a=0$, $b=1$, $c=-1$, and we have

$$1(-1)^4+(-1)(-1^4)=(-1)(2)(-1)\{2l-m\};$$

$$\therefore\ 2=4l-2m. \quad\text{..........................(2)}$$

From (1) and (2), $\quad l=1, \quad m=1,$

and $\quad a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$

$$=(a-b)(b-c)(c-a)\{a^2+b^2+c^2+ab+bc+ca\}.$$

EXERCISES XXVI.

1. Find the remainder when
   - (i) x^2+4x+6 is divided by $x-1$;
   - (ii) x^2+4x+5 is divided by $x+2$;
   - (iii) x^3+5x-3 is divided by $3x-1$;
   - (iv) $(a-b)x^3+(b-c)x^2+(c-a)$ is divided by $x-1$.

2. If x^3+px-3 is divisible by $x+1$, what is the value of p?

3. Find the factors of:
   - (i) $x^3+6x^2+11x+6$;
   - (ii) $x^4-5x^3+5x^2+5x-6$;
   - (iii) x^3-2x+1;
   - (iv) $x^3-15x-4$;
   - (v) $x^5+x^4-5x^3-5x^2+4x+4$.

4. Factorise:

(i) $ab(a-b)+bc(b-c)+ca(c-a)$;

(ii) $(b-c)^3+(c-a)^3+(a-b)^3$;

(iii) $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$;

(iv) $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$;

(v) $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz$.

5. Shew that $(y+z)$, $(z+x)$, $(x+y)$ are factors of

$$(x+y+z)^5-x^5-y^5-z^5.$$

6. Prove that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$.

7. Find the roots of the equations:

(i) $x^3-3x-2=0$;

(ii) $x^3-4x^2-7x+10=0$;

(iii) $x^3-1\cdot7x^2-2\cdot4x+3\cdot6=0$;

(iv) $(b-c)x^3+(c-a)x+(a-b)=0$.

8. What values of x will make the expressions x^3-x^2-4x+4 and x^3-2x^2-x+2 both zero. [I.C.E.]

9. If $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$, shew that the remainder on dividing $f(x)$ by $(x-a)$ is $f(a)$.

Shew that $x+b+c$ is a factor of $x^3(b-c)-x(b^3-c^3)+(b^3c-bc^3)$, and find the remaining factors. [L.U.]

10. Prove that, if $f(x)$ is a rational algebraic function of x, then $x-a$ is a factor of $f(x)$, if $f(a)=0$.

Find the linear factors of $4x^3+12x^2y-13xy^2-21y^3$. [L.U.]

11. By the remainder theorem, or otherwise, shew that $x-1$ is a factor of $(b-c)x^3-(a+2b-3c)x^2-(b+2c-3a)x-2(a-b)$, and find the other two factors.

Factorise $a(b^2-c^2)-b(c^2-a^2)-c(a^2-b^2)$. [L.U.]

12. n being a positive integer and

$$x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\ldots+a_{n-1}x+a_n$$

vanishing when x is put $=a$, prove that $x-a$ is a factor of the expression.

Use this theorem to determine by trial the integral root of $16x^3+31\cdot6x^2-6\cdot8x-12=0$, and find the values of the remaining roots. [L.U.]

13. Find by trial the integral root of $3x^3-6\cdot5x^2-10\cdot5x+9=0$, and solve the equation.

14. Solve the equation $x^3+3\cdot2x^2+3\cdot37x+1\cdot17=0$.

15. Solve the equation

$$4x^4-2\cdot8x^3-3\cdot4x^2+0\cdot7x+0\cdot6=0.$$

16. Solve the equation $8x^4+6x^3-x^2+6x-9=0$.

17. State and prove the remainder theorem.

Write down the remainders when $ax^4+bx^3+cx^2+dx+e$ is divided by (i) $x-1$, (ii) $x+1$.

$7ab73$ is a number of five digits, of which a, b are unknown. Determine a, b so that the number shall be divisible by 99. [S.W.]

18. State and prove the Remainder Theorem. Apply this theorem to find the numerical values of A, B, C, D if

$$\begin{aligned}90=A(x+1)(x+3)(x+6)&+Bx(x+3)(x+6)\\&+Cx(x+1)(x+6)+Dx(x+1)(x+3)\end{aligned}$$

for all values of x. [S.W.]

19. Prove that x^3 can be expressed in the form

$$A+Bx+Cx(x-1)+Dx(x-1)(x-2);$$

and determine the coefficients.

20. Shew that

$$x^4\equiv x+7x(x-1)+6x(x-1)(x-2)+x(x-1)(x-2)(x-3).$$

21. Factorise

$$\begin{aligned}(y+2a+2b)\{(x-3a)^2-4a^2-2by-2b^2\}&\\+(x-3a)\{y^2+2(2a+b)y+8ab+2b^2\}&\end{aligned}$$

by forming and solving a quadratic in $(x-3a)$ or otherwise. [L.U.]

Partial Fractions. When a single fraction is replaced by two or more simpler fractions, it is said to be expressed in partial fractions. In higher mathematics, the process is of more importance than the converse one of combining two or more fractions into a single fraction.

Ex. 1. *Express in partial fractions* $\dfrac{3x+4}{(x+1)(x+2)}$.

Assume that

$$\frac{3x+4}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}.$$

Now

$$\begin{aligned}\frac{A}{x+1}+\frac{B}{x+2}&=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}\\&=\frac{(A+B)x+(2A+B)}{(x+1)(x+2)}.\end{aligned}$$

Now, since this fraction is the same as the original fraction, and the denominators are the same,

$$\therefore\ (A+B)x+(2A+B)\equiv 3x+4,$$

where $\equiv$ means that the expressions are identically equal.

That is to say, the coefficient of x on the right is the same as that on the left, and the constant terms are the same,

$$\text{i.e.}\qquad \left.\begin{aligned} A+B&=3,\\ 2A+B&=4.\end{aligned}\right\}$$

Subtracting the first from the second, we have

$$A=1,$$

and, using the first, $\qquad B=2.$

Hence
$$\frac{3x+4}{(x+1)(x+2)}=\frac{1}{x+1}+\frac{2}{x+2}.$$

Note.—This method only applies when the highest power of x in the denominator is greater than the highest power of x in the numerator.

Ex. 2. *Express in partial fractions* $\dfrac{2x^2+5x-11}{x^2+2x-3}$.

Since the highest power of x in the denominator is not greater than the highest power of x in the numerator, the expression is written

$$\frac{2(x^2+2x-3)+x-5}{(x^2+2x-3)},$$

i.e. the numerator is written so that part is exactly divisible by the denominator.

$\therefore$ the fraction is $\quad 2+\dfrac{x-5}{(x+3)(x-1)}$,

in which the denominator has been factorised.

Assume
$$\frac{x-5}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}$$

whence, as above, $\quad \left.\begin{aligned} B&=-1,\\ A&=2.\end{aligned}\right\}$

Alternatively, $\quad A(x-1)+B(x+3)\equiv x-5.$

Putting $\quad x=1,\quad 4B=-4,\quad$ or $\quad B=-1.$

Putting $\quad x=-3,\quad -4A=-8,\quad$ or $\quad A=2$;

$$\therefore\quad \frac{x-5}{(x+3)(x-1)}=\frac{2}{x+3}+\frac{-1}{x-1},$$

and
$$\frac{2x^2+5x-11}{x^2+2x-3}=2+\frac{2}{x+3}-\frac{1}{x-1}.$$

EXERCISES XXVII.

1. Factorise x^2+x, and express $\dfrac{4x-1}{x^2+x}$ in partial fractions.

2. Express $\dfrac{x^3-x^2}{x+1}$ in partial fractions.

3. Factorise $x^2-0{\cdot}4x-0{\cdot}05$, and express $\dfrac{x}{x^2-0{\cdot}4x-0{\cdot}05}$ in partial fractions.

4. Express in partial fractions:

(i) $\dfrac{1}{x^2-1}$; (ii) $\dfrac{2x^2+3}{2x+1}$; (iii) $\dfrac{x^2+1}{x^2-1}$;

(iv) $\dfrac{6{\cdot}2x+4{\cdot}3}{(x-1)(x+2)}$; (v) $\dfrac{1}{x^2+5{\cdot}2x+6{\cdot}4}$; (vi) $\dfrac{2x+3}{x^2-x-12}$;

(vii) $\dfrac{2x-3}{x^2+x-12}$; (viii) $\dfrac{a^2+1}{a^2+2a-8}$;

(ix) $\dfrac{x+4{\cdot}5}{x^2-2x-15}$; (x) $\dfrac{3x-1{\cdot}74}{x(x-1{\cdot}5)}$.

5. Express
$$\frac{3x+5}{x^2+x-12}$$
as the sum of two simpler fractions. [B.E.]

6. Express $\dfrac{1}{x^2-7x+12}$ as the sum of two simpler fractions. [B.E.]

7. Express $\dfrac{x-13}{x^2-2x-15}$ in partial fractions. [B.E.]

8 Find the partial fractions equal to $\dfrac{3x^2-7}{x^3+2x^2-8x}$. [L.U.]

9. Factorise $x^3+3x^2-22x-24$, and find the partial fractions equivalent to $\dfrac{1-11x}{x^3+3x^2-22x-24}$. [L.U.]

10. If $x^2+2x+3=A+Bx+Cx(x+1)$ for all values of x, find A, B, C.

If
$$\frac{x^2-3x}{(x+1)(x+2)(x+3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3},$$
find A, B, C. [L.U.]

11. Express the fraction $\dfrac{x-6}{3-2x-3x^2+2x^3}$ as the sum of three partial fractions. [L.U.]

**** Determinants.** The expression

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a **determinant** (of the *second* order, since it contains *two* rows and *two* columns) and, written out, means $a_1b_2 - a_2b_1$.

The expression

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{..........................(1)}$$

is a determinant of the third order and means

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

or
$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

The expression

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is a determinant of the fourth order; and so on.

In what follows determinants of the third order are alone considered, but the methods and principles are of general application.

The letters a_1, b_1, c_1, a_2, b_2, ..., etc., which form a determinant, are called **elements**. If a vertical and horizontal line are drawn through any element, the determinant composed of the remaining letters is called the **minor** of that element. Thus, in (1), the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, etc.

The **cofactor** of an element is either *plus* or *minus* its minor. The sign is found as follows:—Start from a_1 (the *leading* element) and pass along the top row and down the proper column until the element under consideration is reached; if the number of steps from letter to letter in this process is even, the sign is positive; if odd, the sign is negative. Thus, in the expression (1) above, in passing from a_1 to b_1 there is one step, and the cofactor of b_1 is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$; in passing from a_1

to c_3 there are four steps, and the cofactor of c_3 is $+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. It is convenient to represent the cofactors of elements by corresponding capitals; *e.g.* the cofactors of a_1, b_1, b_2, c_3 are A_1, B_1, B_2, C_3.

To expand a determinant, take the elements of *any* row or column, multiply each in turn by its corresponding cofactor, and add together.

Thus the determinant (1) is the same as any one of the six expressions

$$a_1A_1+b_1B_1+c_1C_1,\quad a_2A_2+b_2B_2+c_2C_2,\quad a_3A_3+b_3B_3+c_3C_3,$$
$$a_1A_1+a_2A_2+a_3A_3,\quad b_1B_1+b_2B_2+b_3B_3,\quad c_1C_1+c_2C_2+c_3C_3.$$

All these expressions are identical. The first is written out above; the second, $a_2A_2+b_2B_2+c_2C_2$, means

$$-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix},$$

i.e. $$-a_2(b_1c_3-b_3c_1)+b_2(a_1c_3-a_3c_1)-c_2(a_1b_3-a_3b_1),$$

which is the same as the first expansion, the arrangement of terms being different. Each of the remaining four expressions should be written out, in order to verify that they are all identical.

** EXERCISES XXVIII.

1. Evaluate

$$\begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix},\quad \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix},\quad \begin{vmatrix} 7 & 0 \\ 8 & -1 \end{vmatrix}\quad \text{and}\quad \begin{vmatrix} 6 & 10 \\ 3 & 5 \end{vmatrix}.$$

2. Find the cofactors of the elements indicated by a dash and evaluate each determinant.

$$\begin{vmatrix} 1 & 1' & 2 \\ -1 & 2 & 4 \\ 6 & 2 & 1 \end{vmatrix},\quad \begin{vmatrix} 4 & 3 & 7 \\ 6 & 2 & 0' \\ -1 & 5 & 6 \end{vmatrix},\quad \begin{vmatrix} 5 & 1 & 2 \\ 4 & 3 & 8 \\ 9' & -7 & 2 \end{vmatrix},\quad \begin{vmatrix} 8 & 1 & 4 \\ 2 & 4 & 5 \\ 7 & 11' & 0 \end{vmatrix}.$$

3. Evaluate

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},\quad \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}\quad \text{and}\quad \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

** Elementary Properties of Determinants.

I. *A determinant is unaltered in value if the columns are written as rows.*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The left-hand side is

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

and the right-hand side is also, (see p. 232),

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},$$

which are identical.

II. *If any two rows or any two columns are identical, the value of the determinant is zero.*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_3\begin{vmatrix} b_1 & c_1 \\ b_1 & c_1 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_1 & c_1 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = 0.$$

III. *If all the elements in a row or column are multiples of the same number, that number is a factor of the determinant.*

For example,

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Ex. *Shew that*

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} \text{ is equal to } (a-b)(b-c)(c-a)(a+b+c).$$

Let Δ be the value of the determinant.

If b be put equal to a, two rows are exactly alike;

$$\therefore\ \Delta = 0 \text{ when } b = a;$$

$$\therefore\ (a-b) \text{ is a factor of } \Delta.$$

Similarly $(b-c)$ and $(c-a)$ are factors.

Again, Δ is of the fourth degree in a, b and c, and is clearly symmetrical in a, b and c;

$\therefore$, as on p. 225, $\Delta \equiv k(a-b)(b-c)(c-a)(a+b+c)$, ...(1)

where k is a mere number.

By equating coefficients of bc^3 on both sides of (1),

$$k=1;$$

$$\therefore\ \Delta=(a-b)(b-c)(c-a)(a+b+c).$$

** EXERCISES XXIX.

1. Shew by expanding that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

are identical.

2. Shew by expanding that

$$\begin{vmatrix} ka_1 & lb_1 & pc_1 \\ ka_2 & lb_2 & pc_2 \\ ka_3 & lb_3 & pc_3 \end{vmatrix} = klp \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

3. Shew that
$$\begin{vmatrix} 6 & 5 & 3 \\ 8 & 13 & 4 \\ 10 & 71 & 5 \end{vmatrix} = 0.$$

4. Shew that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

5. Shew that

$$\begin{vmatrix} y^2+z^2 & xy & xz \\ xy & z^2+x^2 & yz \\ zx & zy & x^2+y^2 \end{vmatrix} = 4x^2y^2z^2.$$

** IV. *If P, Q, R are the cofactors of the elements p, q, r of any row or column of a determinant and λ, μ, ν are any numbers whatever, then $\lambda P+\mu Q+\nu R$ equals the determinant formed by writing λ, μ, ν for p, q, r in the original determinant.*

For example, A_2, B_2, C_2 being the cofactors of a_2, b_2, c_2 in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then

$$\lambda A_2+\mu B_2+\nu C_2=\begin{vmatrix} a_1 & b_1 & c_1 \\ \lambda & \mu & \nu \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Corollary. *If the elements of any column (or row) are multiplied by the cofactors of the corresponding elements of another column (or row), the sum of the products is zero.*

For example, $a_1A_3+b_1B_3+c_1C_3=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}=0$ by II.

Ex. 1. *Solve the equations*

$$a_1x+b_1y+c_1=0, \quad \text{......................} (1)$$

$$a_2x+b_2y+c_2=0. \quad \text{......................} (2)$$

Multiplying (1) by b_2 and (2) by b_1 and subtracting,

$$x(a_1b_2-a_2b_1)=b_1c_2-b_2c_1,$$

i.e.

$$x\div\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 1\div\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Similarly

$$y\div\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = 1\div\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Consequently the solution of any pair of simultaneous equations such as (1) and (2) can be written down immediately in the form

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Ex. 2. *Solve the equations*

$$a_1x+b_1y+c_1z=d_1, \quad \text{.......................(1)}$$
$$a_2x+b_2y+c_2z=d_2, \quad \text{.......................(2)}$$
$$a_3x+b_3y+c_3z=d_3. \quad \text{.......................(3)}$$

Let $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and let A_1, B_1, C_1, etc., be the cofactors of a_1, b_1, c_1, etc., in Δ.

Multiply (1), (2) and (3) by A_1, A_2 and A_3 respectively and add.

We have

$$x(a_1A_1+a_2A_2+a_3A_3)+y(b_1A_1+b_2A_2+b_3A_3)$$
$$+z(c_1A_1+c_2A_2+c_3A_3)=d_1A_1+d_2A_2+d_3A_3.$$

The coefficient of x is Δ.

The right-hand side is $\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, by IV. above.

The coefficients of y and z are each zero, by the Corollary to IV.;

$$\therefore\ x\Delta=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly, multiplying (1), (2) and (3) by B_1, B_2 and B_3 respectively and adding, we have

$$y\Delta=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}.$$

Similarly
$$z\Delta=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Ex. 3. *Shew how to eliminate x, y, z from the equations*

$$a_1x+b_1y+c_1z=0, \quad \text{.......................(1)}$$
$$a_2x+b_2y+c_2z=0, \quad \text{.......................(2)}$$
$$a_3x+b_3y+c_3z=0. \quad \text{.......................(3)}$$

With the notation of the last example, multiplying (1), (2) and (3) by A_1, A_2, A_3 and adding, we have

$$x\Delta + y\,.\,0 + z\,.\,0 = 0;$$

$$\therefore\ \Delta = 0.$$

The same result might have been arrived at by obtaining

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1},\text{ as in Ex. 1,}$$

from (1) and (2) and substituting in (3).

V. *If each element of any line is the sum of two numbers, the determinant can be expressed as the sum of two determinants.*

For example,

$$\begin{vmatrix} a_1+\alpha & b_1+\beta & c_1+\gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

VI. *A determinant is unaltered if to the elements of any row are added the same multiples of the corresponding elements of any other row or rows. The same is true of columns.*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\kappa a_2+\lambda a_3 & b_1+\kappa b_2+\lambda b_3 & c_1+\kappa c_2+\lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The last determinant, by V., equals

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \kappa a_2 & \kappa b_2 & \kappa c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda a_3 & \lambda b_3 & \lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and in this, each of the last two determinants is zero, by III. and II.

Ex. 1. *Solve*

$$4x + 3y + 2z = 57, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$5x + 2y - z = 3, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$x + y - 3z = -44. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Let

$$\Delta \equiv \begin{vmatrix} 4 & 3 & 2 \\ 5 & 2 & -1 \\ 1 & 1 & -3 \end{vmatrix}.$$

Multiplying (1), (2) and (3) by the cofactors of the elements of the first column and adding, we have

$$\begin{aligned} x\Delta &= \begin{vmatrix} 57 & 3 & 2 \\ 3 & 2 & -1 \\ -44 & 1 & -3 \end{vmatrix} \\ &= 57(-6+1) - 3(-9-44) + 2(3+88) \\ &= -285 + 159 + 182 \\ &= 56, \end{aligned}$$

and $$\begin{aligned} \Delta &= 4(-6+1) - 3(-15+1) + 2(5-2) \\ &= -20 + 42 + 6 \\ &= 28; \end{aligned}$$

$$\therefore\ x = 2.$$

Similarly $$\begin{aligned} y\Delta &= \begin{vmatrix} 4 & 57 & 2 \\ 5 & 3 & -1 \\ 1 & -44 & -3 \end{vmatrix} \\ &= 4(-9-44) - 57(-15+1) + 2(-220-3) \\ &= -212 + 798 - 446 \\ &= 140; \end{aligned}$$

$\therefore\ y = 5$, and, by substituting for x and y in (2),

$$z = 17.$$

Ex. 2. *Shew that* $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0.$

The given determinant, by VI., is equal to

$$\begin{vmatrix} 1 & x & x+y+z \\ 1 & y & y+z+x \\ 1 & z & z+x+y \end{vmatrix}$$

by adding the elements of second and third columns to obtain a new third column.

$\therefore$, by III., the given determinant is equal to

$$(x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix},$$

which is zero, by II.

Ex. 3. *Shew that* $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ca & c^2-ab \end{vmatrix}=0.$

Subtracting the first column from each of the others, the determinant equals

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2-bc & b^2-a^2-c(a-b) & c^2-a^2-b(a-c) \end{vmatrix},$$

i.e. $(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2-bc & b+a+c & c+a+b \end{vmatrix},$

i.e. $(b-a)(c-a)\{(c+a+b)-(b+a+c)\}$;

i.e. $0.$

N.B.—If a determinant can be so transformed that two elements in a row or column are made zero, this should always be done.

Area of Triangle. If the coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3), the area of the triangle is

$$\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

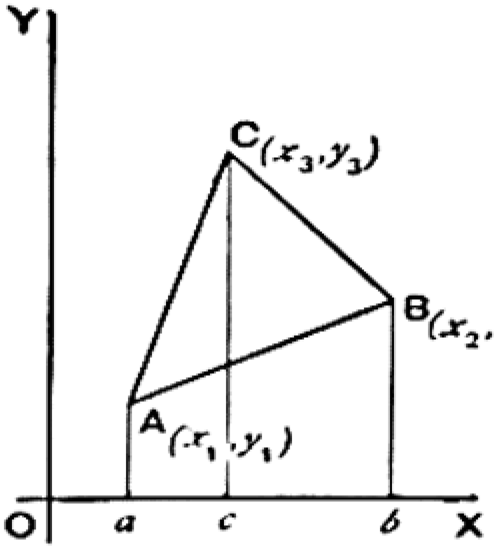


FIG. 116.

For let A, B, C be the points (x_1, y_1), (x_2, y_2), (x_3, y_3) (Fig. 116).

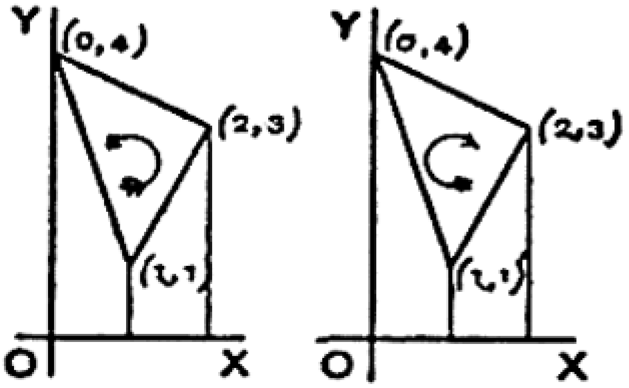
Then, using the figure, the triangle ABC equals

trapezium $ACca$ + trapezium $CBbc$ − trapezium $ABba$.

∴ area of triangle

$$\begin{aligned} &= \tfrac{1}{2}(y_1+y_3)(x_3-x_1)+\tfrac{1}{2}(y_3+y_2)(x_2-x_3)-\tfrac{1}{2}(y_1+y_2)(x_2-x_1) \\ &= \tfrac{1}{2}\{y_1x_3-y_1x_1+y_3x_3-y_3x_1\}+\tfrac{1}{2}\{y_3x_2-y_3x_3+y_2x_2-y_2x_3\} \\ &\quad -\tfrac{1}{2}\{y_1x_2-y_1x_1+y_2x_2-y_2x_1\} \\ &= \tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)] \\ &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \end{aligned}$$

Suppose the boundary of the triangle is traversed by moving from (x_1, y_1) through (x_2, y_2) and (x_3, y_3), and thus back to (x_1, y_1), so that the area of the triangle is always on the left hand, then the determinant formula gives a *positive* result. If the triangle is such that the area is always on the right hand, the determinant formula gives a *negative* result.



(a) (b)

FIG. 117.

Thus in Fig. 117 (a),

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 4 & 1 \end{vmatrix} = \frac{5}{2},$$

which is a positive result for the area of the triangle shewn.

While in Fig. 117 (b),

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -\frac{5}{2},$$

which is a negative result for the area of the same triangle.

**EXERCISES XXX.

1. Evaluate $\begin{vmatrix} a & b & c \\ -a & b & c \\ -a & -b & c \end{vmatrix}$.

2 Evaluate $\begin{vmatrix} x+2y & x+3y & x+4y \\ x+4y & x+5y & x+6y \\ x+6y & x+7y & x+8y \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} -1 & 1 & 1 \\ 5 & 13 & 6 \\ 8 & -10 & 14 \end{vmatrix}$.

4. Evaluate
$$\begin{vmatrix} x+y & x & y \\ x & x+z & z \\ y & z & y+z \end{vmatrix}.$$

5. Shew that
$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$
$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

6. Prove the identity
$$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = \begin{vmatrix} yz & x & x^2 \\ zx & y & y^2 \\ xy & z & z^2 \end{vmatrix}.$$

7. Find the area of each of the triangles whose vertices are :
(*a*) (0, 0), (3, 1), (−5, 6) ; (*b*) (1, 5), (2, 7), (4, 8) ;
(*c*) (1, 1), (−2, −3), (−1, 7) ; (*d*) (8, 9), (13, 27), (41, 63).
Draw rough sketches, and explain the sign of each result.

8. Solve the equations
(*a*) $2x+3y+z=6$, $2x+9y+3z=14$, $4x+y+2z=7$;
(*b*) $x-y+6z=12$, $3x+7y-4z=-6$, $5x+3y+z=16$;
(*c*) $7p-4q+2r=4$, $2p+3q-7r=-5$, $5p-2q-r=-1$.

9. From the equations
$$x+y+z=1, \quad ax+by+cz=d, \quad a^2x+b^2y+c^2z=d^2$$
express x as a quotient of two determinants ; and reduce your answer to the form
$$x(a-b)(a-c)=(d-b)(d-c).$$
[S.W.]

10. Solve the equations
$$x+y+z=0, \quad ax+by+cz=1, \quad bcx+cay+abz=0.$$

11. Solve the equations
$$x+y+z=a+b+c,$$
$$bx+cy-az=ab+bc-ca,$$
$$cx+ay+bz=bc+ca+ab.$$

12. Prove that the area of the triangle, the coordinates of whose angular points are (x_1, y_1), (x_2, y_2), (x_3, y_3), is half the value of the determinant
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Prove that the area of the triangle formed by the straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and the axis of x, is (without regard to sign)
$$\frac{(c_1a_2-c_2a_1)^2}{2a_1a_2(a_1b_2-a_2b_1)}.$$
[S.W.]

13. The straight line $2x+y=5$ meets the axis of x at A, and the perpendicular from $P(6, 3)$ to it at Q. If PA be produced to R so that $PR=3AR$, find the coordinates of Q and R and the area of the triangle PQR. [B.U.]

14. Obtain the equation of the straight line passing through the point (a, b), and perpendicular to the straight line $lx+my+n=0$.

The straight line PQ, joining the points $(1, 2)$ and $(4, 1)$, is cut at right angles by a line RS passing through the point $(3, 5)$. Find the area of the triangle included by the lines PQ, RS and the axis of x. [Man.]

15. Prove that the area of the triangle included between the points $(0, 0)$, (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2-x_2y_1)$; and explain the meaning of the sign of your result.

Find the area of the quadrilateral with one corner at the origin and the others at the points $(5, -1)$, $(4, 3)$ and $(7, 9)$. [S.W.]

16. Shew that $\frac{1}{2}\{(x_1-x_3)(y_2-y_4)+(x_2-x_4)(y_3-y_1)\}$ represents the area of the quadrilateral formed by joining the points $(1, 2, 3, 4)$ in that order.

Explain how the sign is to be interpreted; and give a geometrical interpretation when the result is zero. Apply this theorem, giving a figure, to the points $(3, -5)$, $(7, 14)$, $(7, 1)$, $(-1, 2)$. [L.U.]

17. Solve the equations

$$px+qy+rz=p, \quad qx+ry+pz=q, \quad rx+py+qz=r.$$

18. Solve the equations

$$ax+by+cz=l, \quad a^2x+b^2y+c^2z=l^2, \quad a^3x+b^3y+c^3z=l^3.$$

19. Express as a determinant the condition that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

20. Express as a determinant the condition that the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ are concurrent.

CHAPTER IX.

THE PROGRESSIONS. ANNUITIES. LIMITS.

Arithmetic Progression. In Fig. 118 let the intercept of the straight line A_1A_n on the vertical axis be a and let its slope be d.

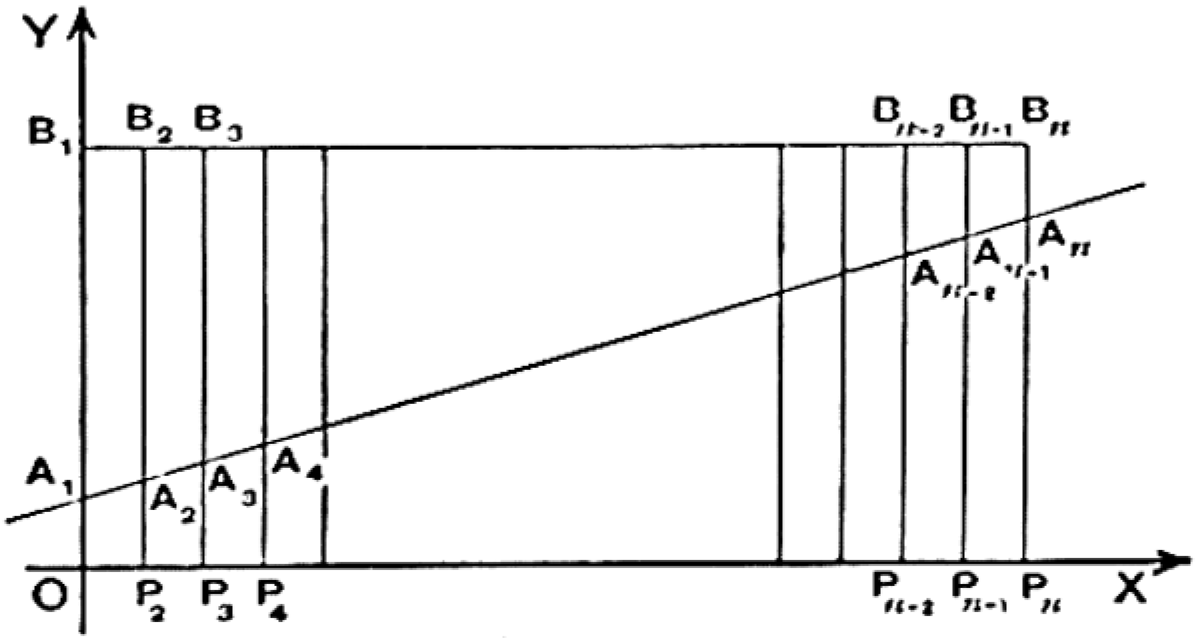


FIG. 118.

The equation of A_1A_n is

$$y = a + dx. \qquad \text{.............................(1)}$$

Along OX mark off distances OP_2, P_2P_3, P_3P_4, ... $P_{n-1}P_n$, each equal to unity.

Erect ordinates P_2A_2, P_3A_3, P_4A_4, ... P_nA_n.

From (1) it is seen that the length of each ordinate exceeds that of the preceding ordinate by the constant length d.

Thus $A_2P_2 = a + d$, $A_3P_3 = a + 2d$, $A_4P_4 = a + 3d$ ... and

$$A_nP_n = a + \overline{n-1}\,d = l \text{ (say)}.$$

If P_nA_n be produced to B_n so that $A_nB_n = OA_1$ and if B_nB_1 be drawn parallel to OX to meet OY in B_1, it is clear from the figure that $A_nB_n = OA_1$, $A_{n-1}B_{n-1} = A_2P_2$, $A_{n-2}B_{n-2} = A_3P_3$, etc., ... $A_2B_2 = A_{n-1}P_{n-1}$ and $A_1B_1 = A_nP_n$.

That is, the sum of any two ordinates equidistant from beginning and end of the series is constant and equal to $a+l$.

Hence the total length of the ordinates OA_1, P_2A_2, ... P_nA_n is half the sum of n ordinates each equal in length to $a+l$, or, symbolically,

$$a+(a+d)+(a+2d)+(a+3d)+\dots+(a+\overline{n-1}\,d)=\frac{n}{2}(a+l).$$

The lengths of the ordinates OA_1, P_2A_2, P_3A_3, ... P_nA_n form a series of numbers in **arithmetic progression**, or, briefly, an A.P.

a is the first term, d is the *common difference.*

$$l\text{, the last or } n^{\text{th}} \text{ term, } =a+(n-1)d\,; \quad \dots\dots\dots\dots(2)$$

and the sum to n terms, S, is given by $S=\frac{1}{2}n(a+l)$, or, substituting for l from (2), $S=\frac{1}{2}n\{2a+\overline{n-1}\,d\}$.

The same relations may be obtained algebraically as follows.

As before, let a be the first term and d the common difference;

$$\begin{aligned}
&\text{the second term} &&=a+d\,;\\
&\text{the third term} &&=a+2d\,;\\
&\text{the fourth term} &&=a+3d\,;\\
&\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\\
&\text{the } r^{\text{th}} \text{ term} &&=a+(r-1)\,d\,;\\
&\dots\dots\dots\dots\dots\dots\\
&\text{the } n^{\text{th}} \text{ term} &&=a+(n-1)d.
\end{aligned}$$

If there are n terms and the n^{th} term be denoted by $\boldsymbol{l}$, we have

$$l=a+(n-1)\,d\,;$$

$$\text{the last but one}=l-d \text{ or } a+(n-2)d\,;$$

$$\text{the last but two}=l-2d \text{ or } a+(n-3)d\,;$$

and so on.

If S denote the sum to n terms, we have

$$S=a+(a+d)+(a+2d)+\dots+(l-2d)+(l-d)+\boldsymbol{l}$$

or, reversing,

$$S=l+(l-d)+(l-2d)+\dots+(a+2d)+(a+d)+\boldsymbol{a}.$$

Adding, $2S = (a+l) + (a+l) + (a+l) + \dots$ to n terms ;

$$\therefore \quad 2S = n(a+l),$$

or $$S = \frac{n}{2}(a+l) ;$$

or, using (2), $$S = \frac{n}{2}(2a + \overline{n-1}\, d).$$

Ex. 1. *Insert m arithmetic means between p and q. That is, form the arithmetic progression of $m+2$ terms which has p as its first and q as its last term.*

Let d be the common difference.

Then $q = p + (m+1)d$, since it is the $(m+2)^{\text{th}}$ term ;

$$\therefore \ d = \frac{q-p}{m+1},$$

and the required means are

$$p + \frac{q-p}{m+1}, \quad p + \frac{2(q-p)}{m+1}, \dots \ p + m\frac{q-p}{m+1}.$$

A graphic method of finding the m arithmetic means between two numbers should be obvious from Fig. 118.

Note.—If $m = 1$, the term between p and q is called *the arithmetic mean.* Its value is $(p+q)/2$.

Ex. 2. *If the p^{th}, q^{th} and r^{th} terms of an arithmetic progression are x, y, z respectively, prove that*

$$(y-z)p + (z-x)q + (x-y)r = 0.$$

Let a be the first term and d the common difference.

Then
$$x = a + (p-1)d,$$
$$y = a + (q-1)d,$$
$$z = a + (r-1)d ;$$

$$\therefore \ x - y = (p-q)d$$

and $$y - z = (q-r)d$$

and $$z - x = (r-p)d ;$$

$$\therefore \ p(y-z) + q(z-x) + r(x-y)$$
$$= d\{p(q-r) + q(r-p) + r(p-q)\} = 0.$$

Ex. 3. *The sum of the first n terms of a series is $3n^2+n$. Find the n^{th} term, and shew that the series is an arithmetic progression.*

Let S_n stand for the sum of the first n terms;

$$\therefore \quad S_n = 3n^2 + n,$$

$$S_{n-1} = \text{the sum of the first } (n-1) \text{ terms}$$

$$= 3(n-1)^2 + (n-1);$$

$$\therefore \quad S_{n-1} = 3n^2 - 5n + 2;$$

$$\therefore \quad S_n - S_{n-1} = 6n - 2.$$

But $$S_n = S_{n-1} + \text{the } n^{\text{th}} \text{ term};$$

$$\therefore \text{ the } n^{\text{th}} \text{ term is } 6n-2;$$

$$\therefore \text{ the } (n-1)^{\text{th}} \text{ term is } 6(n-1)-2$$

and the n^{th} term $-$ the $(n-1)^{\text{th}}$ term is 6.

That is to say, any term is 6 more than the preceding term.

Therefore the series is an A.P., of which the first term is 4 and common difference 6.

Geometric Progression. A series of numbers of the form $a, ar, ar^2, ar^3, \ldots$ where any term is obtained by multiplying the preceding term by the same number, r, is called a geometric progression, or, briefly, a G.P.

r is called the *common ratio.*

The second term is ar.

The third term is ar^2.

The fourth term is ar^3.

....

The n^{th} term is $ar^{n-1} = l$, say. ....(1)

Let S be the sum of n terms.

Then $$S = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}.$$

Multiplying by r,

$$Sr = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting, $$S(1-r) = a - ar^n;$$

$$\therefore \quad S = \frac{a(1-r^n)}{1-r}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Formula (2) can be written in the form $S = a(r^n - 1)/(r-1)$. It is usual to employ the first form when $r<1$, the alternative form when $r>1$.

Ex. 1. *Find the lowest value of n for which the sum of the series* $1+2+2^2+2^3+\ldots+2^{n-1}$ *exceeds* 10,000.

Let S_n stand for the sum of this series.

Using (2) above, $S_n = \dfrac{2^n-1}{2-1} = 2^n - 1.$

We must have $2^n - 1 > 10{,}000,$

i.e. $2^n > 10{,}001;$

$\therefore\ n \log 2 > \log 10{,}001.$

Now $\log 10{,}001$ is slightly greater than 4

and $\log 2$ is slightly greater than $0{\cdot}3$;

$\therefore\ n > p,$

where p is approximately equal to $4/0{\cdot}3$ or $13\frac{1}{3}$.

If $n=13$, $S_n = 2^{13}-1 = (2^5)^2 \times 2^3 - 1$

$= 1024 \times 8 - 1$

$= 8191.$

If $n=14$, $S_n = 2^{14} - 1 = 16383.$

Therefore the lowest value of n which makes $S_n > 10{,}000$ is 14.

Ex. 2. *Insert m geometric means between p and q.*

Let p be the first term and r the common ratio.
Then q, being the $(m+2)^{\text{th}}$ term, is pr^{m+1};

$$\therefore\ r = \left(\frac{q}{p}\right)^{1/(m+1)}.$$

The required means are

$$p\left(\frac{q}{p}\right)^{1/(m+1)},\ p\left(\frac{q}{p}\right)^{2/(m+1)} \ldots p\left(\frac{q}{p}\right)^{m/(m+1)}.$$

If $m=1$, there is only one mean between p and q, namely

$$p\left(\frac{q}{p}\right)^{\frac{1}{2}} \text{ or } \sqrt{pq},$$

and this is called the geometric mean of p and q.

Ex. 3. *If x, y, z are the p^{th}, q^{th}, and r^{th} terms respectively of a geometric progression, prove that*

$$x^{q-r} \,.\, y^{r-p} \,.\, z^{p-q} = 1.$$

Let a be the first term and s the common ratio.

Then $\quad x = as^{p-1}$ or $\log x = \log a + (p-1)\log s$.

Similarly, $\quad \log y = \log a + (q-1)\log s$,

and $\quad \log z = \log a + (r-1)\log s$;

$$\begin{aligned}\therefore\ (q-r)\log x + (r-p)\log y + (p-q)\log z \\ &= (q-r+r-p+p-q)\log a \\ &\quad + \{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\}\log s \\ &= 0,\end{aligned}$$

i.e. $\quad \log x^{q-r} + \log y^{r-p} + \log z^{p-q} = 0 = \log 1$,

or $\quad x^{q-r} .\, y^{r-p} .\, z^{p-q} = 1$.

Ex. 4. *Find the approximate cost of an annuity of* £250 *for* 15 *years if interest is allowed at* 3 *per cent.*

£250 has to be paid at the end of 1, 2, 3, ... 15 years.

Let £P be the present value of £250 due n years hence at 3 per cent.

That is, £P amounts to £250 in n years at 3 per cent.;

$$\therefore\ P\{1{\cdot}03\}^n = £250 \text{ (see p. 103)};$$

$$\therefore\ P = \frac{250}{1{\cdot}03^n}.$$

The sum required to produce £250 at end of 1 year is £$\frac{250}{1{\cdot}03}$.

,, ,, ,, £250 ,, 2 years is £$\frac{250}{1{\cdot}03^2}$.

,, ,, ,, £250 ,, 3 years is £$\frac{250}{1{\cdot}03^3}$.

……

,, ,, ,, £250 ,, 15 years is £$\frac{250}{1{\cdot}03^{15}}$.

If £C is the approximate cost of the annuity, we have

$$C = 250\left\{\frac{1}{1{\cdot}03} + \frac{1}{1{\cdot}03^2} + \frac{1}{1{\cdot}03^3} + \dots + \frac{1}{1{\cdot}03^{15}}\right\}.$$

The series in the bracket is a geometric progression, of which the first term is $1/1{\cdot}03$ and the common ratio $1/1{\cdot}03$. There are fifteen terms.

$$\therefore\ C = 250 \times \frac{1}{1{\cdot}03} \times \frac{1-\left(\frac{1}{1{\cdot}03}\right)^{15}}{1-\frac{1}{1{\cdot}03}}$$

$$= \frac{250}{0{\cdot}03} \times \left\{1-\left(\frac{1}{1{\cdot}03}\right)^{15}\right\}.$$

Using logarithms, $C \doteqdot 2980.$

The cost of the annuity is therefore about £2980.

Ex. 5. *Find the present value of a deferred annuity of £P which commences at the end of n years (the first payment being made after $n+1$ years) and continues for x years, the rate of interest being r per cent.*

£P due $n+1$ years hence has a present value of

$$£P \Big/ \left(1+\frac{r}{100}\right)^{n+1}.$$

If £C is the cost of the annuity, we have, writing R for $1+r/100$,

$$C = P\{R^{-(n+1)} + R^{-(n+2)} + R^{-(n+3)} \ldots \text{to } x \text{ terms}\}$$

$$= PR^{-(n+1)}\left\{\frac{1-R^{-x}}{1-R^{-1}}\right\}$$

$$= \frac{PR^{-(n+1)}}{1-R^{-1}} - \frac{PR^{-(x+n+1)}}{1-R^{-1}}$$

$$= \frac{PR^{-n}}{R-1} - \frac{PR^{-(x+n)}}{R-1}.$$

Ex. 6. *A manufactured article is sold to an agent for £P, and when he sells it again the price is increased by $1/n^{th}$. If, before reaching the consumer, who pays £$2P$ for it, it passes through the hands of 10 agents, each increasing the price by $1/n^{th}$ on selling, find n.*

Compute the sum of the purchase moneys. [L.U.]

The first agent pays £P.

The second agent pays £$P\left(1+\frac{1}{n}\right)$.

The third agent pays $\pounds P\left(1+\frac{1}{n}\right)^2$.

The tenth agent pays $\pounds P\left(1+\frac{1}{n}\right)^9$.

The consumer pays $\pounds P\left(1+\frac{1}{n}\right)^{10}$;

$$\therefore\ P\left(1+\frac{1}{n}\right)^{10}=2P;$$

$$\therefore\ \left(1+\frac{1}{n}\right)^{10}=2;$$

$$\therefore\ \log\left(1+\frac{1}{n}\right)=0{\cdot}03010;$$

$$\therefore\ 1+\frac{1}{n}=1{\cdot}072;$$

$$\therefore\ \frac{1}{n}=0{\cdot}072;$$

$$\therefore\ n \doteqdot 13{\cdot}9.$$

The sum of the purchase moneys in pounds is

$$P+P\left(1+\frac{1}{n}\right)+P\left(1+\frac{1}{n}\right)^2+\ldots+P\left(1+\frac{1}{n}\right)^{10}.$$

This is a geometric series of 11 terms. The sum is

$$P\left\{\frac{1-\left(1+\frac{1}{n}\right)^{11}}{1-\left(1+\frac{1}{n}\right)}\right\},$$

$$\text{i.e.}\qquad nP\left\{\left(1+\frac{1}{n}\right)^{11}-1\right\}.$$

Using $(1+1/n)^{10}=2$, this becomes

$$nP\{2\left(1+\frac{1}{n}\right)-1\},$$

$$\text{i.e.}\qquad P(n+2),$$

or $\quad 16P$, taking $n=14$.

The sum of the purchase moneys is approximately **£16P.**

Continued Proportion. The numbers a, b, c, d, etc., are said to be in continued proportion if $a/b=b/c=c/d=\dots$.

Clearly a, b, c, d, ... form a geometric progression.

Draw any line A_1X (Fig. 119). At A_1 erect a perpendicular $A_1B_1=a$.

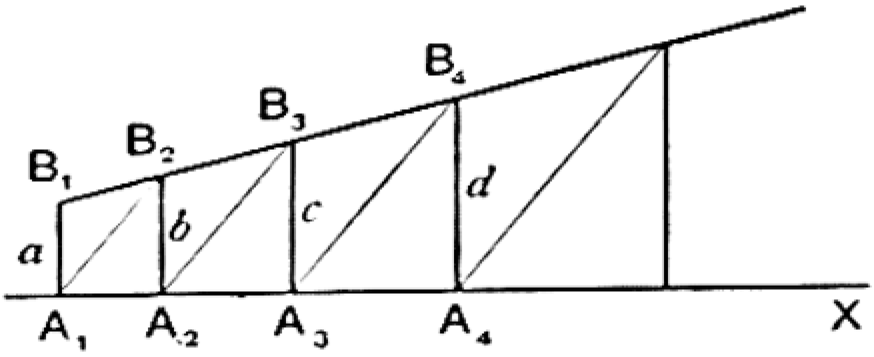


FIG. 119.

At any convenient point A_2, erect a perpendicular $A_2B_2=b$.

Join B_1B_2 and produce.

Join A_1B_2, and through A_2 draw A_2B_3 parallel to A_1B_2 to meet B_1B_2 in B_3.

From B_3, drop a perpendicular B_3A_3 to A_1X.

Similarly, draw A_3B_4 parallel to A_2B_3 and drop a perpendicular B_4A_4 to A_1X.

Then A_3B_3 is c; A_4B_4 is d, and so on, where a, b, c, d.... is a geometric progression.

By construction, the triangles $A_1B_1B_2$ and $A_2B_2B_3$ are similar;

$$\therefore \frac{a}{b}=\frac{A_1B_2}{A_2B_3}. \quad \dots\dots\dots\dots(1)$$

Again, the triangles $A_1B_2A_2$ and $A_2B_3A_3$ are similar;

$$\therefore \frac{A_1B_2}{A_2B_3}=\frac{b}{c}. \quad \dots\dots\dots\dots(2)$$

From (1) and (2), $$\frac{a}{b}=\frac{b}{c}.$$

Similarly, $$\frac{b}{c}=\frac{c}{d},$$

and so on.

Limits. Consider any line AB (Fig. 120).

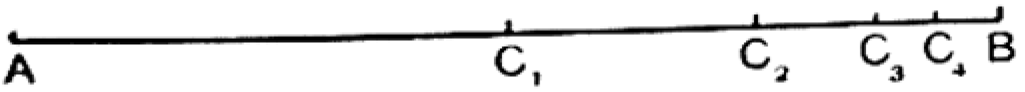


FIG. 120.

Let C_1 be the mid-point of AB; C_2 the mid-point of C_1B; C_3 of C_2B; C_4 of C_3B; and so on. As this process of bisection is continued, it is clear that a series of points C_1, C_2, C_3, ... C_n, etc., is obtained, each of which is nearer B than the preceding. But, however far the process is continued, the point of bisection finally obtained will never actually coincide with B, although it can be as close to B as desired.

The points $C_1, C_2, C_3, \dots C_n$ are said to form a series which has the point B as its limit. It should be noted that this limit is never actually attained; but it is always possible to make as close an approximation to it as desired.

It is convenient to express the foregoing illustration in algebraic form. Let AB be x cm. long.

Then
$$AC_1 = x/2 \text{ cm.},$$
$$C_1C_2 = \tfrac{1}{2}(x/2) = x/2^2,$$
$$C_2C_3 = \tfrac{1}{2}(x/2^2) = x/2^3, \text{ etc.};$$
$$\therefore\ AC_n = x/2 + x/2^2 + x/2^3 + \dots + x/2^n$$
$$= \frac{x/2[1 - 1/2^n]}{[1 - 1/2]} = x[1 - 1/2^n].$$

But clearly AB is the limit to which AC_n tends as n becomes indefinitely great, a result which agrees with that obtained, since $1/2^n$ tends to zero as n tends to ∞.

Two important cases, which should now be clear, are

(i) **if $x > 1$, $1/x^n$ tends to zero as n tends to ∞,**

(ii) **if $x < 1$, x^n tends to zero as n tends to ∞.**

Infinite Series. It has been shewn that the sum of the series $a, ar, ar^2, \dots$ to n terms is given by $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

The **sum to infinity** means the limit to which S_n tends as n is increased indefinitely. Clearly, if $r > 1$, r^n becomes indefinitely great as n is increased indefinitely, and the sum to infinity cannot be found. Again, if $r = 1$,

$$S_n = a + a + a + \dots \text{ to } n \text{ terms}$$
$$= na,$$

and this increases indefinitely with n.

But if $r < 1$, then r^n tends to zero as n is increased, and therefore S_n tends to $\dfrac{a}{1 - r}$.

Thus, it is said that the sum to infinity of the series $a, ar, ar^2, \dots$ when $r < 1$ is $\dfrac{a}{1 - r}$.

This may be written

$$\lim_{n\to\infty}\{a+ar+ar^2+\ldots+ar^n\}=\frac{a}{1-r}\text{ if } r<1.$$

There are many infinite series the sums of which are finite in this sense.

Ex. 1. *Shew graphically that the infinite series*

$$1+0{\cdot}9+(0{\cdot}9)^2+(0{\cdot}9)^3+(0{\cdot}9)^4+\ldots$$

has 10 *for its " sum to infinity."*

Let S_n denote the sum of n terms of the given series.

Then $$S_n=\frac{1-(0{\cdot}9)^n}{1-0{\cdot}9}=10\{1-(0{\cdot}9)^n\}.$$

Fig. 121 shews the graph of $y=10\{1-(0{\cdot}9)^x\}$.

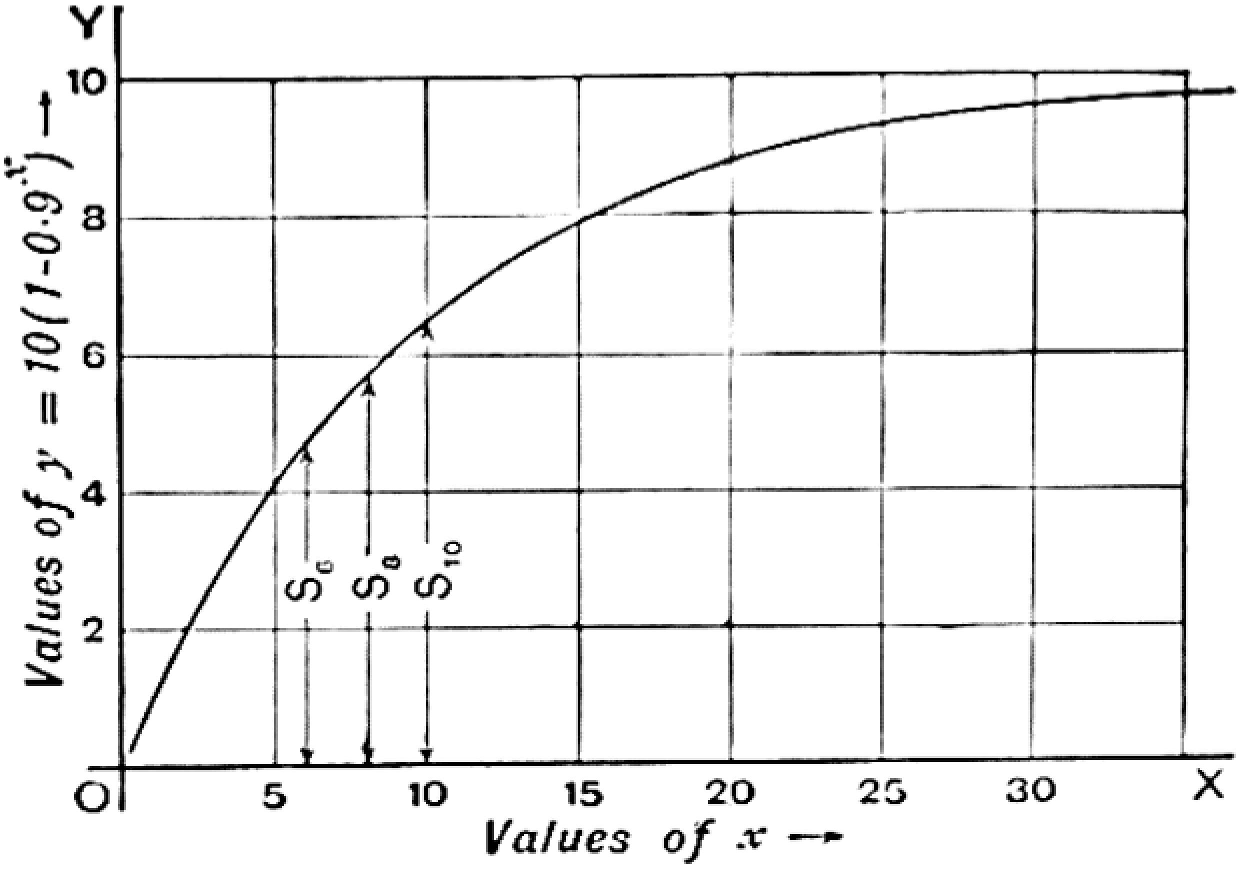


FIG. 121.

The ordinates at $x=1, 2, 3, \ldots n$ are $S_1, S_2, S_3, \ldots S_n$.

It is clear from the graph that S_n approximates to 10 as n increases, *i.e.* $\lim\limits_{n\to\infty} S_n=10$.

It is important to see that S_n can be made to differ from its limit 10 by any number, however small. For example, suppose S_n is to differ from 10 by not more than 5.

Inspection of the graph shows that n must be 7, or any greater number.

If the difference is to be <2, n must be 16, or more.

If the difference is to be $<0{\cdot}5$, n must be 29, or more.

Ex. 2. *How many terms of the series in the preceding example must be taken in order that their sum may differ from 10 by less than 0·01?*

We have $$S_n = 10\{1-(0{\cdot}9)^n\}.$$

Then $$10-[10\{1-(0{\cdot}9)^n\}] < 0{\cdot}01,$$
$$\textit{i.e.}\qquad 10(0{\cdot}9)^n < 0{\cdot}01;$$
$$\therefore\ 0{\cdot}9^n < 0{\cdot}001;$$
$$\therefore\ n\log(0{\cdot}9) < \log 0{\cdot}001;$$
$$\therefore\ n(\bar{1}{\cdot}9542) < -3;$$
$$\therefore\ n(0{\cdot}0458) > 3^*,\ \text{or } n > 65.$$

Hence the sum of 66 or any greater number of terms differs from 10 by less than 0·01.

Harmonic Progression. The numbers x, y, z, ... form a **harmonic progression,** or H.P., if their reciprocals $1/x$, $1/y$, $1/z$, ... form an arithmetic progression.

Ex. 1. *Find the harmonic mean between p and q.*

Let H be the harmonic mean.

Then $1/p$, $1/H$, $1/q$ is an arithmetic progression;

$$\therefore\ \frac{1}{H} = \left(\frac{1}{p}+\frac{1}{q}\right)\Big/2 \quad \text{(see p. 245)}$$
$$= \frac{p+q}{2pq};$$
$$\therefore\ H = \frac{2pq}{p+q}.$$

Ex. 2. *If p, q, r are in harmonic progression, prove that*

$$\frac{p}{r} = \frac{p-q}{q-r}.$$

By Ex. 1, $$q = \frac{2pr}{p+r};$$
$$\therefore\ qp+qr = 2pr;$$
$$\therefore\ qp-pr = pr-qr;$$
$$\therefore\ p(q-r) = r(p-q),$$
$$\text{or}\qquad \frac{p}{r} = \frac{p-q}{q-r}.$$

---

*If $x=y$, $-x=-y$; but if $x>y$, $-x<-y$, and if $x<y$, $-x>-y$. Numerical examples will make this clear.

Graphic Construction for a Harmonic Progression. (Fig. 122.)

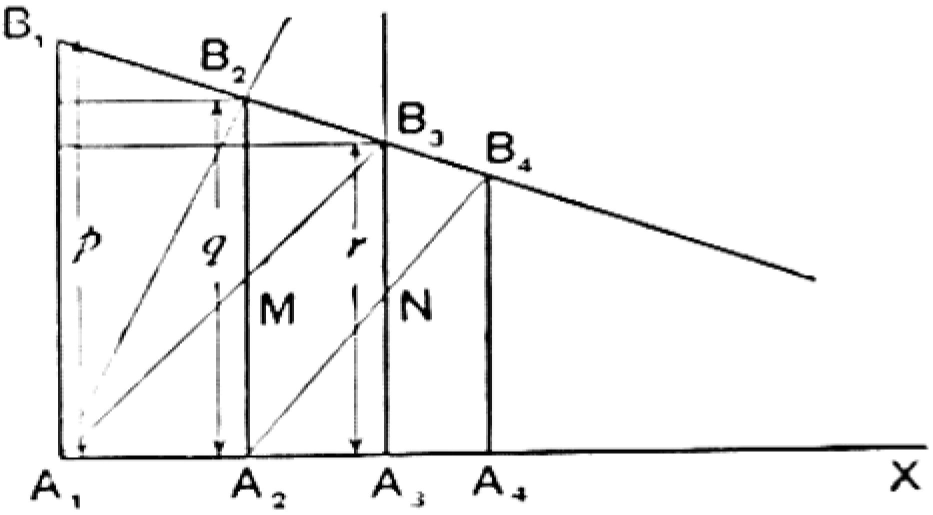


FIG. 122.

Draw any line A_1X. At A_1 erect a perpendicular

$$A_1B_1 = p,$$

the first term.

At any convenient point A_2 erect a perpendicular

$$A_2B_2 = q,$$

the second term.

Bisect A_2B_2 at M and let A_1M and B_1B_2 meet in B_3.

From B_3 draw B_3A_3 perpendicular to A_1X. Then B_3A_3 is r, the third term of the progression.

We have,

$$\begin{aligned} \frac{q}{2p} &= \frac{B_2M}{B_1A_1} \\ &= \frac{B_3M}{B_3A_1} \left\{ \begin{array}{l} \text{from similar triangles} \\ \quad B_3B_2M,\ B_3B_1A_1 \end{array} \right. \\ &= \frac{A_2A_3}{A_1A_3}. \end{aligned}$$

Again
$$\begin{aligned} \frac{q}{2r} &= \frac{A_2M}{A_3B_3} \\ &= \frac{A_1A_2}{A_1A_3}; \end{aligned}$$

$$\begin{aligned} \therefore\ \frac{q}{2p} + \frac{q}{2r} &= \frac{A_2A_3 + A_1A_2}{A_1A_3} \\ &= 1; \end{aligned}$$

$$\therefore\ \frac{2}{q} = \frac{1}{r} + \frac{1}{p}.$$

This shews that r is the third term of the harmonic progression, of which p, q are the first two terms.

Similarly, if B_3A_3 is bisected at N and A_2N be produced to meet B_1B_2 in B_4, then the perpendicular B_4A_4 is the fourth term of the H.P., and so on.

Ex. *If a, b are two positive numbers, shew that their arithmetic mean $>$ their geometric mean $>$ their harmonic mean.*

Let A, G, H, be the arithmetic, geometric and harmonic means.

$$A = \frac{a+b}{2},$$

$$G = \sqrt{ab},$$

$$H = \frac{2ab}{a+b}.$$

$$\begin{aligned} A - G &= \tfrac{1}{2}(a + b - 2\sqrt{ab}) \\ &= \tfrac{1}{2}(\sqrt{a} - \sqrt{b})^2, \text{ which is positive;} \\ &\therefore\ A > G. \end{aligned}$$

$$\begin{aligned} G - H &= \sqrt{ab} - \frac{2ab}{a+b} \\ &= \frac{\sqrt{ab}}{a+b}\{a + b - 2\sqrt{ab}\} \\ &= \frac{\sqrt{ab}}{a+b}(\sqrt{a} - \sqrt{b})^2, \text{ which is positive;} \\ &\therefore\ G > H. \end{aligned}$$

EXERCISES XXXI.

1. If p^2, q^2, r^2 are in arithmetic progression, prove that so also are $1/(q+r)$, $1/(r+p)$, and $1/(p+q)$.

2. The sum of three numbers in arithmetic progression is 21, the sum of their squares is 179: find them.

3. If four numbers are in arithmetic progression, prove that the excess of the product of the second and third over that of the first and fourth is twice the square of the common difference.

4. If the roots of the equation $(q-r)x^2 + (r-p)x + p - q = 0$ are equal, prove that p, q, r are in arithmetic progression.

5. If the sum of $2n+1$ terms of an arithmetic progression is S, and that of the arithmetic progression formed by taking every alternate term, starting with the first, be S', find $S : S'$.

6. Shew that $\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \dots$ if $a < x$. Shew that reciprocals of numbers may be rapidly calculated by using this result, and find $\frac{1}{97}$ to 6 decimal places.

7. How many terms of the series $1 + 1/3 + 1/3^2 + 1/3^3 + \dots$ must be taken in order that the sum may be greater than $1{\cdot}4995$?

8. If p, q, r are in geometric progression, shew that $\log_a p$, $\log_a q$ and $\log_a r$ are in arithmetic progression.

9. If p, q, r are in geometric progression and $p-q$, $r-p$ and $q-r$ are in harmonic progression, shew that $p+4q+r=0$.

10. Evaluate $0{\cdot}3\dot{2}\dot{7}$, $0{\cdot}8\dot{2}\dot{5}$ and $0{\cdot}\dot{5}7142\dot{8}$.

11. If P be the r^{th} term when n arithmetic means are inserted between x and y and Q be the r^{th} term when n harmonic means are inserted between x and y, shew that $P/x + y/Q$ is independent of n and r.

12. If a, b, c, d are in harmonic progression, prove that

(1) $(a-c)(b-d) = 4(a-b)(c-d)$,

(2) $ab + bc + cd = 3ad$,

(3) $\dfrac{b+c}{a}$, $\dfrac{c+a}{b}$, $\dfrac{a+b}{c}$ are in arithmetic progression,

(4) $2a-b$, b, $2c-b$ are in geometric progression.

13. If p, q, r are in arithmetic progression and q, r, p in geometric progression, shew that r, p, q are in harmonic progression.

14. If the p^{th} term of a harmonic progression is q and the q^{th} term is p, find the $(p+q)^{\text{th}}$ term.

15. If $p(q-r)x^2 + q(r-p)xy + r(p-q)y^2$ is a perfect square, prove that p, q, r are in harmonic progression.

16. If p, q, r, s are in geometric progression, prove that $px^2 + r$ is a factor of $px^3 + qx^2 + rx + s$.

17. Find the sum of the numbers in the n^{th} group of the series 1, (2, 3), (4, 5, 6), (7, 8, 9, 10), (11, 12, 13, 14, 15), etc.

Examples 18-30 *are taken from papers set at the Inter. B.Sc.* [*L.U.*].

18. A company sets aside £5000 out of its profits at the end of each year to form a Reserve Fund and invests the amount at 4% per annum compound interest. What is the value of the fund at the end of ten years ?

19. War Savings Certificates used to be sold at 15*s.* 6*d.*, to be paid off at £1 in 5 years' time. Find the rate of compound interest.

If a certain sum is set aside at the *beginning* of each year to accumulate at the same compound rate, determine the sum necessary to meet the final payment.

20. Find the present value of an annuity of £A to continue for n years, interest being reckoned at $r\%$ per annum.

A man owns the lease of a house which has 45 years to run: the rent he receives for the house is £150 per annum, the repairs and other expenses amount to £25 per annum, and the ground rent is £20 per annum. Assuming interest at 6%, find the present value of the house.

21. In a works the initial value of the plant is C_0 and the rate of depreciation per annum is r per cent. Shew that the value of the plant after n years is

$$C_0\left(1-\frac{r}{100}\right)^n,$$

The initial value being £28,000 and the rate of depreciation 4%, find the value at the end of 6 years.

By expending a sum £A at the end of the sixth year and an equal sum at the end of the seventh, the plant is restored to its initial value. Find A.

22. The sides A_1A_2, A_2A_3, A_3A_4, ... A_9A_{10} of a ten-sided polygon all subtend angles of 20° at a point O on the side A_1A_{10}; the angles OA_1A_2, OA_2A_3, OA_3A_4 ... OA_9A_{10} are all right angles. Prove that the lengths of the above-mentioned 9 sides are in geometrical progression, and find the perimeter of the polygon to the nearest hundredth of an inch if $OA_1=1$ inch.

23. Find the two values of x which make 4, $9+x$, $33+x$, three consecutive terms of a geometrical progression. Find, in each of the two progressions, the sum of the 19 terms which precede the number 4.

Find the lowest value of n for which the sum of n terms of the series

$$(1{\cdot}035)^{-1}+(1{\cdot}035)^{-2}+(1{\cdot}035)^{-3}+\dots$$

exceeds 15.

24. A series of straight lines AB, BC, CD, DE, ... are drawn so that each is at right angles to the one before it. Their lengths are connected thus:

$$AB=BC=2CD=2DE=4EF=4FG=8GH=8HK=\dots,$$

so that the points A, C, E, G, ... are in the same straight line.

If $AB=\sqrt{2}$ inch, find the distance from A of the further end of the $2n^{\text{th}}$ line, and find to what limit this tends as the number of lines is increased indefinitely.

25. A circle of one inch radius is drawn on a piece of paper, and with the same centre, $3n-1$ other circles are drawn of radii 2 in., 3 in., 4 in., ... The inner circle is painted blue, the ring between that and the next circle is painted red, the next ring yellow, then the other rings blue, red, yellow, and so on in this order. Shew that the successive areas of each colour are in A.P., and find the total areas painted in each colour.

26. The 3rd term of an arithmetical progression is 10, the n^{th} term is 8 and the $2n^{\text{th}}$ term is 4·5; find the progression. Draw a graph shewing the connection between the number of terms and the sum of these terms.

27. In order to make a pump rod equally strong throughout, the cross-sectional areas at equal distances apart must form a geometrical progression. If one end of the rod has a cross-section of $\frac{1}{4}$ sq. in. and the one a foot above it has a cross-section of $\frac{5}{16}$ sq. in., find the cross-section at the top of a rod 30 ft. long.

What is the average area of these 31 sections?

28. Prove that the geometric series $a + ar + ar^2 + \ldots$ is convergent if r is numerically less than unity.

If S_n and S denote the sum to n terms and to infinity respectively of the geometric series $2 + 1\frac{1}{2} + 1\frac{1}{8} +$ etc., find the value of n such that $S - S_n$ is less than $\frac{1}{1000}$.

29. If equal weights are suspended from a string the ends of which are fixed at two points not in the same vertical straight line, it is known that the tangents of the angles of inclination to the horizontal of the portions of the string form a series in arithmetical progression. If the lowest link is horizontal and the next makes an angle A with the horizontal, and the verticals through the weights are at a constant distance $2a$ apart, shew that the height y of the point of attachment of any weight above the lowest link is connected with the horizontal distance x from the vertical through the middle point of that link by the equation

$$4ay \cot A = x^2 - a^2.$$

30. What is meant by the sum to infinity of a geometrical progression, and determine the condition that such a sum should exist?

How many terms of such a progression, of which the first two terms are 8 and 6, must be taken that their sum may differ from the sum to infinity by less than 0·1% of this sum?

31. A tank of capacity 480 gallons can be filled in 3 hours: the medium tap delivers 40 gallons an hour: the deliveries of the other taps form a harmonic progression with the medium one. Find the deliveries of the other two.

32. Find the present value of n equal annual payments, the first being made one year hence, allowing compound interest at 4 per cent.

If a debt of £1600 is to be settled by 20 such annual payments each of £x, calculate the value of x to two decimal places.

33. Find the sum of the progression

$$a + ar + ar^2 + \dots + ar^{n-1}.$$

Shew that £273·5 will buy an annuity, payable half-yearly, commencing in six months' time, and lasting for twenty years at the rate of twenty pounds per annum; the value of money being taken as four per cent. per annum.

34. Find a formula for the present value of a sum of £N to be paid every p^{th} year, starting q years hence, allowing compound interest at r per cent. per annum.

A and B have equal rights in an investment which produces an income of £100 per annum, and they agree that A shall receive the first, third, fifth, ... dividends while B takes the second, fourth, sixth, ... Find how much A should pay B at once to make the agreement equitable, if interest is calculated at 4 per cent. per annum. [S.W.]

35. A man wishes to build a house costing £1100. Out of his income of £600 per annum, paid quarterly, he is able to save one-sixth, which he invests at the times of payment at 5 per cent. compound interest. How long will it be before he can build the house? [B.U.]

36. A man contributes £150 a year to a pension fund, in which the money accumulates at 4 per cent. compound interest. What is the amount of the fund at the end of 30 years if the payments are made at the beginning of each year? [B.U.]

37. Two sons have equal shares in an estate which yields £500 per annum. One of them buys his brother out by paying him an annuity for twenty years. Reckoning interest at 5 per cent. per annum, find the amount of the annuity (per annum). [B.U.]

38. A cottage at the beginning of a year was worth £250, but on account of dilapidations it was found that at the end of each year it had lost ten per cent. of its value at the beginning of each year. After how many years would its value be reduced below £25?

Take $\log_{10} 3 = 0{\cdot}4771213$. [B.U.]

39. Find the sum of n terms of a geometric progression. Explain what is meant by 'sum to infinity' of a G.P., and give an example.

Prove that the difference between the sum of the first n odd terms and the sum of the first n even terms of the G.P. of which the first term is 1 and the common ratio 3 is $\frac{1}{4}(3^{2n} - 1)$. [S.W.]

CHAPTER X.

PERMUTATIONS AND COMBINATIONS. THE BINOMIAL THEOREM. SUMMATION OF SERIES.

EACH of the *groups* which can be made by taking some, or all, of a number of things is called a **Combination** of these things.

Thus *ab, bc, abc, a, abcd,* are all combinations of the four letters a, b, c, d.

If there are n different things and each group contains r of them, the symbol ${}_nC_r$ is used to denote the number of these groups. Or, in other words, ${}_nC_r$ is the number of combinations of n different things taken r at a time.

Ex. 1. *Shew that* ${}_4C_2=6$ *and that* ${}_5C_3=10$.

Consider the four letters a, b, c, d. They can be grouped two at a time as follows:

ab, ac, ad, bc, bd, cd.

That is, there are six groups in all;

$$\therefore\ {}_4C_2=6.$$

Consider the five letters a, b, c, d, e. They can be grouped three at a time as follows:

abc, abd, abe, acd, ace, ade ; bcd, bce, bde ; cde.

That is, 10 groups in all;

$$\therefore\ {}_5C_3=10.$$

If any one combination, such as *abc*, be taken and the order of the letters changed, any such possible arrangement is called a **Permutation.**

Thus the combination *abc* gives rise to the following permutations:

abc, acb ; bac, bca ; cab, cba.

If there are n different things and each *arrangement* contains r of them, the symbol ${}_nP_r$ is used to denote the number of these arrangements;

Or, in other words, ${}_nP_r$ is the number of permutations of n different things taken r at a time.

Ex. 2. *Shew that* ${}_4P_2=12$ *and that* ${}_5P_3=60$.

Consider the four letters a, b, c, d.

Example 1 shews that the number of combinations two at a time is six.

Each of these groups gives rise to two permutations;

$$\therefore\ {}_4P_2={}_4C_2\times 2=12.$$

Consider the five letters a, b, c, d, e.

These are shewn *grouped* three at a time in Ex. 1.

The group abc can be *arranged* as

$$abc,\ acb;\quad bac,\ bca;\quad cab,\ cba.$$

That is, the group abc gives rise to six permutations, and so do all the other groups;

$$\therefore\ {}_5P_3={}_5C_3\times 6=60.$$

Theorem : $\mathbf{{}_nP_r=n(n-1)(n-2)\ldots(n-r+1)}$.

Suppose there are n different parcels and a row of r pigeon-holes, each of which has to receive one parcel. The number of ways of filling up the pigeon-holes is the number of ways of arranging n things r at a time, *i.e.* ${}_nP_r$.

The first pigeon-hole can be filled in n ways.

Having filled it, there are $n-1$ parcels left, and therefore the second pigeon-hole can be filled in $n-1$ ways.

Hence the first two pigeon-holes can be filled in $n(n-1)$ ways, as *each* way of filling the first can be associated with *every* way of filling the second.

When the first two pigeon-holes are filled, there are $n-2$ parcels left, so that the third pigeon-hole can be filled in $n-2$ ways.

Therefore the first three pigeon-holes can be filled in $n(n-1)(n-2)$ ways.

Similarly, the first four pigeon-holes can be filled in $n(n-1)(n-2)(n-3)$ ways; and the r pigeon-holes can be filled in $n(n-1)(n-2)\ldots\{n-(r-1)\}$ ways;

$$\therefore\ {}_nP_r=n(n-1)(n-2)\ldots(n-r+1).$$

Evidently

$$\begin{aligned}{}_nP_n&=n(n-1)(n-2)\ldots(n-n+1)\\&=n(n-1)(n-2)\ldots 3.2.1.\end{aligned}$$

The product of the n integers from n to 1 is written $\underline{|n}$ or $n\,!$ and called " factorial n."

Clearly

$$ {}_nP_r = \frac{n(n-1)(n-2)\ldots(n-r+1)\times(n-r)(n-r-1)\ldots 2\,.\,1}{(n-r)(n-r-1)\ldots 2\,.\,1} $$

$$ \text{or } {}_nP_r = \frac{\underline{|n}}{\underline{|n-r}}. $$

Ex. 1. *In how many ways can the letters of the word* postage *be arranged ? In how many of these permutations do the vowels occupy the even places ? In how many do the consonants occur together ?*

(1) The total number of arrangements is $\underline{|7} = 5040$.

(2) The three even places can be filled with vowels in $\underline{|3}$ ways : the remaining four places can be filled with consonants in $\underline{|4}$ ways.

Hence the number of arrangements in which the vowels occupy the even places is $\underline{|3} \times \underline{|4}$, *i.e.* 144.

(3) Consider the four consonants *pstg*. Regarding the arrangement as a single symbol, this symbol with the three vowels *o*, *a*, *e* can be arranged in $\underline{|4}$ ways. But in each of these, the four consonants can be arranged in $\underline{|4}$ ways amongst themselves without altering the positions of the vowels.

Hence the number of arrangements in which the consonants occur together is $\underline{|4} \times \underline{|4}$, *i.e.* 576.

Ex. 2. *Find the number of permutations of n different things r at a time in which two specified things occur together.*

First deduct the two specified things ; then the number of permutations of $(n-2)$ things taken $(r-2)$ at a time is ${}_{n-2}P_{r-2}$.

Now add the two specified things, regarded temporarily as a single thing, to each of these permutations, then the double thing may occupy any of the $(r-1)$ places ; hence, the number of permutations becomes $(r-1)_{n-2}P_{r-2}$.

Finally, as the places of the two specified things may be interchanged, the required number of permutations is

$$ 2(r-1)_{n-2}P_{r-2} = 2(r-1)\,\underline{|n-2}\,/\,\underline{|n-r}. $$

Theorem: $\mathbf{{}_nC_r = {}_nP_r/\underline{|r}}$.

${}_nC_r$ means the number of groups or selections which can be made by taking n different things r at a time. The r things in each of these groups may be arranged among themselves in ${}_rP_r$, or $\underline{|r}$, ways.

One group gives $\underline{|r}$ permutations.

Two groups give $2\underline{|r}$ permutations.

Three groups give $3\underline{|r}$ permutations;

and ${}_nC_r$ groups give ${}_nC_r \times \underline{|r}$ permutations.

But the number of permutations of n different things r at a time is ${}_nP_r$;

$$\therefore \quad {}_nP_r = {}_nC_r \times \underline{|r}$$

$$\text{or} \qquad {}_nC_r = {}_nP_r/\underline{|r}.$$

This may be written

$$ {}_nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$$

$$\text{or} \qquad {}_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. \qquad \dots\dots\dots\dots(1)$$

It should be noticed that each time a group of r things is selected from n, $n-r$ things are left; or, symbolically,

$$ {}_nC_r = {}_nC_{n-r}. \qquad \dots\dots\dots\dots(2)$$

This is also evident from equation (1) above, for

$$\begin{aligned} {}_nC_{n-r} &= \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-\overline{n-r}}} \\ &= \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}} \\ &= {}_nC_r. \end{aligned}$$

The fact stated by equation (2) often reduces the labour of calculation.

Ex. 1. *Find* ${}_{100}C_{98}$.

$$\begin{aligned} {}_nC_r &= {}_nC_{n-r}; \\ \therefore \quad {}_{100}C_{98} &= {}_{100}C_2 \\ &= \frac{100.99}{1.2} \\ &= 4950. \end{aligned}$$

Ex. 2. *Shew that* ${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}$.

We have
$$ {}_{n-1}C_r = \frac{\underline{|n-1}}{\underline{|r}\ \underline{|n-1-r}}, $$

$$ {}_{n-1}C_{r-1} = \frac{\underline{|n-1}}{\underline{|r-1}\ \underline{|n-1-\overline{r-1}}}; $$

$$ \therefore\ {}_{n-1}C_r + {}_{n-1}C_{r-1} = \underline{|n-1}\left\{\frac{1}{\underline{|r}\ \underline{|n-r-1}} + \frac{1}{\underline{|r-1}\ \underline{|n-r}}\right\} $$

$$ = \underline{|n-1}\left\{\frac{(n-r)+r}{\underline{|r}\ \underline{|n-r}}\right\} $$

$$ = \frac{n\ \underline{|n-1}}{\underline{|r}\ \underline{|n-r}} $$

$$ = \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} $$

$$ = {}_nC_r. $$

The interpretation of the identity above is that, of the number of groups or selections of r things which may be made from n things, ${}_{n-1}C_{r-1}$ contain one specified thing and ${}_{n-1}C_r$ do not contain that specified thing.

Thus, suppose we have n letters a, b, c, ... from which a selection of r is to be made.

The number containing a specified letter, say a, is the number of ways of choosing $r-1$ letters (to put with a) from the $n-1$ other letters;

i.e. ${}_{n-1}C_{r-1}$. ....(1)

The number of selections which do not contain a is the number of ways of choosing r letters from the group b, c, d, ..., *i.e.* from $n-1$ things, which is

${}_{n-1}C_r$. ....(2)

Now the total number of selections of r letters which may be made from the given n letters is ${}_nC_r$, and these include (1) and (2);

$$ \therefore\ {}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r. $$

The number of permutations of n things n at a time when p are alike of one kind, q alike of a second, r alike of a third, and so on, is

$$\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\ldots}$$

Let N be the required number of permutations. From any one of these, if the p like things were different, we could make $\underline{|p}$ new permutations.

Thus if the p like things were all different, we should get $N\,\underline{|p}$ permutations.

Similarly, if the q like things were different, we should get $\underline{|q}$ new permutations from each of the second set of permutations.

Thus, if the p like things and the q like things were all different, we should get $N\,\underline{|p}\,\underline{|q}$ permutations in all.

The process is continued until all the sets of like things are different, and we then get the number of permutations of n things n at a time when they are all different. This is $\underline{|n}$.

$$\therefore\ N\,\underline{|p}\,\underline{|q}\,\underline{|r}\ldots=\underline{|n};$$

$$\therefore\ N=\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}\ldots}$$

The number of ways in which a selection may be made from p + q + r + ... things when p are alike of one kind, q alike of a second, r alike of a third, and so on, is

$$(p+1)(q+1)(r+1)\ldots-1.$$

For of the p things we may take 0, 1, 2, 3, ... or p, *i.e.* the p things may be selected in $p+1$ ways.

Similarly, of the q things we may take 0, 1, 2, 3, ... or q, *i.e.* the q things may be selected in $q+1$ ways; and so on for the others.

Hence the $p+q+r\ldots$ things may be selected in

$$(p+1)(q+1)(r+1)\ldots\text{ ways.}$$

But this includes the case when none of the p things, none of the q things, none of the r things, etc., are taken, which is inadmissible.

Hence the total number of ways is

$$(p+1)(q+1)(r+1)\ldots-1.$$

If all the things are different, $p = q = r = \ldots = 1$, and the total number of combinations which can be formed of n different things, taking any or all of them at a time, is $2^n - 1$.

The number of permutations of n different things r at a time when repetitions are allowed is n^r.

For suppose we have r places, each of which is to be filled by one letter from the n letters a, b, c, etc. The first place may be filled in n ways. Since repetitions of the letters are allowed, the second may also be filled in n ways.

Therefore two places may be filled in $n \times n$ or n^2 ways.

Similarly, the third may be filled in n ways, since repetitions are allowed.

Therefore three places may be filled in $n^2 \times n$ or n^3 ways.

Proceeding in this way, it is clear that the r places may be filled in n^r ways.

Ex. 1. *How many permutations can be formed by taking all the letters of the word* Exercises ? *How many of these begin with 3 e's ?*

There are 9 letters, of which 3 are e's and 2 are s's.

$$\therefore \text{ the number of permutations is } \frac{\underline{|9}}{\underline{|3}\ \underline{|2}} = 30240.$$

Excluding the 3 e's, there are 6 letters left, of which 2 are s's.

$\therefore$ the number of permutations which begin with 3 e's is the number of permutations of 6 things, 6 at a time, when 2 are alike:

$$i.e. \quad \underline{|6} / \underline{|2} \quad \text{or} \quad 360.$$

Ex. 2. *In how many ways can $p + q$ different things be divided into 2 groups, one containing p and the other q things?*

For every group of p things selected from the $p + q$ given things a group of q things is left.

Hence the number of ways of dividing the $p + q$ things into the two groups is the same as the number of combinations of $p + q$ things p at a time or ${}_{p+q}C_p$.

$$\text{This is equal to } \frac{\underline{|p+q}}{\underline{|p}\ \underline{|p+q-p}} \quad \text{or} \quad \frac{\underline{|p+q}}{\underline{|p}\ \underline{|q}}.$$

Ex. 3. *Find the number of ways in which* (i) *a selection,* (ii) *an arrangement of* 5 *letters can be made from the letters of the word* Mississippi.

The letters of the given word are 4 s's, 4 i's, 2 p's and 1 m.

The *selections* of 5 letters may be made up as follows :

The group may contain

(i) 4 like letters and 1 different, viz. $ssss+i$, p or m ; $iiii+s$, p or m ; giving 6 groups.

(ii) 3 like letters and 2 different, viz. $sss+ip$, im or mp ; $iii+sp$, sm or mp ; giving 6.

(iii) 3 like letters and 2 like letters of another kind, viz. $sss+ii$ or pp ; $iii+ss$ or pp ; giving 4.

(iv) 2 like letters and 3 different letters, viz. $ss+ipm$; $ii+spm$; $pp+sim$; giving 3.

(v) 2 like letters, 2 like letters of another kind and one different letter, viz. $ss+ii+p$ or m ; $pp+ss+i$ or m ; $pp+ii+m$ or s ; giving 6.

$\therefore$ total number of selections of five letters

$$=6+6+4+3+6$$
$$=25.$$

To find the number of arrangements that can be made, note that (i) gives $6\,\underline{|5}/\underline{|4}$ permutations

(ii) ,, $6\,\underline{|5}/\underline{|3}$,,

(iii) ,, $4\,\underline{|5}/(\underline{|3}\,\underline{|2})$,,

(iv) ,, $3\,\underline{|5}/\underline{|2}$,,

(v) ,, $6\,\underline{|5}/(\underline{|2}\,\underline{|2})$,,

$\therefore$ total number of permutations is

$$6\,\underline{|5}/\underline{|4}+6\,\underline{|5}/\underline{|3}+4\,\underline{|5}/(\underline{|3}\,\underline{|2})+3\,\underline{|5}/\underline{|2}+6\,\underline{|5}/(\underline{|2}\,\underline{|2})$$
$$=30+120+40+180+180$$
$$=550.$$

Ex. 4. *Shew that the number of combinations of the* 4 *letters* a, b, c, d, *taken three at a time, when repetitions are allowed, is* 20.

These combinations will be of the following kinds:

(i) all letters alike;
(ii) two letters alike, the third different;
(iii) all letters different.

There are 4 combinations of type (i): aaa, bbb, etc.
,, ,, 3×4 ,, type (ii): aab, aac, aad, etc.
,, ,, ${}_4C_3$,, type (iii).

The total number is $4+12+4$, *i.e.* 20.

The number of combinations of n things, r at a time, when repetitions are allowed, is written ${}_nH_r$.

Ex. 5. *Shew that of the* ${}_nH_r$ *combinations of* n *letters* a, b, c, d, *etc., taken* r *at a time, repetitions being allowed, there are* ${}_nH_{r-1}$ *in which the letter* a *occurs.*

The ${}_nH_{r-1}$ groups in which the n letters can be combined $r-1$ at a time can be classified as follows:

Those containing no letter a.
,, ,, a once.
,, ,, a twice.
....
The last containing a $(r-1)$ times.

If now another letter a is inserted in each of the above groups, we obtain combinations of the n letters r at a time (repetitions being allowed) grouped as follows:

Those containing the letter a once.
,, ,, ,, a twice.
,, ,, ,, a three times.
....
The last containing the letter a r times.

That is, the letter a occurs in ${}_nH_{r-1}$ of the ${}_nH_r$ combinations.

Ex. 6. *Find a formula for* ${}_nH_r$.

Consider n letters a, b, c, d, ..., and, for brevity, denote the combinations of these n letters r at a time, repetitions being allowed, as the ${}_nH_r$ groups.

If the ${}_nH_r$ groups were all written out, it is clear that each letter would occur the same number of times.

As each group contains r letters, the total number of letters is ${}_nH_r \times r$, and so the letter a occurs ${}_nH_r \times r/n$ times.

Next, consider all the ${}_nH_{r-1}$ groups written out. The letter a would occur ${}_nH_{r-1} \times (r-1)/n$ times. If now the letter a were inserted in each of the ${}_nH_{r-1}$ groups, the letter a would occur ${}_nH_{r-1} \times (r-1)/n + {}_nH_{r-1}$ times, for there are ${}_nH_{r-1}$ additional appearances, and, as in Ex. 5, these groups would become exactly those of the ${}_nH_r$ groups in which a occurs.

Therefore $${}_nH_r \times \frac{r}{n} = {}_nH_{r-1} \times \frac{r-1}{n} + {}_nH_{r-1} = {}_nH_{r-1} \cdot \frac{n+r-1}{n};$$

$$\therefore \quad H_r = {}_nH_{r-1} \times \frac{n+r-1}{r}. \qquad \text{..................(1)}$$

Writing $r-1$ for r, (1) becomes

$${}_nH_{r-1} = {}_nH_{r-2} \times \frac{n+r-2}{r-1}. \qquad \text{..................(2)}$$

Similarly,

$${}_nH_{r-2} = {}_nH_{r-3} \times \frac{n+r-3}{r-2}, \qquad \text{..................(3)}$$

....

$${}_nH_3 = {}_nH_2 \times \frac{n+2}{3}, \qquad \text{......................}(r-2)$$

$${}_nH_2 = {}_nH_1 \times \frac{n+1}{2}. \qquad \text{......................}(r-1)$$

Multiplying equations (1) to $(r-1)$ together,

$${}_nH_r = {}_nH_1 \times \frac{(n+r-1)(n+r-2)(n+r-3)\ldots(n+2)(n+1)}{r\,.\,(r-1)\,(r-2)\,\ldots\,(3)\,.\,(2)}.$$

Now ${}_nH_1$ is clearly n;

$$\therefore \quad {}_nH_r = \frac{(n+r-1)\,(n+r-2)\ldots(n+1)\,.\,n}{\underline{|r}} = {}_{n+r-1}C_r.$$

EXERCISES XXXII.

1. How many permutations can be made from the letters of the word *German*, (i) three at a time, (ii) five at a time?

2. How many different numbers of five digits can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, (i) if no digit occurs more than once in each number, (ii) if any digit can be repeated as often as desired?

3. In how many ways (i) can 3 letters be posted in 4 letter boxes, (ii) can n letters be posted in k letter boxes?

4. A party of three is to be chosen from six Englishmen, seven Americans and five Frenchmen, the party containing one of each nationality. In how many ways can this be done? And in how many of the different groupings are a particular Englishman included and a particular American excluded?

5. Nine boxes are to be filled with nine parcels, one in each box. In how many ways can this be done if four boxes are too small for five of the parcels?

6. In how many ways can five ladies and five men make couples for a dance?

7. In how many ways can four ladies and four men form two doubles for tennis, each side consisting of a lady and a man?

8. In how many different ways can seven people be arranged at a round table?

9. Answer Ex. 8 if two particular people may not sit together.

10. How many numbers of (i) 3 digits (ii) not more than 3 digits can be made from 0, 1, 2, 3, 4, no digit occurring more than once in any number?

11. A compositor drops the letters of the word *advertise*, picks them up and replaces them at random. How many possible mistakes can he make?

12. In how many ways can all the letters of the word *Constantinople* be arranged? How many of these begin with two t's? How many end with s? How many have s in the third place and e at the end?

13. In how many ways can n letters be arranged in a row so that two particular letters do not occur at either beginning or end of the row?

14. If p, q, r, s, t are primes, in how many ways can the product pq^2r^3st be expressed as the product of two factors?

15. In how many ways can five girls and five boys form themselves in a circle, so that each boy is between two girls?

16. A party of nine persons are to travel in two vehicles, one of which will not hold more than seven and the other not more than four: in how many ways can the party travel?

17. Find the number of permutations of n things taken r at a time.

How many words of three letters can be formed from the word *Zeppelin*, a vowel being always in the middle? [S.W.]

18. Find the formula for the number of combinations of n different things taken r at a time.

In how many ways can any four letters be selected from the word *Degree*? [S.W.]

19. If ${}_nC_r$ denote the number of combinations of n things taken r at a time, prove, without quoting formulae, that

$$ {}_{n+1}C_r = {}_nC_{r-1} + {}_nC_r \,; $$

and verify this by using the formula for ${}_nC_r$.

In how many ways can three sovereigns be given away, when there are four applicants, and any applicant may have either 0, 1, 2, or 3 sovereigns? [Man.]

20. Shew that the number of selections of r things which can be made from n different things is

$$ \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} . $$

Explain why the number is the same if each selection contains $n-r$ things instead of r.

Among the twenty members of a cricket club, there are two wicket-keepers and five members who can bowl. In how many ways can an eleven be chosen to include only one of the wicket-keepers and at least three bowlers? [Durham.]

21. A cricket team of eleven men is to be chosen from fourteen men, of whom five can bowl and two others can keep wicket. If the team is to include at least one wicket-keeper and three bowlers, in how many ways may the choice be made? [B.U.]

22. Find the number of ways in which the six letters A, B, C, D, E, F can be arranged in a row so that the two vowels never come together.

In how many ways can five sovereigns be distributed among six applicants, each applicant receiving none, one, or more?

[B.U.]

23. A golfer wishes to play the first three holes of a nine-hole course with a heavy ball, and the remaining holes with a light ball. If he have three heavy balls and two light balls, all different, shew that the number of different ways he may go round the course is 1728.

[B.U.]

24. Find the number of permutations of $a+b+c$ things of which a are alike of one kind, b alike of another kind, and c alike of a third kind, taken all together.

How many words of four letters, each consisting of two consonants and two vowels, can be formed out of the letters occurring in the word *Universal*, if either the vowels and consonants can occur alternately or the consonants can come together in the middle of the word but not at the beginning or end of the word?

[B.U.]

Examples 25-28 *are taken from papers set at the Inter. B.Sc.* [*L.U.*]

25. Find from first principles the number of selections r at a time that can be made from n different things.

A committee of four is to be chosen from five Englishmen and three Americans. In how many ways can this be done so that the committee contains (*a*) at least one Englishman, (*b*) at least one of each nationality ?

26. Find the number of permutations of n different things taken r at a time.

At an examination there are eight candidates from one school and six from another. In how many ways can they be seated at two tables each holding seven so that no candidates from the same school shall sit next to each other, the seats being on one side of the table only ?

27. Prove the formulae for the number of permutations and of combinations of n things taken r at a time.

A mechanism consists of five parts : there are three sets of these mechanisms, and the fifteen components have all got mixed together. The five parts are easily distinguished the one from the other, but similar parts require a skilled person to distinguish between them. What is the greatest number of different and unsuccessful trials that an unskilled person could make before getting the three sets disentangled ?

28. Find the number of permutations of n things taken all together when p are of one kind, q of another kind and r of a third kind.

The crew of an eight-oar boat is to be chosen from twelve men, of whom three can row on the stroke side only ; find the number of ways in which the crew can be selected.

Product of Linear Factors. In the expansion of a continued product like

$$(a+b)(x+y)(p+q+r+s)(l+m+n)\ldots,$$

each term must contain one letter from each of the brackets.

Consider $\quad (a+b)(x+y)=ax+ay+bx+by.$

Each term on the right contains one letter from each of the two brackets $(a+b)$, $(x+y)$.

Again, multiplying by $p+q+r+s$,

$$(a+b)(x+y)(p+q+r+s)=(ax+ay+bx+by)(p+q+r+s).$$

On expanding the right-hand side it will be found that each term of the expansion contains one letter from each of the three brackets

$$(a+b),\ (x+y)\ \text{ and }\ (p+q+r+s).$$

The process can evidently be extended indefinitely.

Ex. *Find the coefficient of x^2 and of x in the expansion of*

$$(x+a)(x-2b)(x+3c).$$

The terms involving x^2 are obtained by taking x from any two of the brackets and not from the third.

These terms are $x^2(3c)$, $x^2(-2b)$ and $x^2(a)$.

The required coefficient is therefore $(a-2b+3c)$.

The terms involving x are obtained by taking x from any one bracket and not from the remaining two.

These terms are $x(-2b \times 3c)$, $x(a \times 3c)$ and $x(a \times -2b)$.

The required coefficient is therefore $(3ac-2ab-6bc)$.

The Binomial Theorem. Consider the expansion of

$$(a+b)(a+b)(a+b)\ldots$$

to n factors.

If a is chosen from each bracket, we get the term a^n.

If a is chosen from $(n-1)$ brackets and b from the remaining bracket, we get $a^{n-1}b$. But the number of ways of doing this is equal to the number of ways of choosing 1 bracket from n brackets, *i.e.* ${}_nC_1$.

Therefore there are ${}_nC_1$ terms equal to $a^{n-1}b$.

If a is chosen from $(n-2)$ brackets and b from the remaining two, we get $a^{n-2}b^2$. But the number of ways of doing this is equal to the number of ways of choosing 2 brackets from n brackets, *i.e.* ${}_nC_2$.

Therefore there are ${}_nC_2$ terms equal to $a^{n-2}b^2$.

If a is chosen from $n-r$ brackets and b from the remaining r, we get $a^{n-r}b^r$. But the number of ways of doing this is equal to the number of ways of choosing r brackets from n brackets, *i.e.* ${}_nC_r$.

Therefore there are ${}_nC_r$ terms equal to $a^{n-r}b^r$.

Finally, choosing b from all the brackets, we get the term b^n.

$$\therefore\ (a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_ra^{n-r}b^r + \ldots + b^n.$$

This is the **Binomial Theorem** for a positive integral index—the argument is only intelligible when n is a positive integer.

The term ${}_nC_ra^{n-r}b^r$ may be called the **general term** of the expansion, for it is typical of all the rest.

The *form* of the general term should be noticed.

Let a and b be called the **elements** and n the **exponent**. It is clear that the sum of the powers of the elements in the general term, and consequently in every term of the expansion, is equal to the exponent. In other words, the **degree** of each term is equal to n.

Further, the coefficient of the general term is ${}_nC_r$ or ${}_nC_{n-r}$ (see p. 264),

$$\text{or} \qquad {}_{\text{exponent}}C_{\text{power of either element}}.$$

It should be clear that the coefficients of $a^{n-r}b^r$ and of a^rb^{n-r} are equal.

Ex. 1. *Expand* $(3y-2)^4$.

Using the Binomial Theorem, the expansion is

$$(3y)^4 + {}_4C_1(3y)^3(-2) + {}_4C_2(3y)^2(-2)^2 + {}_4C_3(3y)(-2)^3 + (-2)^4,$$

$$i.e. \qquad 81y^4 - 216y^3 + 216y^2 - 96y + 16.$$

Ex. 2. *What is the coefficient of* x^{13} *in the expansion of* $(x-2)^{19}$?

The term involving x^{13} will contain $(-2)^6$, and the multiplying factor is ${}_{19}C_6$.

The required coefficient is therefore ${}_{19}C_6(-2)^6$, or ${}_{19}C_6 \times 2^6$.

On multiplying out, this coefficient is 1,736,448.

When the coefficient is a large number like this, it is usual not to multiply out. Thus, in this example, the required coefficient may be left in the form ${}_{19}C_6 \times (2)^6$.

Ex. 3. *Find the term independent of x in* $\left(2x^2 - \frac{1}{3x^2}\right)^6$.

$$\left(2x^2 - \frac{1}{3x^2}\right)^6 = \left(\frac{6x^4-1}{3x^2}\right)^6 = \frac{1}{3^6x^{12}}(6x^4-1)^6.$$

The term independent of x in $\left(2x^2 - \frac{1}{3x^2}\right)^6$ is the term involving x^{12} in $\frac{1}{3^6}(6x^4-1)^6$.

$$(6x^4-1)^6 = (6x^4)^6 + {}_6C_1(6x^4)^5(-1) + {}_6C_2(6x^4)^4(-1)^2 + \ldots + (-1)^6.$$

The term containing x^{12} is ${}_6C_3(6x^4)^3(-1)^3$, *i.e.* $-{}_6C_3 \times 6^3 \times x^{12}$.

The required term is therefore $-\frac{1}{3^6} \times {}_6C_3 \times 6^3$;

$$i.e. \qquad -\frac{2^3}{3^3} \cdot \frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3} \text{ or } -\frac{160}{27}.$$

Ex. 4. *Find the coefficient of* x^6 *in the expansion of* $(1+3x+2x^3)^7$.

In the formula for the expansion of $(a+b)^n$, write $(1+3x)$ for a and $2x^3$ for b.

Then
$$(1+3x+2x^3)^7=(1+3x)^7+{}_7C_1(1+3x)^6(2x^3)$$
$$+{}_7C_2(1+3x)^5(2x^3)^2+\ldots+(2x^3)^7 \ldots (1)$$

The coefficient of x^6 in $(1+3x)^7$ is ${}_7C_6\,3^6$;

$$i.e. \qquad 7\times 729 \quad \text{or} \quad 5103.$$

The coefficient of x^6 in ${}_7C_1(1+3x)^6(2x^3)$ is the coefficient of x^3 in ${}_7C_1(1+3x)^6\,.\,2$, and is ${}_7C_1\times{}_6C_3\times 3^3\times 2$;

$$i.e. \qquad 7\times 20\times 27\times 2, \quad \text{or} \quad 7560.$$

The coefficient of x^6 in ${}_7C_2(1+3x)^5(2x^3)^2$ is the coefficient of x^0 in ${}_7C_2(1+3x)^5\,.\,2^2$, and is ${}_7C_2\times 1\times 2^2$;

$$i.e. \qquad 21\times 4 \quad \text{or} \quad 84.$$

The remaining terms in (1) all contain powers of x higher than the sixth.

Therefore the required coefficient is $5103+7560+84$ or 12747.

Ex. 5. *Find the ratio of the* $(r+1)^{th}$ *term to the* r^{th} *term in the expansion of* $(x+y)^n$.

The expansion is

$$x^n+{}_nC_1x^{n-1}y+{}_nC_2x^{n-2}y^2+\ldots+{}_nC_rx^{n-r}y^r+\ldots+y^n.$$

The first term is x^n.

The second term is ${}_nC_1x^{n-1}y$.

The third term is ${}_nC_2x^{n-2}y^2$; and so on.

The $(r+1)^{th}$ term is ${}_nC_rx^{n-r}y^r$.

$\therefore$ the r^{th} term is ${}_nC_{r-1}x^{n-\overline{r-1}}y^{r-1}$ or ${}_nC_{r-1}x^{n-r+1}y^{r-1}$;

$$\therefore \text{ the required ratio is } \frac{{}_nC_rx^{n-r}y^r}{{}_nC_{r-1}x^{n-r+1}y^{r-1}};$$

$$i.e. \qquad \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1\,.\,2\,.\,3\ldots(r-1)\,.\,r}$$

$$\times\frac{1\,.\,2\,.\,3\ldots(r-1)}{n(n-1)(n-2)\ldots(n-r+2)}\times\frac{y}{x};$$

$$i.e. \qquad \frac{n-r+1}{r}\,.\,\frac{y}{x}.$$

Ex. 6. *Find the greatest term in the expansion of* $(2x+3y)^{23}$ *when* $x=5$ *and* $y=3$.

Let T_r denote the r^{th} term.

Using Example 5 above, we have

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot\frac{3y}{2x}=\frac{24-r}{r}\cdot\frac{9}{10}.$$

Now
$$T_{r+1}>T_r, \quad\text{......................(1)}$$
$$\text{if}\quad \frac{24-r}{r}\cdot\frac{9}{10}>1,$$
$$\text{if}\quad 216-9r>10r,$$
$$\text{if } 216>19r, \quad \textit{i.e.} \text{ if } r<11\tfrac{7}{19}.$$

Therefore the inequality (1) holds if $r=1, 2, 3, \ldots 11$, and the terms $T_1, T_2, T_3, \ldots T_{12}$ increase in magnitude.

In the same way, if $r>11\frac{7}{19}$, we shall have $T_{r+1}<T_r$.

Therefore the terms $T_{12}, T_{13}, \ldots T_{24}$ decrease in magnitude.

Therefore T_{12} is the greatest term.

Its value is $_{23}C_{11}\,(2x)^{12}\,(3y)^{11}$ or $_{23}C_{11}\,10^{12}\,.\,9^{11}$.

Ex. 7. *In the expansion of* $(3x-4y)^{21}$, *find the numerically greatest coefficient of* x^r *when* $y=\frac{1}{2}$.

Let $(3x-4y)^{21}=N_1x^{21}+N_2x^{20}+N_3x^{19}+\ldots+N_{21}x+N_{22}$; then

$N_1=3^{21}$, $N_2={}_{21}C_1\times3^{20}\times(-4y)$, $N_3={}_{21}C_2\times3^{19}\times(-4y)^2$, etc.

The question means: find the greatest of the numbers N_1, N_2, N_3, etc., when $y=\frac{1}{2}$.

Now
$$N_{r+1}={}_{21}C_r\times3^{n-r}\times(-4y)^r,$$
$$N_r={}_{21}C_{r-1}\times3^{n-r+1}\times(-4y)^{r-1};$$
$$\therefore\ \frac{N_{r+1}}{N_r}=\frac{21-r+1}{r}\times\frac{-4y}{3}. \quad \text{(See Ex. 5.)}$$

As the numerically greatest coefficient is required, we can neglect the minus sign, and, using $y=\frac{1}{2}$,

$$N_{r+1}>N_r,$$
$$\text{if}\quad \frac{22-r}{r}\times\frac{2}{3}>1,$$
$$\textit{i.e.}\quad \text{if}\quad 44-2r>3r,$$
$$\textit{i.e.}\quad \text{if}\quad 5r<44, \quad \textit{i.e.} \text{ if } r<8\tfrac{4}{5}.$$

As in Example 6, if $r > 8\frac{4}{5}$, $N_{r+1} < N_r$.

Thus $N_1, N_2, N_3, \dots N_9$ increase in magnitude,
and $N_9, N_{10}, N_{11}, \dots N_{22}$ decrease in magnitude.

Therefore N_9 is the numerically greatest coefficient.

Its value is $${}_{21}C_8\, 3^{21-8}(-4y)^8;$$
i.e. $${}_{21}C_8 \,.\, 3^{13}(2)^8, \text{ using } y = \tfrac{1}{2}.$$

Properties of the Coefficients in the Binomial Expansion.

The expansion of $(1+x)^n$ is
$$1 + {}_nC_1x + {}_nC_2x^2 + {}_nC_3x^3 + \dots + {}_nC_rx^r + \dots + x^n.$$

For convenience, this is sometimes written
$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_rx^r + \dots + C_nx^n,$$
where $C_0 = C_n = 1$, $C_1 = C_{n-1} = {}_nC_1$, $C_2 = C_{n-2} = {}_nC_2$, and so on.

Ex. 1. *Find the value of* $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$, *where*
$$(1+x)^n = C_0 + C_1x + \dots + C_nx^n.$$

We have $(1+x)^n = C_0 + C_1x + C_1x^2 + \dots + C_rx^r + \dots + C_nx^n$;
$$\therefore \left(1+\frac{1}{x}\right)^n = C_0 + C_1/x + C_2/x^2 + \dots + C_r/x^r + \dots + C_n/x^n.$$

The term independent of x in the product of the expressions on the right is $C_0^2 + C_1^2 + C_2^2 + \dots + C_r^2 + \dots + C_n^2$.

Therefore $C_0^2 + C_1^2 + C_2^2 + \dots + C_r^2 + \dots + C_n^2 =$ the term independent of x in
$$(1+x)^n\left(1+\frac{1}{x}\right)^n, \qquad \text{i.e. in } \frac{(1+x)^{2n}}{x^n}.$$

This is the coefficient of x^n in $(1+x)^{2n}$, i.e. ${}_{2n}C_n$,

and $${}_{2n}C_n = \frac{\underline{|2n}}{\underline{|n}\;\underline{|n}};$$
$$\therefore\ C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{\underline{|2n}}{(\underline{|n})^2}.$$

Ex. 2. *Shew that* $C_0 - \frac{1}{2}C_1 + \frac{1}{3}C_2 - \dots + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$.

The expression on the left is
$$1 - \frac{n}{2} + \frac{n(n-1)}{1\,.\,2\,.\,3} - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4} + \dots \text{ to } (n+1) \text{ terms,}$$

i.e.
$$\frac{1}{n+1}\left[(n+1)-\frac{(n+1)n}{1\,.\,2}+\frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}\right.$$
$$\left.-\frac{(n+1)n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}+\ldots \text{ to } (n+1) \text{ terms}\right],$$

i.e.
$$\frac{1}{n+1}[{}_{n+1}C_1-{}_{n+1}C_2+{}_{n+1}C_3-\ldots+(-1)^n{}_{n+1}C_{n+1}]\ \ldots\ldots(1)$$

Now $(1-1)^{n+1}=1-{}_{n+1}C_1+{}_{n+1}C_2-\ldots+(-1)^{n+1}{}_{n+1}C_{n+1}$;

$$\therefore\ 0=1-[{}_{n+1}C_1-{}_{n+1}C_2+\ldots+(-1)^n{}_{n+1}C_{n+1}];$$

$$\therefore \text{ from (1), } C_0-\tfrac{1}{2}C_1+\tfrac{1}{3}C_2-\ldots+(-1)^n\frac{C_n}{n+1}=\frac{1}{n+1}.$$

Ex. 3. *In the expansion of* $(px+qy)^n$, *find the excess of the odd terms over the even terms when* $x=1$ *and* $y=2$.

The expansion is $T_1+T_2+\ldots+T_r+\ldots+T_{n+1}$ where T_r is the r^{th} term.

We have to find $T_1-T_2+\ldots+(-1)^nT_{n+1}$ when $x=1$ and $y=2$.

That is, the value of the expansion of $(px-qy)^n$ when $x=1$ and $y=2$.

This is $(p-2q)^n$.

Ex. 4*. *Express in partial fractions*

$$\frac{\underline{|n}}{x(x+1)(x+2)\ldots(x+n)}.$$

Let
$$\frac{\underline{|n}}{x(x+1)(x+2)\ldots(x+n)}$$
$$\equiv\frac{A_0}{x}+\frac{A_1}{x+1}+\frac{A_2}{x+2}+\ldots+\frac{A_r}{x+r}+\ldots+\frac{A_n}{x+n}.$$

Multiplying by $(x+r)$, we have

$$\frac{\underline{|n}}{x(x+1)\ldots(x+r-1)(x+r+1)\ldots(x+n)}$$
$$\equiv\left(\frac{A_0}{x}+\frac{A_1}{x+1}+\ldots+\frac{A_{r-1}}{x+r-1}\right)(x+r)+A_r$$
$$+\left(\frac{A_{r+1}}{x+r+1}+\ldots+\frac{A_n}{x+n}\right)(x+r).$$

Now, as this is an identity, we may put $x=-r$.

The left-hand side becomes

$$\frac{\underline{|n}}{(-r)(-r+1)\ldots(-2)(-1).1(2)\ldots(n-r)},$$

which is $$(-1)^r\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}, \qquad \textit{i.e.}\ (-1)^r.\,C_r.$$

All the right-hand side vanishes, except the term A_r.

$$\therefore\ A_r=(-1)^r.\,C_r.$$

Hence

$$\begin{aligned} A_0&=(-1)^0C_0=C_0,\\ A_1&=(-1)^1C_1=-C_1,\\ A_2&=(-1)^2C_2=C_2,\\ &\text{etc.,}\qquad \text{etc.} \end{aligned}$$

$$\therefore\ \frac{\underline{|n}}{x(x+1)\ldots(x+n)}=\frac{C_0}{x}-\frac{C_1}{x+1}+\frac{C_2}{x+2}-\ldots+\frac{(-1)^nC_n}{x+n}.$$

Approximations.

Ex. 1. *Evaluate* $(0{\cdot}98)^6$ *to five decimal places.*

$$\begin{aligned}(0{\cdot}98)^6&=(1-x)^6 \text{ where } x=0{\cdot}02\\ &=1-6x+15x^2-20x^3+15x^4-6x^5+x^6.\end{aligned}$$

Now $$x^4=(0{\cdot}02)^4=0{\cdot}00000016;$$

$$\therefore\ 15x^4=0{\cdot}0000024.$$

Evidently terms beyond x^4 will not affect the required result;

$$\begin{aligned}\therefore\ (0{\cdot}98)^6 &\doteqdot 1-6(0{\cdot}02)+15(0{\cdot}02)^2-20(0{\cdot}02)^3+15(0{\cdot}02)^4\\ &\text{or } 1-0{\cdot}12+0{\cdot}006-0{\cdot}00016+0{\cdot}0000024\\ &\text{or } 1{\cdot}0060024-0{\cdot}12016\\ &\text{or } 0{\cdot}88584 \text{ to five decimal places.}\end{aligned}$$

Ex. 2. *Shew that, if x is so small that its square and higher powers can be neglected, the approximate value of*

$$(1+3x)^{15}(1-2x)^{12}\ \textit{is}\ 1+21x.$$

As x^2, x^3, ... and higher powers of x are negligible,

$(1+3x)^{15}$ is approximately $1+{}_{15}C_1(3x)$, *i.e.* $1+45x$,
and $(1-2x)^{12}$ is approximately $1+{}_{12}C_1(-2x)$, *i.e.* $1-24x$.

The approximate value of the given expression is, therefore, $(1+45x)(1-24x)$, which is $1+45x-24x$, since x^2 is negligible. This reduces to $1+21x$.

Binomial Theorem for any Index. It can be proved that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3 + \dots \quad \dots\dots(1)$$

for all values of n, provided the following conditions are satisfied :—The expression on the left which is raised to the power of n must contain two terms, (i) one of which must be *unity* and (ii) the other, x, must *lie between* -1 *and* $+1$.

An expression of the form $(a+b)^n$ can be put in the required form by writing it as $a^n(1+b/a)^n$ if $b<a$, or as $b^n(1+a/b)^n$ if $b>a$.

It should be noticed that if n is not a positive integer, the right-hand side of (1) is an infinite series. The meaning of identity (1) is that $(1+x)^n$ is the limit to which the sum of p terms of the series on the right tends as p tends to infinity.

Ex. 1. *Find series for* $(1-x)^{-1}$ *and* $(1-x)^{-2}$ *where* $-1<x<1$.

We have
$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{1\,.\,2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\,.\,2\,.\,3}(-x)^3 + \dots$$
$$= 1 + x + x^2 + x^3 + \text{etc.}$$
$$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{1\,.\,2}(-x)^2 + \frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}(-x)^3 + \dots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \dots\,.$$

These expansions are very useful and should be remembered.

Similarly $(1+x)^{-1} = 1 - x + x^2 - x^3 + \text{etc.}$

and $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \text{etc.}$

Ex. 2. *Find* $\sqrt{99{\cdot}8}$ *to* 8 *decimal places.*

$$\sqrt{99{\cdot}8} = (100 - 0{\cdot}2)^{\frac{1}{2}} = 100^{\frac{1}{2}}(1 - 0{\cdot}002)^{\frac{1}{2}}$$

i.e. $\sqrt{99{\cdot}8} = 10(1-x)^{\frac{1}{2}}$ where $x = 0{\cdot}002$.

Now, by the Binomial Theorem, since $x<1$,

$$(1-x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(-x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\,.\,2}(-x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}(-x)^3 + \text{etc.}$$
$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \dots\,.$$

Substituting 0·002 for x,

$$\tfrac{1}{2}x = 0{\cdot}001, \quad \tfrac{1}{8}x^2 = 0{\cdot}0000005, \quad \tfrac{1}{16}x^3 = 0{\cdot}0000000005,$$

and succeeding terms are smaller still;

$$\therefore \sqrt{99{\cdot}8} \doteqdot 10\{1 - 0{\cdot}0010005005\}$$

$$\text{or } 9{\cdot}998994995$$

$$\text{or } 9{\cdot}99899500 \text{ to 8 decimal places.}$$

Ex. 3. *Prove that, if x is positive and small,*

$$\frac{(2-3x^2)^2}{(1-x)^2(1+2x)} \textit{ is approximately equal to } 4 - \frac{8x^3}{(1-x)^2}.$$

$$\frac{(2-3x^2)^2}{(1-x)^2(1+2x)} = \frac{1}{(1-x)^2}\{(2-3x^2)^2(1+2x)^{-1}\}$$

$$= \frac{1}{(1-x)^2}\{(4 - 12x^2 + 9x^4)(1 - 2x + 4x^2 - 8x^3 + \ldots)\}$$

$$= \frac{1}{(1-x)^2}\{4(1 - 2x + 4x^2 - 8x^3 + \ldots)$$

$$- 12x^2(1 - 2x + 4x^2 - \ldots)\},$$

neglecting powers of x higher than the third.

$$\therefore \frac{(2-3x^2)^2}{(1-x)^2(1+2x)} = \frac{1}{(1-x)^2}\{4 - 8x + 4x^2 - 8x^3 + \ldots\}$$

$$\doteqdot \frac{1}{(1-x)^2}\{4(1-x)^2 - 8x^3\}$$

$$\text{or } 4 - \frac{8x^3}{(1-x)^2}.$$

EXERCISES XXXIII.

1. Expand

$(1+x)^5$, $(a-b)^4$, $(x+1/x)^7$, $(3x-y/3)^6$, $(\sqrt{3}+\sqrt{2})^4+(\sqrt{3}-\sqrt{2})^4$, and $(a+b)^{13}$ to six terms.

2. Find $(1{\cdot}01)^3$, $(0{\cdot}9)^5$, $(0{\cdot}999)^2$, $(3{\cdot}03)^4$.

3. Find the fifth term of $(1+x)^8$: the third term of $(2x-\frac{1}{3})^{10}$; the seventh term of $(2x^2+1/2x)^9$; the eighth term of $(p-a/3)^{21}$; the middle term of $(1+x)^{10}$; the middle term of $(y-x/3)^{12}$.

4. Find the two middle terms of $(x-1)^9$, $(x+1/y)^{13}$ and $(0{\cdot}1x+2)^7$.

5. Find the coefficient of x^6 in $(2+x)^{11}$; of x^{11} in $(x-b)^{17}$; of x^{15} in $(3x^2-x)^9$; of $\frac{1}{x}$ in $\left(2x-\frac{3}{x^2}\right)^5$; of x^4 in $\left(ax^2+\frac{b}{x^3}\right)^{17}$.

6. Find the term independent of x in $(x-1/x)^8$; in $(2x^3+1/x)^{16}$.

7. Find the coefficient of x^5 in $(1+x+x^3)^9$; of x^{11} in

$$(1-2x^5)(1+x)^{10};$$

and of x^6 in $(1-x+x^2-x^3)^5$.

8. Expand $(2a+3b)^5$. Shew that if $a=\frac{3}{2}$ and $b=\frac{2}{3}$, the sum of the odd terms in the expansion exceeds the sum of the even terms by 1. [I.C.E.]

9. Find the term which does not contain x in the expansion of $(2x+1/6x)^{10}$. [I.C.E.]

10. If x is so small that x^2, x^3 and higher powers of x may be neglected, find the approximate value of

$$(1+x)^4(1-3x)^5; \quad \frac{(1-x)^4(1+2x)^3}{1-3x} \text{ and of } \frac{(2+x)^2(x-2)^4}{(1-\frac{1}{2}x)^2}.$$

11. Shew that the approximate value of

$$(1+x)^5(1-x)^3(1-2x) \text{ is } 1-6x^2,$$

if powers of x above the second are negligible.

12. Find the numerically greatest coefficient in the expansion of $(x+y)^{11}$, $(2-\frac{1}{3}x)^{14}$ and $(1+3x)^{25}$.

13. Find the numerically greatest term in the expansion of $(4-3x)^7$ when $x=\frac{2}{3}$ and $(2x+5y)^{13}$ when $x=10$ and $y=2$.

14. Find the sum of the coefficients in the expansion of

$$\left(2x-\frac{1}{2}y\right)^7.$$

15. The 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal. Find x.

16. Shew that the sum of the coefficients of the odd powers of a is equal to the sum of the coefficients of the even powers of a in $(1+a)^n$.

17. Prove ${}_nC_r={}_{n-2}C_r+2\,.\,{}_{n-2}C_{r-1}+{}_{n-2}C_{r-2}$, by using the Binomial Theorem to expand both sides of the identity

$$(1+x)^n=(1+x)^2(1+x)^{n-2}.$$

18. Prove ${}_nC_r={}_{n-3}C_r+3\,.\,{}_{n-3}C_{r-1}+3\,.\,{}_{n-3}C_{r-2}+{}_{n-3}C_{r-3}$.

19. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_rx^r + \dots + C_nx^n$,

prove that

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n,$$

$$C_0 - C_1 + C_2 - \dots + (-1)^nC_n = 0,$$

$$C_1 + 2C_2 + 3C_3 + \dots + nC_n = n2^{n-1},$$

$$C_0C_1 + C_1C_2 + C_2C_3 + \dots + C_{n-1}C_n = {}^{2n}C_{n-1},$$

$$C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^nC_n^2$$

$$= 0, \text{ if } n \text{ is odd},$$

$$= \frac{(-1)^n \lfloor\underline{n}}{\lfloor\underline{n/2}\,\lfloor\underline{n/2}}, \text{ if } n \text{ is even},$$

$$1.2C_2 + 2.3C_3 + 3.4C_4 + \dots + (n-1)nC_n = n(n-1)2^{n-2},$$

$$\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \dots = 2^n/(n+1),$$

Examples 20-30 *are taken from papers set at the Inter. B.Sc.* [*L.U.*].

20. Prove the Binomial Theorem for a positive integral index.
Sum to n terms the series ${}^nC_1 \cdot \frac{1}{3} + {}^nC_2 \cdot \frac{1}{9} + {}^nC_3 \cdot \frac{1}{27} + \dots$.
Expand $\left(1 - x^2 + \frac{x^3}{3}\right)^5$ as far as the term containing x^6.

21. Prove the truth of the Binomial Theorem for a positive integral exponent.
If x be small, shew that

$$\frac{(1+x)^3\left(1 - \frac{x}{3}\right)^7}{\left(1 + \frac{x}{12}\right)^8} = 1 - \tfrac{67}{36}x^2 \text{ approximately.}$$

22. Prove the Binomial Theorem when the index is positive and integral.
Write down the coefficient of x^r in the expansion of $(a+x)^n$.
Find the coefficient of y^3 in the expansion of

$$\left(1 + x^2 - 3\frac{y^2}{x}\right)^7.$$

23. Prove the Binomial Theorem for a positive integral index.
Explain how the numerically greatest term in the expansion of $(a+b)^n$ may be found, and find it when $a = 100$, $b = -15$, $n = 15$.

24. Expand $(2x - 3y)^{17}$ to 5 terms by the Binomial Theorem.
If $y = \frac{1}{2}$, find the coefficient of x^3, and determine the numerically greatest term or terms when $x = 1$ and $y = \frac{4}{3}$.

25. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ where n is an integer, prove that

$$a_0{}^2 + a_1{}^2 + a_2{}^2 + \ldots + a_n{}^2 = {}_{2n}C_n,$$

and that $$a_0 - 2a_1 + 3a_2 - \ldots + (-1)^n(n+1)a_n = 0.$$

26. Prove that when n is a positive integer,

$$1 + n + \frac{n(n-1)}{\underline{|2}} + \frac{n(n-1)(n-2)}{\underline{|3}} + \ldots + n + 1 = 2^n.$$

Find also the value of the expression when the signs are alternately positive and negative.

27. Expand $(a+b)^n$ by the Binomial Theorem, n being a positive integer.

Shew that each term may be obtained from the preceding by multiplication by $(n-r+1)b/(ra)$, where r takes all integral values from 1 to n.

Hence shew that each term is less than the preceding term if

$$n < a/b.$$

Evaluate $(1{\cdot}00458)^{12}$ correct to five significant figures.

28. What is the sum of

$$1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9\ ?$$

Give its values when $x = 99$ and when $x = -0{\cdot}99$.

Prove the general law by which the coefficients of powers of x can be successively determined the one from the other in the general case.

29. In the expansion of $(2x+3y)^{17}$ by the Binomial Theorem, find (i) the greatest coefficient, (ii) the greatest coefficient of y^r when $x = \frac{1}{3}$, (iii) the greatest coefficient of x^r when $y = \frac{1}{4}$.

Find an approximate value of

$$(1+0{\cdot}002)^7(1-0{\cdot}003)^3 \div (1{\cdot}0005)^{12}.$$

30. Shew how to find the greatest term in $(x+y)^n$, and when $n = 17$, $x = 2$, $y = \frac{1}{3}$, compute its value. By expanding the binomials as far as x^2 obtain an approximate value of

$$(1+2x)^{16} + (1-3x)^{19}$$

when $x = 0{\cdot}02$.

31. Expand in ascending powers of x, as far as the 5th term,

$$\left(1 - \frac{x}{6}\right)^{\frac{1}{2}}.$$

State the conditions in which the expansion is legitimate.

Shew that $1 - \frac{1}{6} - \frac{1}{6}\cdot\frac{1}{12} - \frac{1}{6}\cdot\frac{1}{12}\cdot\frac{3}{18} -$ etc. to $\infty = \sqrt{\frac{2}{3}}$.

32. Explain the symbol ${}_nC_r$, and establish the formula by which it can be evaluated.

Prove $${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}.$$

Write down the 4th term, and also the rth term, of the expansion of $(2x - 3a)^n$, where $n > r$. [S.W.]

33. Write down the coefficient of x^4 in the expansion of

$$\left(x^2 + \frac{1}{x}\right)^{20}.$$

By means of the Binomial Theorem evaluate $(0{\cdot}995)^6$ correct to five places of decimals.

34. Expand $(1 - 3x + x^2)^4$ in powers of x as far as the term containing x^4.

What is the algebraic sum of the coefficients in the complete expansion? [B.U.]

35. State and prove the Binomial Theorem for a positive integral index.

The coefficients of three consecutive terms in the expansion of $(1+x)^m$ are in the ratio 1 : 7 : 42. Find m. [B.U.]

36. Find the term independent of x in the expansion of

$$\left(2x - \frac{1}{4x^2}\right)^{12}.$$

Evaluate $(4{\cdot}08)^{-\frac{1}{2}}$ correct to four places of decimals by means of the Binomial Theorem, and verify the result by the use of logarithms. [Durham.]

37. Expand $(8 - 6x)^{-\frac{1}{3}}$ in a series of ascending powers of x, stating the condition under which the expansion is legitimate. Hence shew that

$$1 + \frac{1}{8} + \frac{1\,.\,4}{8\,.\,16} + \frac{1\,.\,4\,.\,7}{8\,.\,16\,.\,24} + \dots \text{ to } \infty = \frac{2}{\sqrt[3]{5}}.$$ [Man.]

38. Prove that, if x is positive and small,

$$\frac{(2 - 3x^2)^2}{(1-x)^2(1+2x)} \text{ lies between 4 and } 4 - \frac{8x^3}{(1-x)^2}.$$

Hence, taking $x = 1/8$, prove that $\sqrt{5}$ is approximately equal to 125/56. [Man.]

39. Shew that the coefficient of x^n in the expansion of $(1 + ax + x^2)^n$ is

$$a^n + n(n-1)a^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(\underline{|2})^2}a^{n-4} + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-2r+1)}{(\underline{|r})^2}a^{n-2r} + \dots.$$

Find a definite expression for this series if $a = 2$. [L.U.]

Summation of Series. If the n^{th} term of a series is a simple function of n, its sum to any number of terms can often be found.

Ex. 1. *Find the sum of the first n natural numbers.*

The series is $1+2+3+\dots+n$.

This is clearly an arithmetic progression with common difference unity.

The sum is $\frac{n}{2}(n+1)$.

This formula should be tested and remembered.

Ex. 2. *Find the sum of the squares of the first n natural numbers.*

The series is $1^2+2^2+3^2+4^2+\dots+n^2$.

Now $(n+1)^3-n^3=3n^2+3n+1$.

Writing $n-1$ for n, we have

$$n^3-(n-1)^3=3(n-1)^2+3(n-1)+1.$$

Similarly,

$$(n-1)^3-(n-2)^3=3(n-2)^2+3(n-2)+1\ ;$$

and so on.

Finally, $2^3-1^3=3\,.\,1^2+3\,.\,1+1$.

There are n of these identities.

Adding them, we have

$$(n+1)^3-1^3=3(1^2+2^2+\dots+n^2)+3(1+2+\dots+n)+n. \quad\dots(1)$$

If S_2 denote the sum of the squares of the first n natural numbers, and S_1 the sum of the first n natural numbers, (1) becomes $(n+1)^3-1=3S_2+3S_1+n$;

or, using Example 1,

$$(n+1)^3-1-n=3S_2+\tfrac{3}{2}n(n+1)\ ;$$

$$\therefore\ 3S_2=(n+1)\{(n+1)^2-1-\tfrac{3}{2}n\}\ ;$$

$$\therefore\ S_2=\frac{n+1}{6}\{2n^2+n\}=\frac{n(n+1)(2n+1)}{6}.$$

Therefore the sum of the squares of the first n natural numbers is

$$\frac{n(n+1)(2n+1)}{6}.$$

This result should be tested and remembered.

Ex. 3. *Find the sum of the cubes of the first n natural numbers.*

We have $(n+1)^4-n^4=4n^3+6n^2+4n+1$.

Writing $(n-1)$ for n,

$$n^4-(n-1)^4=4(n-1)^3+6(n-1)^2+4(n-1)+1.$$

Similarly,

$$(n-1)^4-(n-2)^4=4(n-2)^3+6(n-2)^2+4(n-2)+1;$$

and so on.

Finally, $2^4-1^4=4.1^3+6.1^2+4.1+1$.

Adding, and denoting the sum of the cubes by S_3, we have

$$\begin{aligned}(n+1)^4-1&=4S_3+6S_2+4S_1+n\\&=4S_3+n(n+1)(2n+1)+2n(n+1)+n,\end{aligned}$$

using Examples 1 and 2.

$$\begin{aligned}\therefore\ 4S_3&=(n+1)^4-1-n-n(n+1)(2n+1)-2n(n+1)\\&=(n+1)\{(n+1)^3-1-2n^2-n-2n\}\\&=(n+1)\{n^3+n^2\};\end{aligned}$$

$$\therefore\ S_3=\left\{\frac{n(n+1)}{2}\right\}^2=S_1^2.$$

This result should be tested and remembered.

N.B.—Many series can be summed rapidly by using the formulae for S_1, S_2, S_3.

It is useful to denote the 1st, 2nd, ... nth terms of a series by the symbols $u_1, u_2, \dots u_n$.

Ex. 4. *Sum to n terms* $3.7+7.11+11.15+\dots$.

Note that the first factors of the terms form the arithmetic progression 3, 7, 11, of which the nth term is $3+4(n-1)$, *i.e.* $4n-1$. The second factor of any term is always 4 more than the first.

The given series is, therefore, $u_1+u_2+u_3+\dots+u_n$, where

$$u_n=(4n-1)(4n+3).$$

This formula for u_n can be tested as follows.

Putting $n=1$, $u_1=(4-1)(4+3)$, which is the first term.

Putting $n=2$, $u_2=(8-1)(8+3)$, which is the second term; and so on.

Now
$$u_n = 16n^2 + 8n - 3,$$
$$u_{n-1} = 16(n-1)^2 + 8(n-1) - 3.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$u_1 = 16(1)^2 + 8(1) - 3;$$
$$\therefore\ u_1 + u_2 + \dots + u_n = 16S_2 + 8S_1 - 3n$$
$$= \frac{16n(n+1)(2n+1)}{6} + 8 \cdot \frac{n(n+1)}{2} - 3n$$
$$= \frac{n}{3}[8(2n^2+3n+1) + 12(n+1) - 9]$$
$$= \frac{n}{3}[16n^2 + 36n + 11].$$

Ex. 5. *Let X_1 stand for the sum of n terms of an arithmetic progression whose common difference is d, and let X_2 stand for the sum of the squares of the same terms. Prove that*
$$nX_2 - X_1^2 = \tfrac{1}{12}n^2d^2(n^2-1).$$

Let a be the first term of the arithmetic progression.

Then $X_1 = \frac{n}{2}\{2a + (n-1)d\} = na + \frac{n(n-1)d}{2}$,
$$X_1^2 = n^2a^2 + n^2(n-1)da + \frac{n^2(n-1)^2d^2}{4}. \quad \dots\dots\dots\dots\dots\dots(1)$$

Again, $X_2 = a^2 + (a+d)^2 + (a+2d)^2 + \dots + \{a + (n-1)d\}^2$
$$= n(a^2) + 2ad\{1 + 2 + 3 + \dots + (n-1)\}$$
$$+ d^2\{1^2 + 2^2 + 3^2 + \dots + (n-1)^2\}$$
$$= na^2 + 2ad\frac{(n-1)n}{2} + d^2\frac{(n-1)(n)(2n-1)}{6};$$
$$\therefore\ nX_2 = n^2a^2 + adn^2(n-1) + d^2\frac{n^2(n-1)(2n-1)}{6}. \quad \dots\dots\dots(2)$$

From (1) and (2)
$$\frac{nX_2 - X_1^2}{d^2} = \frac{n^2(n-1)(2n-1)}{6} - \frac{n^2(n-1)^2}{4}$$
$$= \frac{n^2(n-1)\{4n - 2 - 3n + 3\}}{12}$$
$$= \frac{n^2(n^2-1)}{12}.$$

Ex. 6. *Sum to* 14 *terms the series*

$$1-\frac{13}{4}+\frac{13\,.\,12}{4\,.\,8}-\frac{13\,.\,12\,.\,11}{4\,.\,8\,.\,12}+\dots.$$

The series may be written

$$1+13\left(-\frac{1}{4}\right)+\frac{13\,.\,12}{1\,.\,2}\left(-\frac{1}{4}\right)^2+\frac{13\,.\,12\,.\,11}{1\,.\,2\,.\,3}\left(-\frac{1}{4}\right)^3+\dots,$$

which are the first four terms of the expansion of $\left(1-\frac{1}{4}\right)^{13}$

Hence the sum to 14 terms is $\left(1-\frac{1}{4}\right)^{13}$ or $\left(\frac{3}{4}\right)^{13}$.

Ex. 7. *Sum to infinity the series*

$$1-\frac{1}{5}+\frac{1\,.\,4}{5\,.\,10}-\frac{1\,.\,4\,.\,7}{5\,.\,10\,.\,15}+\dots.$$

The coefficient of any term in a binomial expansion is of the form

$$\frac{n(n-1)(n-2)\dots}{1\,.\,2\,.\,3\dots},$$

in which the successive factors of the numerator differ by unity, while the denominator is a factorial.

The given series may be written

$$1-\frac{1}{5}+\frac{\frac{1}{3}\cdot\frac{4}{3}}{1\,.\,2}\cdot\frac{3^2}{5^2}-\frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{1\,.\,2\,.\,3}\cdot\frac{3^3}{5^3}+\dots;$$

$$\textit{i.e.}\quad 1+\left(-\frac{1}{3}\right)\frac{3}{5}+\frac{(-\frac{1}{3})(-\frac{4}{3})}{1\,.\,2}\left(\frac{3}{5}\right)^2+\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{1\,.\,2\,.\,3}\cdot\left(\frac{3}{5}\right)^3+\dots,$$

which is $\left(1+\frac{3}{5}\right)^{-\frac{1}{3}}$, or $(8/5)^{-\frac{1}{3}}$, or $\sqrt[3]{5}/2$.

EXERCISES XXXIV.

1. Write down formulae for the sum, the sum of the squares, and the sum of the cubes of the first $n-1$ natural numbers.

2. Sum to n terms the following series:

(i) $1\,.\,2+2\,.\,3+3\,.\,4+\dots$;
(ii) $1\,.\,3+5\,.\,7+9\,.\,11+\dots$;
(iii) $1^2\,.\,2+2^2\,.\,3+3^2\,.\,4+\dots$;
(iv) $1^2+3^2+5^2+\dots$.

3. Find the sum of the series

$$1^2+2^2+\ldots+n^2.$$

In a geometrical progression the sum of the first n terms is s_1, the sum of the first $2n$ terms is s_2, and the sum of the first $3n$ terms is s_3; prove that

$$s_1{}^2+s_2{}^2=s_1(s_2+s_3).$$

[B.U.]

4. Find the sum of the squares of the first n positive numbers. If s_1, s_2 are respectively the sum, and the sum of the squares, of n successive positive integers beginning with a, shew that $ns_2-s_1{}^2$ is independent of a. [Man.]

5. If s be the sum of the first n natural numbers, prove that s^2 is the sum of their cubes.

Sum to n terms the series whose n^{th} term is $n(2n^2-1)$. If this sum be equal to the sum of the first m natural numbers, prove that m must be an odd number and that the sum of the first n natural numbers is $\frac{1}{2}(m+1)$. [B.U.]

6. Sum the series :

(i) $\frac{3}{4}-\frac{1}{2}+\frac{1}{3}-\ldots$ to 7 terms, and to infinity ;

(ii) $2.3+4.5+6.7+\ldots$ to n terms. [Man.]

7. Find the sum of the cubes of all the integers from 1 up to n. Sum the series :

(i) $1.2^2+2.3^2+3.4^2+\ldots+n(n+1)^2$;

(ii) $1^2.2+2^2.3+3^2.4+\ldots+n^2(n+1)$. [B.U.]

8. Sum to $2n$ terms the series $1^2-2^2+3^2-4^2+\ldots$.

9. Find the sum of the 4^{th} powers of the first n natural numbers.

10. Find the sum of the geometrical progression

$$a+ar+\ldots+ar^{n-1}.$$

If $$S=a+2ar+3ar^2+\ldots+nar^{n-1},$$
prove that $$S(1-r)+nar^n=a+ar+\ldots+ar^{n-1},$$
and hence find the value of S. [B.U.]

11. Sum the series :

(i) $\frac{3}{4}-1-1\frac{1}{4}-\ldots$ to $2n$ terms.

(ii) $12-\sqrt{72}+6-\ldots$ to infinity.

(iii) $1-\dfrac{n}{5}+\dfrac{n(n-1)}{5.10}-\dfrac{n(n-1)(n-2)}{5.10.15}+\ldots$ to $n+1$ terms.

[Durham.]

12. Sum to 8 terms the series $1+\frac{7}{4}+\frac{7.6}{4.8}+\frac{7.6.5}{4.8.12}+\ldots$.

13. Sum to infinity the series $1+\frac{1}{18}+\frac{1.3}{18.36}+\frac{1.3.5}{18.36.54}+\ldots$.

14. Sum to infinity the series

$$1-\tfrac{1}{6}-\tfrac{1}{6}\cdot\tfrac{1}{12}-\tfrac{1}{6}\cdot\tfrac{1}{12}\cdot\tfrac{3}{18}-\tfrac{1}{6}\cdot\tfrac{1}{12}\cdot\tfrac{3}{18}\cdot\tfrac{5}{24}-\ldots.$$

15. Sum to infinity the series

$$1+\tfrac{1}{12}+\tfrac{1}{12}\cdot\tfrac{5}{24}+\tfrac{1}{12}\cdot\tfrac{5}{24}\cdot\tfrac{9}{36}+\ldots.$$

16. Sum to infinity the series

$$\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{1}{8}+\tfrac{1}{2}\cdot\tfrac{1}{8}\cdot\tfrac{4}{12}+\tfrac{1}{2}\cdot\tfrac{1}{8}\cdot\tfrac{4}{12}\cdot\tfrac{7}{16}+\ldots.$$

CHAPTER XI.

VECTORS. APPLICATIONS TO TRIGONOMETRY. INVERSE NOTATION.

Vectors and Scalars. Quantities like displacements, which are not determined completely unless their *directions* as well as their *magnitudes* are specified, are called **directed quantities or vectors.**

When a body moves, its velocity at any instant has a definite direction ; when a body undergoes change of motion, the change is in a definite direction ; in all cases of motion *ideas of magnitude and direction are associated.* So, too, in all cases when forces are considered. If we are told that a body is being acted on by a force of 50 lb., the statement is incomplete and indefinite. We may ask if the body is pushed or pulled ; if it is being urged to the north-east or lifted vertically ; and only if the direction as well as the magnitude of the force are stated, is the statement complete, precise and definite.

Among vectors are classified : displacements, velocities, accelerations, forces, moments, magnetic and electric fields, and so on. On the other hand, such quantities as time, temperature, mass, work, etc., are essentially non-directive ; they are called **scalar quantities** or **scalars.**

Representation of Vectors. Since a vector is a quantity in which a direction is associated with a magnitude, it can be represented by a line whose direction is parallel to the direction of the vector and whose length measures the magnitude of the vector. To avoid ambiguity, an arrowhead should be placed on the line to indicate the **sense** of the vector.

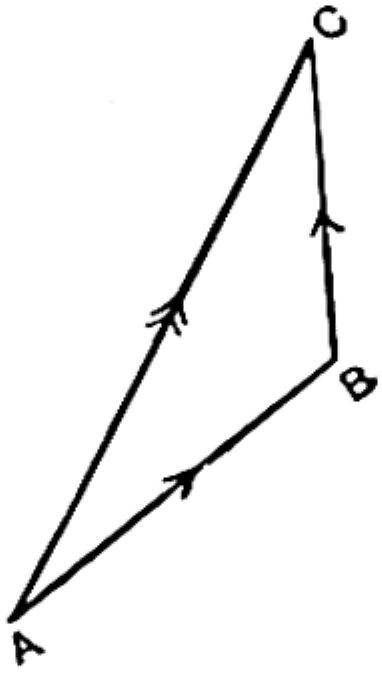


FIG. 123.
Addition of Vectors.

In Fig. 123, AC is a side of the triangle ABC, which has length only.

But $\overline{AC}$ means the vector whose magnitude is AC and whose direction is shewn by the arrowhead on the line AC.

Vectors may also be written as a_α, b_β, etc., in which the suffixes α, β, etc., give the directions referred to some **standard line** and a, b, etc., give the magnitudes.

Ex. 1. *Illustrate by a figure the vector* 2_{20°.

Let OZ be the standard direction (Fig. 124).

Take $ZOP = 20^\circ$, and from O set off 2 units along OP.

Then $\overline{OP}$ is 2_{20°.

If Q_1R_1, Q_2R_2, etc., be drawn equal and parallel to OP as in the figure, then

$$\overline{Q_1R_1} = \overline{Q_2R_2} = \overline{Q_3R_3} = \text{etc.} = 2_{20^\circ}.$$

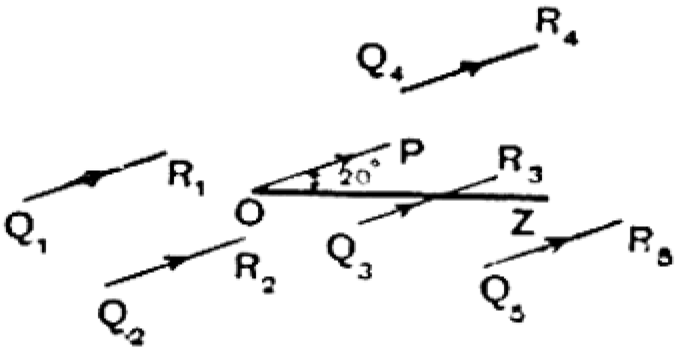


FIG. 124.

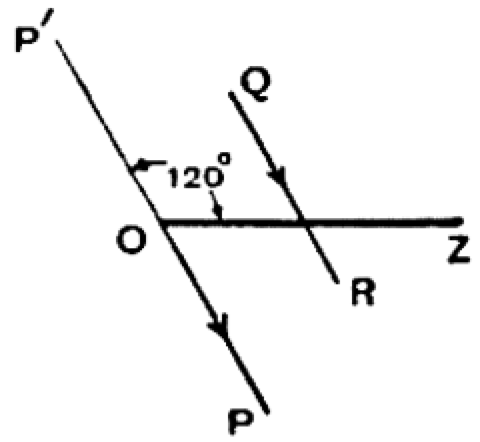


FIG. 125.

Ex. 2. *Illustrate by a figure the vector* -3_{120°.

Let OZ be the standard direction (Fig. 125).

Take $ZOP' = 120^\circ$, and from O set off 3 units *in the direction opposite to* OP', thus reaching the point P.

Then $\overline{OP} = -3_{120^\circ} = \overline{QR}$, where QR is any line equal and parallel to OP.

It should now be clear that -3_{120°, $3_{120^\circ+180^\circ}$, $3_{120^\circ-180^\circ}$ are different ways of writing the vector $\overline{OP}$.

In general, $\qquad -a_\alpha = a_{\alpha+180^\circ} = a_{\alpha-180^\circ}$.

Addition and Subtraction of Vectors. From Fig. 123, it is clear that a displacement represented by $\overline{AB}$, together with a displacement represented by $\overline{BC}$, are equivalent to the single displacement represented by $\overline{AC}$. Further, if $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ represent vectors of any other type than displacement, the first two are together equivalent to the third.

This may be expressed as follows:

$$\overline{AB} + \overline{BC} = \overline{AC}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where the symbol + may be translated as "followed by" or "together with."

Identity (1) at first sight appears to violate the theorem that two sides of a triangle are greater than the third. But this theorem, which would be expressed as $AB+BC>AC$, refers to the magnitudes of the lines only, and the symbol + has its usual arithmetical significance; whereas in identity (1) the directions of the lines are taken into account as well as their magnitudes, and the symbol + has an extended meaning.

The left-hand side of identity (1) is an example of vector addition, and the right-hand side, $\overline{AC}$, is the vector sum of $\overline{AB}$ and $\overline{BC}$.

It is customary to call $\overline{AB}$ and $\overline{BC}$ the **components** of $\overline{AC}$: while $\overline{AC}$ is the **resultant** of $\overline{AB}$ and $\overline{BC}$.

Ex. 1. *Evaluate* $7{\cdot}5_{45^\circ}+4{\cdot}3_{100^\circ}$ and $4{\cdot}3_{100^\circ}+7{\cdot}5_{45^\circ}$.

Let OZ be the standard direction. $\overline{OA}$ represents the vector $7{\cdot}5_{45^\circ}$ and $\overline{OB}$ the vector $4{\cdot}3_{100^\circ}$ (Fig. 126).

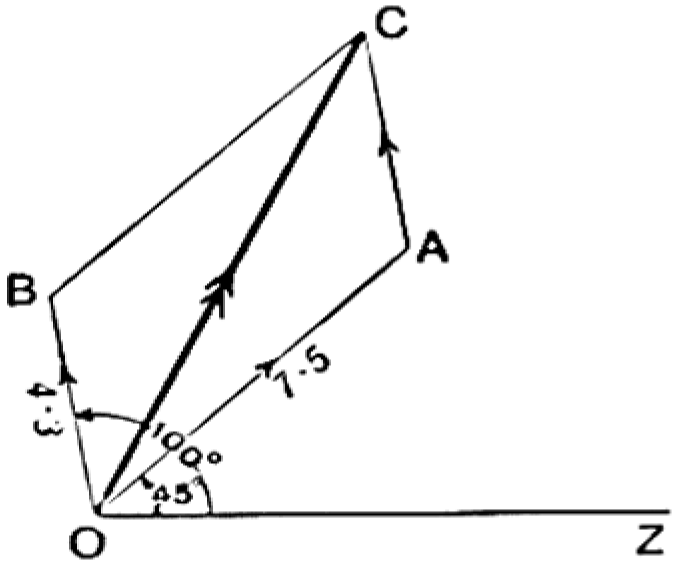


FIG. 126.

Completing the parallelogram $OACB$, $\overline{AC}$ is evidently $\overline{OB}$;

$$\therefore\ 7{\cdot}5_{45^\circ}+4{\cdot}3_{100^\circ}=\overline{OA}+\overline{AC}=\overline{OC}.$$

Again, $4{\cdot}3_{100^\circ}+7{\cdot}5_{45^\circ}=\overline{OB}+\overline{OA}=\overline{OB}+\overline{BC}=\overline{OC}$.

By measurement $OC\doteqdot 10{\cdot}6$ and $Z\hat{O}C\doteqdot 64^\circ$;

$$\therefore\ 7{\cdot}5_{45^\circ}+4{\cdot}3_{100^\circ}=4{\cdot}3_{100^\circ}+7{\cdot}5_{45^\circ}=10{\cdot}6_{64^\circ}.$$

In general, $a_\alpha+b_\beta+c_\gamma+\text{etc.}=b_\beta+a_\alpha+c_\gamma+\text{etc.}$

$$=c_\gamma+b_\beta+a_\alpha+\text{etc., etc.}$$

In words, the resultant of a number of vectors is independent of the order in which the vectors are added together.

Ex. 2. *Evaluate* $8{\cdot}6_{50^\circ} - 5{\cdot}4_{110^\circ}$.

The expression means a displacement of $8{\cdot}6_{50^\circ}$ together with a displacement of $-5{\cdot}4_{110^\circ}$.

In Fig. 127,

$$\overline{OA} = 8{\cdot}6_{50^\circ} \quad \text{and} \quad \overline{OB} = 5{\cdot}4_{110^\circ}.$$

BO is produced to C, so that $OC = BO$.

Then $$\overline{OC} = -5{\cdot}4_{110^\circ}.$$

From A, AD is drawn parallel and equal to OC;

$$\therefore \; \overline{AD} = -5{\cdot}4_{110^\circ};$$

$$\therefore \; 8{\cdot}6_{50^\circ} - 5{\cdot}4_{110^\circ} = \overline{OA} + \overline{AD} = \overline{OD} = 7{\cdot}52_{11^\circ 30'}.$$

From the above examples on addition and subtraction of vectors, it should be clear that $a_\alpha + b_\alpha = (a+b)_\alpha$,

and that $a_\alpha - b_\alpha = (a-b)_\alpha$.

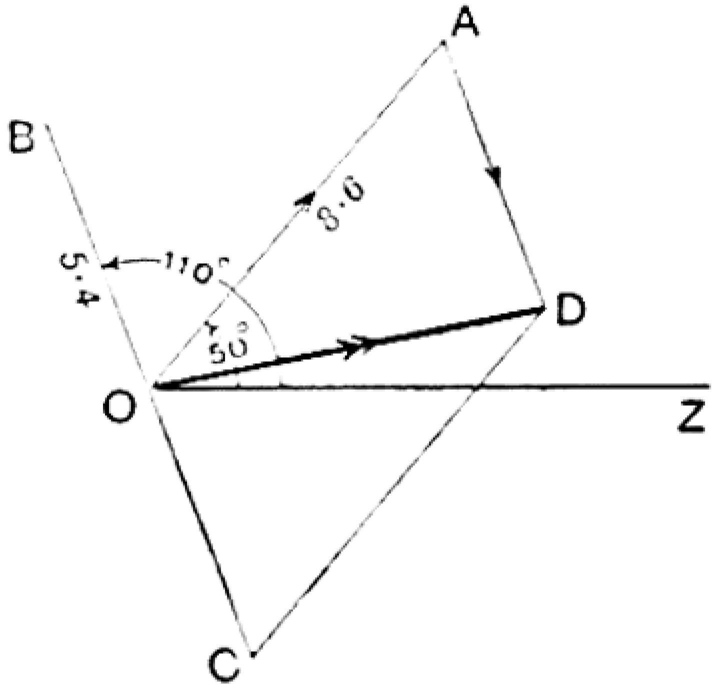


FIG. 127.

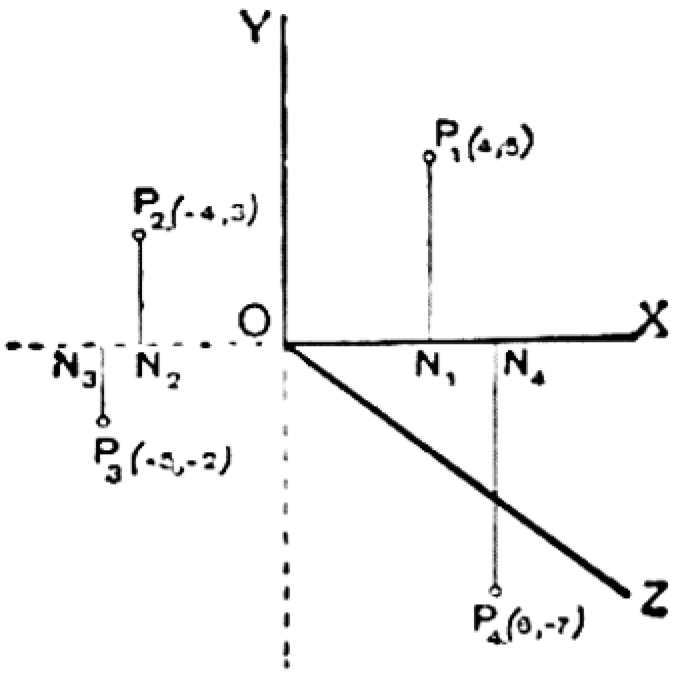


FIG. 128.

Coordinates and Vectors. Let (x, y) be the rectangular coordinates of a point P referred to axes OX, OY, and let OZ be the standard direction. Draw PN perpendicular to OX.

Then if $Z\hat{O}X = \phi$, $\quad \overline{ON} = x_\phi$, $\quad \overline{NP} = y_{\phi + \frac{\pi}{2}}$.

This is illustrated in Fig. 128.

For the point P_1, $x = 4$;

$\therefore$ x_ϕ = a vector in direction OX with magnitude 4, *i.e.* $\overline{ON}_1$.

For the point P_2, $x = -4$;

$\therefore$ x_ϕ = a vector in direction OX with magnitude -4, *i.e.* $\overline{ON}_2$.

For the point P_3, $x = -5$;

$\therefore$ x_ϕ = a vector in direction OX with magnitude -5, *i.e.* $\overline{ON_3}$.

For the point P_4, $x = 6$;

$\therefore$ x_ϕ = a vector in direction OX with magnitude 6, *i.e.* $\overline{ON_4}$.

Again, for the point P_1, $y = 5$;

$\therefore$ $y_{\phi+\frac{\pi}{2}}$ = a vector in direction OY with magnitude 5, *i.e.* $\overline{N_1P_1}$.

For the point P_2, $y = 3$;

$\therefore$ $y_{\phi+\frac{\pi}{2}}$ = a vector in direction OY with magnitude 3, *i.e.* $\overline{N_2P_2}$.

For the point P_3, $y = -2$;

$\therefore$ $y_{\phi+\frac{\pi}{2}}$ = a vector in direction OY with magnitude -2,

i.e. $\overline{N_3P_3}$.

For the point P_4, $y = -7$;

$\therefore$ $y_{\phi+\frac{\pi}{2}}$ = a vector in direction OY with magnitude -7,

i.e. $\overline{N_4P_4}$.

Projection of Vectors. The projection of a vector $\overline{AB}$ in any direction is the vector $\overline{ab}$, where a, b, are the feet of the perpendiculars let fall from A, B, upon any line parallel to the given direction.

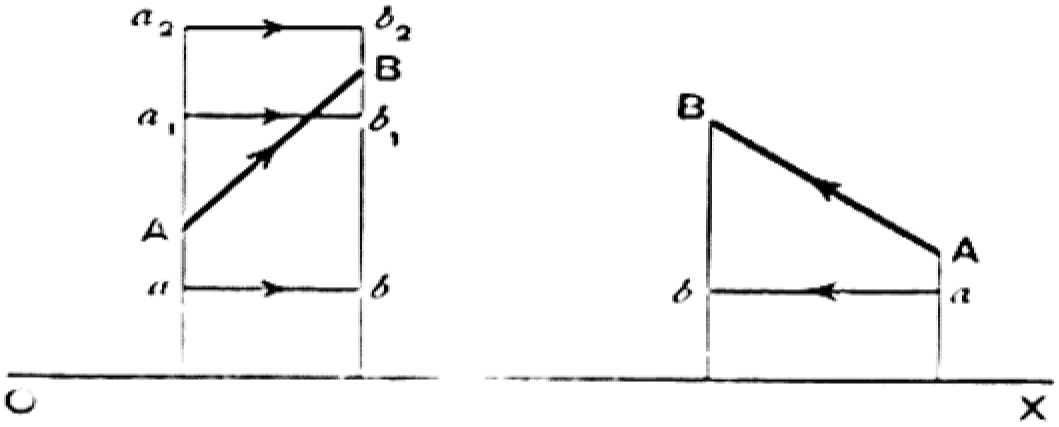


FIG. 129.

Thus in Fig. 129 the projection of $\overline{AB}$ in the direction OX is $\overline{ab}$, $\overline{a_1b_1}$ or $\overline{a_2b_2}$.

Again, in Fig. 128,

$$x_\phi = \overline{ON} = \text{the projection of } \overline{OP} \text{ in the direction } OX.$$

$$y_{\phi+\frac{\pi}{2}} = \overline{NP} = \text{the projection of } \overline{OP} \text{ in the direction } OY.$$

Projection and the Circular Functions. Let OX be any line and let OY make $+90°$ with OX (Fig. 130). Let (x, y) be the rectangular coordinates of P referred to OX and OY as axes, and let (r, θ) be the polar coordinates referred to O as origin and OX as initial line.

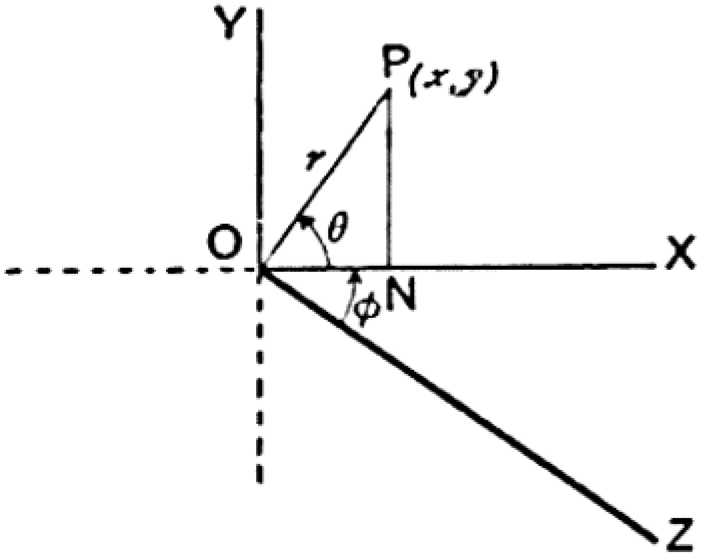


FIG. 130.

Then, by definition (p. 152), $\cos\theta = x/r$,
$\sin\theta = y/r$.

Let OZ be the standard direction, and let $Z\hat{O}X = \phi$.

Then $\overline{OP} = r_{(\theta+\phi)}$.

Now the projection of $\overline{OP}$ in the direction $OX = \overline{ON}$

$$= x_{\phi}$$
$$= (r\cos\theta)_{\phi},$$

i.e. the projection of $r_{\theta+\phi}$ in the direction ϕ is the vector $(r\cos\theta)_{\phi}$. **Thus the projection of a vector in any direction is another vector whose direction coincides with the given direction and whose magnitude is the magnitude of the given vector multiplied by the cosine of the angle between the two directions.**

Again, the projection of $\overline{OP}$ in the direction OY

$$= \overline{NP}$$
$$= y_{\phi+\frac{\pi}{2}}$$
$$= (r\sin\theta)_{\phi+\frac{\pi}{2}}.$$

Thus if OX coincides with the standard direction, $\phi = 0$, the projection of $\overline{OP}$ in the direction OX is $r\cos\theta_0$ and the projection of $\overline{OP}$ in the direction OY is $r\sin\theta_{\frac{\pi}{2}}$.

Ex. 1. *If OZ be the standard direction, verify the formula for the projection in the direction OZ of a vector in the following cases :* 2_{50°, 3_{150°.

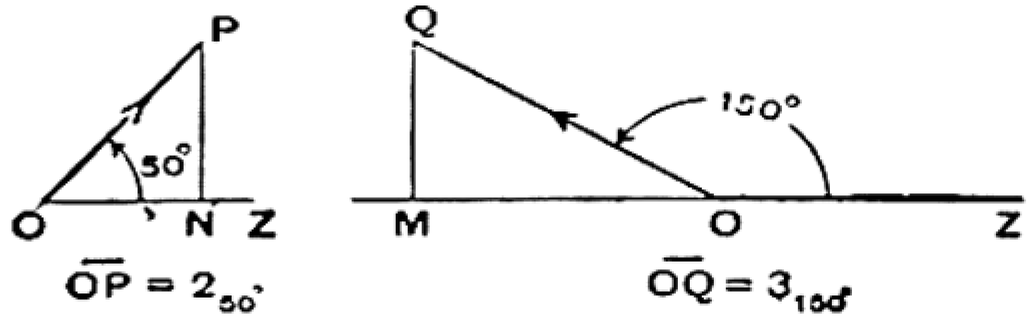

P
50°
O
N
Z
$\overline{OP} = 2_{50^\circ}$
Q
150°
M
O
Z
$\overline{OQ} = 3_{150^\circ}$

FIG. 131.

In Fig. 131 $\overline{OP}$ is the vector 2_{50°, $\overline{OQ}$ is the vector 3_{150°.

By the above formula, the projection of $\overline{OP}$ in the direction OZ is $2 \cos 50^\circ_{0^\circ}$, *i.e.* $1{\cdot}2856_{0^\circ}$.

By drawing, this projection is $\overline{ON}$. The direction of $\overline{ON}$ is $\overline{OZ}$ and its length is 1·2856.

Again, by formula, the projection of $\overline{OQ}$ in the direction OZ is $2 \cos 150^\circ_{0^\circ}$, *i.e.* $-1{\cdot}7320_{0^\circ}$.

By drawing, this projection is $\overline{OM}$. The direction of $\overline{OM}$ is *opposite* to that of $\overline{OZ}$ and its length is 1·7320.

Ex. 2. *Verify the formula for the projection of a vector in the following cases :* 2_{230° *in the direction* 40° ; -3_{300° *in the direction* 20°.

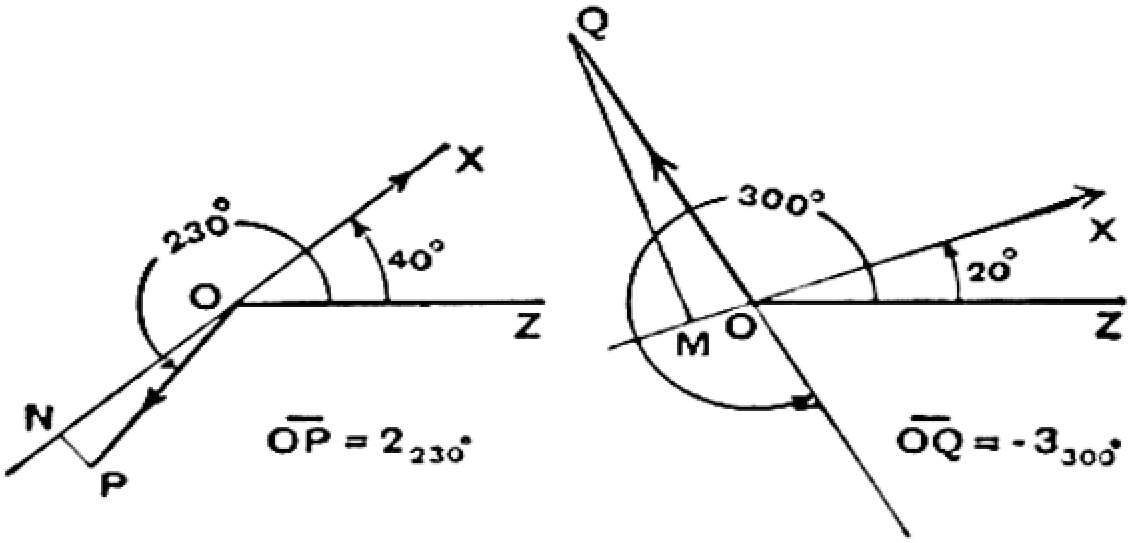

X
230°
40°
O
Z
N
P
$\overline{OP} = 2_{230^\circ}$
Q
300°
X
20°
M
O
Z
$\overline{OQ} = -3_{300^\circ}$

FIG. 132.

In Fig. 132 $\overline{OP}$ is the vector 2_{230° and OX shews the direction 40°.

The angle between OX and OP is $230^\circ - 40^\circ$, *i.e.* 190°.

By formula, the projection of $\overline{OP}$ in direction OX is $2 \cos 190^\circ_{40^\circ}$, *i.e.* $-1{\cdot}9696_{40^\circ}$.

By drawing, the projection is $\overline{ON}$, of which the direction is $40^\circ + 180^\circ$ and the length is 1·97.

Again, $\overline{OQ}$ is the vector -3_{300° and OX' shews the direction 20°.

The angle between the direction of the given vector, viz. -3_{300°, and the direction of OX' is $300^\circ - 20^\circ$, *i.e.* 280°.

By formula, the projection of $\overline{OQ}$ in the direction OX' is $-3\cos 280^\circ{}_{20^\circ}$, *i.e.* $-0{\cdot}5208_{20^\circ}$.

By drawing, the projection is $\overline{OM}$, of which the direction is $20^\circ + 180^\circ$ and the length is $0{\cdot}5208$.

The sum of the projections in a given direction of any number of vectors is equal to the projection of their resultant in that direction.

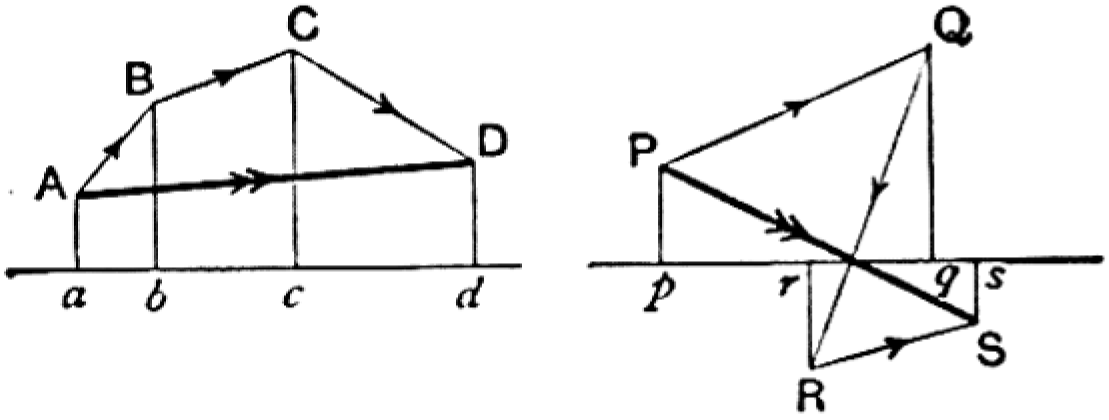


FIG. 133.

In Fig. 133 $\overline{AD}$ is the resultant of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$.

The sum of the projections of these components is

$$\overline{ab} + \overline{bc} + \overline{cd}, \quad i.e.\ \overline{ad},$$

which is the projection of $\overline{AD}$.

Again, $\overline{PS}$ is the resultant of $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$.

The sum of the projections of these components is

$$\overline{pq} + \overline{qr} + \overline{rs}, \quad i.e.\ \overline{ps},$$

which is the projection of $\overline{PS}$.

Clearly the sum of the projections in any direction of a number of vectors represented by the sides of any polygon taken in order is zero.

Ex. 1. *Calculate the vector sum*

$$5{\cdot}2_{10^\circ} + 3{\cdot}7_{50^\circ} + 4{\cdot}1_{110^\circ}.$$

Let $\overline{OP_1}$, $\overline{OP_2}$, $\overline{OP_3}$ be the three vectors, and let OZ be the standard direction (Fig. 134). If r_θ is their resultant, then the projection of r_θ in the direction OZ is the sum of the projections of the given vectors;

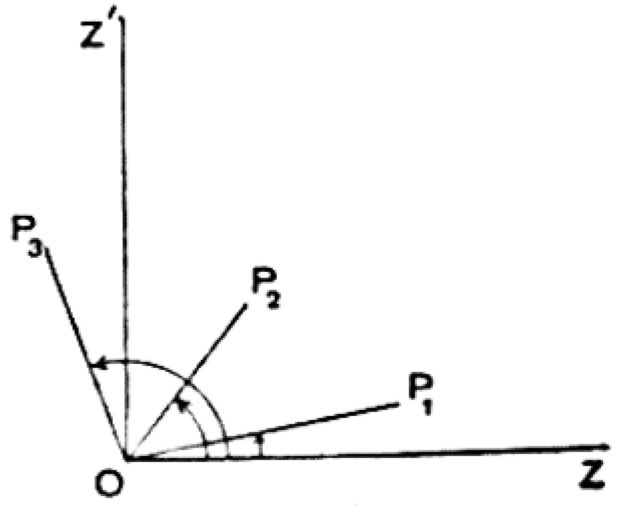


FIG. 134.

$$\therefore\ r\cos\theta = 5{\cdot}2\cos 10^\circ + 3{\cdot}7\cos 50^\circ + 4{\cdot}1\cos 110^\circ \doteqdot 6{\cdot}097.$$

Again, the projection of r_θ in the direction OZ' = the sum of the projections of the given vectors ;

$$\therefore\ r \sin\theta = 5{\cdot}2 \sin 10^\circ + 3{\cdot}7 \sin 50^\circ + 4{\cdot}1 \sin 110^\circ \doteqdot 7{\cdot}590 ;$$

$$\therefore\ \tan\theta \doteqdot \frac{7{\cdot}590}{6{\cdot}097} ; \text{ whence } \theta \doteqdot 51^\circ\, 13',$$

$$\text{and} \qquad r \doteqdot \sqrt{6{\cdot}097^2 + 7{\cdot}590^2} \doteqdot 9{\cdot}73.$$

Therefore the vector sum is $9{\cdot}73_{51^\circ\, 13'}$.

Ex. 2. *Find* P_θ *if* $P_\theta = 4{\cdot}1_{30^\circ} + 2{\cdot}8_{80^\circ} - 3{\cdot}0_{150^\circ}$.

As in the last example,

$$P \cos\theta = 4{\cdot}1 \cos 30^\circ + 2{\cdot}8 \cos 80^\circ + (-3{\cdot}0 \cos 150^\circ) \doteqdot 6{\cdot}635.$$

$$P \sin\theta = 4{\cdot}1 \sin 30^\circ + 2{\cdot}8 \sin 80^\circ + (-3{\cdot}0 \sin 150^\circ) \doteqdot 3{\cdot}307.$$

$$\therefore\ \tan\theta \doteqdot \frac{3{\cdot}307}{6{\cdot}635} ; \text{ whence } \theta \doteqdot 26^\circ\, 30',$$

$$\text{and} \qquad r \doteqdot \sqrt{6{\cdot}635^2 + 3{\cdot}307^2} \doteqdot 7{\cdot}41 ;$$

$$\therefore\ P_\theta \doteqdot 7{\cdot}41_{26^\circ\, 30'}.$$

Ex. 3. *Find the resultant of the following system of forces acting at a point :* 5·2 *lb. in the direction E.* 10° *N.*, 3·7 *lb. in the direction E.* 50° *N.*, *and* 4·1 *lb. in the direction N.* 20° *W.*

Taking the standard direction as due E., the problem is to find the sum of the vectors $5{\cdot}2_{10^\circ}$, $3{\cdot}7_{50^\circ}$ and $4{\cdot}1_{110^\circ}$. (See Fig. 134.)

As in Example 1, the sum is the vector $9{\cdot}73_{51^\circ\, 13'}$.

The resultant force is therefore 9·73 lb. in the direction E. 51° 13′ N.

Ex. 4. *A ship sails* 10 *miles to the South-West, but it is in a current which simultaneously carries it* 3 *miles due North. Find the actual motion of the ship in magnitude and direction.*

Let the motion of the ship be r miles in the direction θ° West of North ; then, taking the standard direction as due N., we have

$$r_\theta = 3_0 + 10_{135^\circ}.$$

As in Example 1,

$$r \cos\theta = 3 + 10 \cos 135^\circ \doteqdot -4{\cdot}07,$$

$$r \sin\theta = 0 + 10 \sin 135^\circ \doteqdot 7{\cdot}07 ;$$

$$\therefore\ r \doteqdot \sqrt{4{\cdot}07^2 + 7{\cdot}07^2}, \quad i.e.\ 8{\cdot}2$$

$$\text{and} \qquad \tan\theta = -\frac{7{\cdot}07}{4{\cdot}07}, \quad \text{whence } \theta \doteqdot 119^\circ\, 56'.$$

The actual motion of the ship is therefore 8·2 miles in a direction 29° 56′ South of West.

EXERCISES XXXV.

1. Find the resultant of $2{\cdot}4_{120^\circ}$ and $3{\cdot}7_{30^\circ}$; $5{\cdot}2_{25^\circ}$ and $4{\cdot}5_{60^\circ}$; $2{\cdot}0_{0^\circ}$ and $4{\cdot}3_{180^\circ}$; $3{\cdot}7_{140^\circ}$ and $2{\cdot}1_{85^\circ}$.

2. Shew that the resultant of $3{\cdot}7_{140^\circ}$, $2{\cdot}1_{90^\circ}$ and $1{\cdot}9_{0^\circ}$ is independent of the order in which they are taken.

3. Shew that the resultant of $2{\cdot}7_{250^\circ}$, $3{\cdot}1_{65^\circ}$ and $-3{\cdot}2_{87^\circ}$ is independent of the order in which they are taken.

4. Evaluate (i) $3{\cdot}1_{60^\circ} - 2{\cdot}5_{80^\circ}$; (ii) $-4{\cdot}1_{100^\circ} + 6{\cdot}2_{30^\circ}$.

5. A ship sails 8·4 miles to the North-East, but it is in a current which simultaneously carries it 4·2 miles due West. Find the actual motion of the vessel in magnitude and direction.

6. Find the resultant speed of the components 3·8 ft. per sec. E. 10° N., and 4·5 ft. per sec. E. 70° N.

7. What single force, and in what direction, is equivalent to the following components?

(*a*) 20 lb. E. and 30 lb. N.
(*b*) 5·7 lb. E. 10° N. and 8·3 lb. N. 30° W.
(*c*) 8·2 tons S. 33° E. and 10·2 tons W. 17° S.
(*d*) 9·8 kg. W. 13° N. and 7·7 kg. S. 22° E.

8. One component of the vector $7{\cdot}5_{38^\circ}$ is $10{\cdot}7_{84^\circ}$. Find the other component.

9. One component of the vector $10{\cdot}7_{84^\circ}$ is $7{\cdot}5_{38^\circ}$. Find the other component.

10. What are the vertical and horizontal components of a force of 5·8 tons which makes an angle of 27° with the vertical?

11. Three forces of 6·8 lb., 5·4 lb. and 8·4 lb. act upon a point in such a way that their resultant is zero. Determine the angles between the directions of the forces.

12. Find the resultant of the following system of forces: 3·7 lb. due E., 3·5 lb. N.E., 3 lb. N.W., 1·5 lb. due W. and 2·7 lb. due S.

13. There are three vectors in one plane, $A = 2{\cdot}5_{28^\circ}$, $B = 3{\cdot}1_{98^\circ}$, $C = 2{\cdot}7_{117^\circ}$. Find

(i) $A+B+C$; (ii) $B+C-A$; (iii) $C-B+A$;
(iv) $A-B$; (v) $C-B$.

Prove by drawing that $A-B-C=A-(B+C)$, also that

$$A-B+C=A-(B-C).$$

14. Find P_θ if $P_\theta = 3{\cdot}7_{35^\circ} - 5{\cdot}2_{90^\circ} - 4{\cdot}5_{180^\circ}$.

15. Express the following vectors in terms of vectors of which the directions are 0° and 90° referred to the standard direction.

(i) $5{\cdot}7_{53^\circ}$. (ii) $17{\cdot}8_{128^\circ}$. (iii) $67{\cdot}0_{230^\circ}$.
(iv) $116{\cdot}0_{320^\circ}$. (v) $0{\cdot}87_{25^\circ}$. (vi) $1{\cdot}56_{90^\circ}$.
(vii) $8{\cdot}45_{180^\circ}$. (viii) $47{\cdot}4_{270^\circ}$. (ix) $0{\cdot}084_{165^\circ}$.
(x) 783_{295°. (xi) $7{\cdot}83_{56^\circ}$. (xii) $7{\cdot}25_{135^\circ}$.

16. Express as vectors the projections in the standard direction of

$$17{\cdot}8_{35^\circ},\quad 67{\cdot}0_{25^\circ},\quad 1{\cdot}56_{120^\circ},\quad 7{\cdot}83_{156^\circ},\quad 72{\cdot}5_{90^\circ},$$
$$84{\cdot}5_{270^\circ},\quad 15{\cdot}6_{180^\circ},\quad 116_{295^\circ},\quad 783_{32^\circ},\quad 0{\cdot}085_{60^\circ}.$$

17. Express as vectors in a direction making 90° with the standard direction the projections of the vectors of Ex. 16.

18. The coordinates of a point P are x, y referred to perpendicular axes OX, OY through the origin O. Express $\overline{OP}$ in terms of vectors of which the directions are OX and OY.

Give results when the coordinates of P are $2{\cdot}3$, $1{\cdot}7$; $-2{\cdot}5$, $2{\cdot}4$; $-3{\cdot}1$, $-2{\cdot}0$; and $2{\cdot}1$, $-1{\cdot}8$.

19. If the axis OX in Exercise 18 makes an angle α with the standard direction OZ, find the projections of $\overline{OP}$ in the direction OZ and in a direction perpendicular to OZ for each of the given points.

20. Plot the following vectors $\overline{OP}$ on squared paper, and find the coordinates of P in each case : $7{\cdot}2_{55^\circ}$, $1{\cdot}4_{100^\circ}$, $5{\cdot}8_{130^\circ}$, $4{\cdot}3_{200^\circ}$, $3{\cdot}7_{285^\circ}$. $\overline{OX}$ is the standard direction.

21. Prove that $\cos\alpha+\cos\left(\alpha+\frac{2\pi}{3}\right)+\cos\left(\alpha+\frac{4\pi}{3}\right)=0$ and

$$\sin\alpha+\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)=0$$

by projecting $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ in a direction making α with $\overline{AB}$ and in a perpendicular direction, where ABC is an equilateral triangle.

22. $ABCDE$ is a regular pentagon. By projecting $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ in a direction making α with AB and in a perpendicular direction, obtain the following identities :

(i) $\cos\alpha+\cos(\alpha+\tfrac{2}{5}\pi)+\cos(\alpha+\tfrac{4}{5}\pi)+\cos(\alpha+\tfrac{6}{5}\pi)+\cos(\alpha+\tfrac{8}{5}\pi)=0$;

(ii) $\sin\alpha+\sin(\alpha+\tfrac{2}{5}\pi)+\sin(\alpha+\tfrac{4}{5}\pi)+\sin(\alpha+\tfrac{6}{5}\pi)+\sin(\alpha+\tfrac{8}{5}\pi)=0$.

23. Obtain identities similar to those in Exercise 22 by considering a regular hexagon.

Circular Functions of Compound Angles. If A, B are any angles,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \text{..............(1)}$$

and
$$\sin(A+B) = \sin A \cos B + \cos A \sin B. \quad \text{..............(2)}$$

In Fig. 135 let OZ be the standard direction; $Z\hat{O}X = A$ and $X\hat{O}P = B$. Then $Z\hat{O}P = A + B$, and if $OP = r$, the polar coordinates of P referred to OX are r, B, and the rectangular coordinates of P referred to OX and a line making $+90°$ with OX are x, y, where $x = r\cos B$ and $y = r\sin B$ whatever the magnitude of B may be.

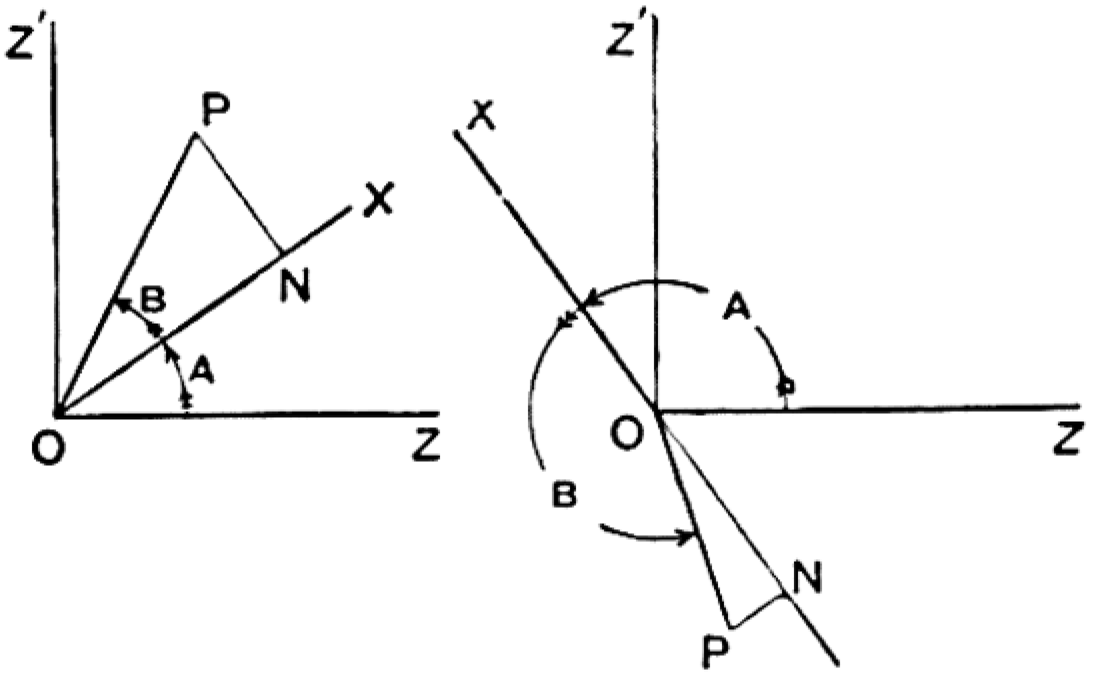


FIG. 135.

As OZ is the standard direction,

$$\overline{OP} = r_{A+B}; \quad \overline{ON} = x_A; \quad \overline{NP} = y_{A+\frac{\pi}{2}};$$

and the projection of $\overline{OP}$ in any direction is equal to the sum of the projections of $\overline{ON}$ and $\overline{NP}$ in that direction.

I. Projecting in the direction OZ, we have

$$\begin{aligned} r\cos(A+B) &= x\cos A + y\cos(A+\tfrac{1}{2}\pi) \\ &= r\cos B\cos A + r\sin B\cos(A+\tfrac{1}{2}\pi); \end{aligned}$$

$$\therefore \cos(A+B) = \cos B\cos A + \sin B\cos(A+\tfrac{1}{2}\pi).$$

Now
$$\cos(A+\tfrac{1}{2}\pi) = -\sin A, \text{ (see p. 168)};$$

$$\therefore \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

II. Projecting in the direction OZ', which makes $90°$ with OZ, we have

$$\begin{aligned} r\sin(A+B) &= x\sin A + y\sin(A+\tfrac{1}{2}\pi), \\ &= r\cos B\sin A + r\sin B\sin(A+\tfrac{1}{2}\pi); \end{aligned}$$

$$\therefore \sin(A+B) = \cos B\sin A + \sin B\cos A.$$

In the preceding, no restriction whatever has been placed on the values of A and B; they may be angles of any magnitude, whether positive or negative.

In (1) write $A=\theta$ and $B=-\phi$.

Then $\cos(\theta-\phi)=\cos\theta\cos(-\phi)-\sin\theta\sin(-\phi)$;

$$\therefore \text{ (see p. 166), } \cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi. \quad \ldots(3)$$

Similarly (2) gives

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi. \quad \ldots\ldots\ldots\ldots(4)$$

These formulae may also be proved directly as follows:

In Fig. 136, let OZ be the standard direction; $Z\hat{O}X=A$ and $X\hat{O}P=-B$. Then $Z\hat{O}P=A-B$, and if $OP=r$, the polar coordinates of P referred to OX are $(r, -B)$ and the rectangular coordinates of P are (x, y) referred to OX and a line OY making 90° with OX.

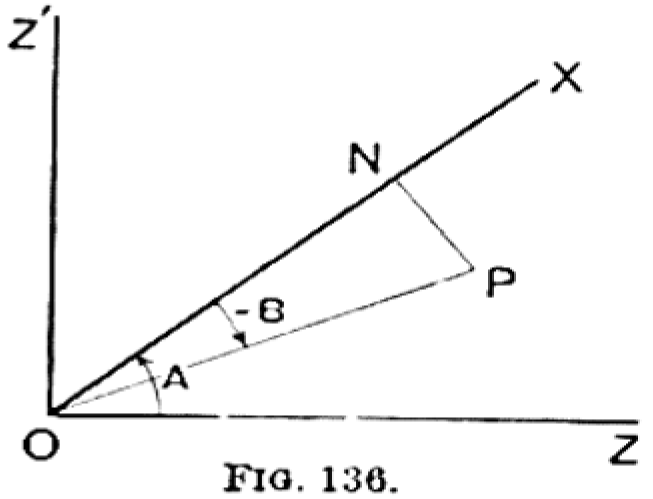


FIG. 136.

As OZ is the standard direction,

$$\overline{OP}=r_{A-B}, \quad \overline{ON}=x_A \quad \text{and} \quad \overline{NP}=y_{A+\frac{\pi}{2}}.$$

Further, the projection of $\overline{OP}$ in any direction is equal to the sum of the projections of $\overline{ON}$ and $\overline{NP}$ in that direction.

I. Projecting in the direction OZ, we have

$$\begin{aligned} r\cos(A-B) &= x\cos A+y\cos(A+\tfrac{1}{2}\pi), \\ &= r\cos(-B)\cos A+r\sin(-B)\cos(A+\tfrac{1}{2}\pi); \\ \therefore \cos(A-B) &= \cos(-B)\cos A+\sin(-B)(-\sin A) \\ &= \cos B\cos A+\sin B\sin A. \end{aligned}$$

II. Projecting in the direction OZ', which makes 90° with OZ,

$$\begin{aligned} r\sin(A-B) &= x\sin A+y\sin(A+\tfrac{1}{2}\pi) \\ &= r\cos(-B)\sin A+r\sin(-B)\cos A; \\ \therefore \sin(A-B) &= \cos(-B)\sin A+\sin(-B)\cos A \\ &= \sin A\cos B-\cos A\sin B. \end{aligned}$$

Ex. 1. *If* $\sin\alpha = \frac{4}{5}$ *and* $\cos\beta = \frac{12}{13}$, *find* $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$, $\sin(\alpha-\beta)$ *and* $\cos(\alpha-\beta)$, *where* α *is an obtuse angle and* β *is an acute angle.*

Since $\sin\alpha = \frac{4}{5}$, $\quad \cos^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$.

As α is obtuse, $\quad \cos\alpha = -\frac{3}{5}$.

Since $\cos\beta = \frac{12}{13}$, $\quad \sin^2\beta = 1 - \frac{144}{169} = \frac{25}{169}$.

As β is acute, $\quad \sin\beta = \frac{5}{13}$.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \tfrac{4}{5}\cdot\tfrac{12}{13} - \tfrac{3}{5}\cdot\tfrac{5}{13} = \tfrac{33}{65}.$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\tfrac{3}{5}\cdot\tfrac{12}{13} - \tfrac{4}{5}\cdot\tfrac{5}{13} = -\tfrac{56}{65}.$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \tfrac{4}{5}\cdot\tfrac{12}{13} + \tfrac{3}{5}\cdot\tfrac{5}{13} = \tfrac{63}{65}.$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = -\tfrac{3}{5}\cdot\tfrac{12}{13} + \tfrac{4}{5}\cdot\tfrac{5}{13} = -\tfrac{16}{65}.$$

Ex. 2. *Expand* $\sin(A+B+C)$.

$$\begin{aligned}\sin(A+B+C) &= \sin A\cos(B+C) + \cos A\sin(B+C)\\ &= \sin A(\cos B\cos C - \sin B\sin C)\\ &\qquad + \cos A(\sin B\cos C + \cos B\sin C)\\ &= \sin A\cos B\cos C + \sin B\cos C\cos A\\ &\qquad + \sin C\cos A\cos B - \sin A\sin B\sin C.\end{aligned}$$

Important Example.

Ex. 3. *Express* $a\cos\theta + b\sin\theta$ *in the form* $r\cos(\theta-\alpha)$.

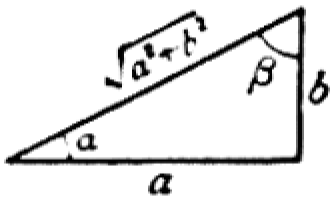


FIG. 137.

In the right-angled triangle (Fig. 137) the sides containing the right angle are a, b; the hypotenuse is $\sqrt{a^2+b^2}$.

Evidently $\cos\alpha = a/\sqrt{a^2+b^2}$, $\sin\alpha = b/\sqrt{a^2+b^2}$;

$$\therefore\ a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right\}$$

$$= \sqrt{a^2+b^2}\{\cos\alpha\cos\theta + \sin\alpha\sin\theta\};$$

$\therefore$ $\mathbf{a}\cos\theta + \mathbf{b}\sin\theta = \sqrt{\mathbf{a}^2+\mathbf{b}^2}\cos(\theta-\alpha)$, where $\tan\alpha = b/a$.

Similarly

$\mathbf{a}\cos\theta + \mathbf{b}\sin\theta = \sqrt{\mathbf{a}^2+\mathbf{b}^2}\sin(\theta+\beta)$, where $\tan\beta = a/b$.

Ex. 4. *Express* $15 \sin\theta + 8 \cos\theta$ *in the form* $r\cos(\theta - \beta)$, *and hence, or otherwise, find all the values of* θ *between* $0°$ *and* $360°$, *which satisfy the equation* $15\sin\theta + 8\cos\theta = -8{\cdot}5$. [L.U.]

Using Fig. 138, the hypotenuse is $\sqrt{15^2+8^2}=17$, and

$$\sin\beta = \tfrac{15}{17},\quad \cos\beta = \tfrac{8}{17},$$

$$\begin{aligned}15\sin\theta + 8\cos\theta &= 17\left(\tfrac{15}{17}\sin\theta + \tfrac{8}{17}\cos\theta\right)\\ &= 17(\sin\beta\sin\theta + \cos\beta\cos\theta)\\ &= 17\cos(\theta-\beta),\end{aligned}$$

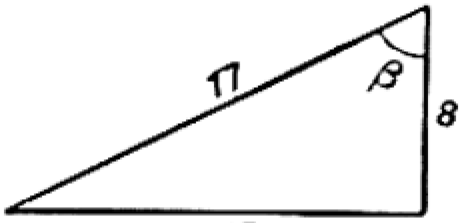


FIG. 138.

where $\tan\beta = \tfrac{15}{8}$, *i.e.* where $\beta = 61°\ 56'$.

The given equation can be written

$$17\cos(\theta - 61°\ 56') = -8{\cdot}5;$$

$$\therefore\ \cos(\theta - 61°\ 56') = -0{\cdot}5 = \cos(180° - 60°);$$

$$\therefore\ \theta - 61°\ 56' = n\,360° \pm 120°.\quad \text{(See p. 171.)}$$

$$\therefore\ \theta = n\,360° + 181°\ 56' \text{ or } n\,360° - 58°\ 4'.$$

The only solutions between 0 and 360° are found by putting $n=0$ in the first value for θ and $n=1$ in the second.

The solutions are $\theta = 181°\,58'$ and $301°\,58'$

Ex. 5. *Prove* $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B},$$

and dividing numerator and denominator by $\cos A\cos B$, we have

$$\mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}.$$

Similarly, $$\mathbf{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}.$$

Application to Algebraic Geometry.

Angle between the lines $\mathbf{y = mx + c}$ and $\mathbf{y = m'x + c'}$.

If the given lines make the angles θ and θ' respectively with the axis of X, the angle between them is $\theta - \theta'$, where

$$\tan\theta = m \quad\text{and}\quad \tan\theta' = m'.$$

Now $\tan(\theta - \theta') = (\tan\theta - \tan\theta')/(1 + \tan\theta\tan\theta')$;

$\therefore\ \theta - \theta'$ is the angle whose tangent is $\dfrac{m - m'}{1 + mm'}$.

COROLLARY. If the lines are given by $ax+by+c=0$ and $a'x+b'y+c'=0$, the angle between them is the angle whose tangent is

$$\left(-\frac{a}{b}+\frac{a'}{b'}\right)\Big/\left(1+\frac{aa'}{bb'}\right) \quad \text{or} \quad \frac{a'b-ab'}{aa'+bb'}.$$

***Angle between the lines $ax^2+2hxy+by^2=0$.**

If the given lines make angles θ and θ' with the axis of X, $ax^2+2hxy+by^2=0$ represents the same lines as

$$(y-x\tan\theta)(y-x\tan\theta')=0,$$

$$\textit{i.e.} \quad y^2-xy(\tan\theta+\tan\theta')+x^2\tan\theta\tan\theta'=0.$$

Comparing coefficients, we have

$$\tan\theta\tan\theta'=\frac{a}{b}, \quad \tan\theta+\tan\theta'=-\frac{2h}{b}.$$

Squaring the second and subtracting four times the first, we have

$$(\tan\theta-\tan\theta')^2=\frac{4h^2-4ab}{b^2};$$

$$\therefore \tan\theta-\tan\theta'=\pm\frac{2\sqrt{h^2-ab}}{b}.$$

Therefore the tangent of one of the angles between the given lines is

$$\frac{2\sqrt{h^2-ab}}{b}\div\left(1+\frac{a}{b}\right), \quad \textit{i.e.} \quad \frac{2\sqrt{h^2-ab}}{a+b},$$

the angle being acute or obtuse according as this expression is positive or negative.

Ex. 1. *The base BC of a triangle ABC is fixed. Find the locus of the point A if $B-C=\beta$, a constant.*

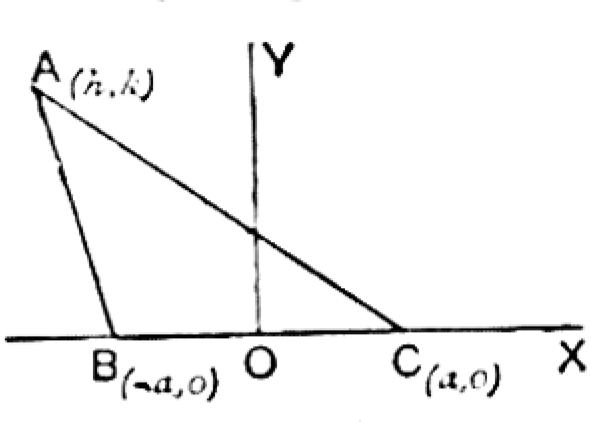


FIG. 139.

Let O be the mid-point of BC, and take axes OX, OY as shewn in Fig. 139. If $BC=2a$, C is the point $(a, 0)$, B is $(-a, 0)$.

Let the equation of AC be

$$y=m(x-a)$$

and the equation of AB be

$$y=m'(x+a).$$

Then $\tan B=m'$ and $\tan C=-m$;

$$\therefore \tan\beta=\frac{\tan B-\tan C}{1+\tan B\tan C}=\frac{m+m'}{1-mm'}. \quad \text{............(1)}$$

If A is the point (h, k),

then $\quad k = m(h-a) \quad$ and $\quad k = m'(h+a)$. ....(2)

Substituting for m and m' from (2), equation (1) becomes

$$\tan\beta = \frac{\dfrac{k}{h-a} + \dfrac{k}{h+a}}{1 - \dfrac{k^2}{h^2-a^2}} = \frac{2kh}{h^2-a^2-k^2};$$

$$\therefore \ (h^2-k^2)\tan\beta - 2hk = a^2\tan\beta;$$

and this is the condition that the point $(h, k,)$ lies on the locus whose equation is

$$(x^2-y^2)\tan\beta - 2xy = a^2\tan\beta.$$

EXERCISES XXXVI.

1. If $\cos\alpha = \frac{4}{5}$ and $\cos\beta = \frac{12}{13}$, find $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$, where α and β are both acute angles.

2. If $\sin\theta = \frac{8}{17}$ and $\cos\phi = \frac{9}{41}$, find $\sin(\theta+\phi)$, $\cos(\theta+\phi)$, $\sin(\theta-\phi)$ and $\cos(\theta-\phi)$, where θ is an obtuse angle and ϕ is an acute angle.

3. Expand $\cos(A+B+C)$.

4. Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$ and $\tan C$.

5. Putting $A = 45°$ and $B = 30°$ in formulae (1) and (2), page 304, find $\sin 75°$ and $\cos 75°$.

6. Find $\cos 15°$, $\sin 15°$ and $\tan 15°$.

7. Prove that $\sin(60°-\theta) + \sin(60°+\theta) = \sqrt{3}\cos\theta$.

8. Prove that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$, and that

$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B.$$

9. Simplify

(i) $\sin 4A\cos 2A + \cos 4A \sin 2A$;

(ii) $\cos(\theta+\phi)\cos\phi + \sin(\theta+\phi)\sin\phi$:

(iii) $\cos(A+B) + \sin A\sin B$.

10. Prove that $\quad \tan\theta + \tan\phi = \dfrac{\sin(\theta+\phi)}{\cos\theta\cos\phi}$,

and that $\quad \cot A - \cot B = \dfrac{\sin(B-A)}{\sin B\sin A}$.

11. If $\sin(\theta+\alpha)=n\sin(\theta-\alpha)$, shew that $\cot\theta=\frac{n-1}{n+1}\cot\alpha$.

12. If $\tan\alpha=\frac{1}{3}$ and $\tan\beta=\frac{3}{4}$, calculate $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$, where α lies between 180° and 270° and β is an acute angle. Verify your result by using the tables.

13. Shew that

$$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}, \quad \text{and that } \tan\left(\frac{\pi}{4}-\theta\right)=\frac{1-\tan\theta}{1+\tan\theta}.$$

14. Simplify $\cos\theta+\sin\phi\sin(\theta-\phi)$ and $\sin\theta+\sin\phi\cos(\theta+\phi)$.

15. (i) Prove that $\cos 2A=2\cos^2 A-1$, and hence or otherwise prove that $\tan 15°=2-\sqrt{3}$.

(ii) Prove that

$$\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos C\cos A$$
$$+\sin C\cos A\cos B-\sin A\sin B\sin C. \quad \text{[I.C.E.]}$$

16. Find all the angles between 0° and 360° which satisfy

(i) $2\sin\theta+3\cos\theta=2$; (ii) $3\cos\theta-\sin\theta=2{\cdot}5$;

(iii) $3\sin\theta-2\cos\theta=-1$; (iv) $5\tan\theta+1=2\sec\theta$.

Examples 17-24 *are taken from papers set at the Inter. B.Sc.* [*L.U.*]

17. If $r\cos\alpha=-7$ and $r\sin\alpha=4$ and r is positive, find r and α. Hence or otherwise solve completely the equation

$$4\sin\theta-7\cos\theta=5.$$

18. If $r\cos\alpha=a$ and $r\sin\alpha=b$ and r is positive, shew that there is only one value of α between 0° and 360°. Find the value of α and the positive value of r if $a=2$ and $b=-3$.

Hence or otherwise find all the solutions of $2\cos\theta-3\sin\theta=1{\cdot}5$ which lie between 0° and 360°.

19. Shew that the expression $a\cos x+b\sin x$ is never numerically greater than $\sqrt{(a^2+b^2)}$, and find the values of x for which it assumes this value.

Find the angles lying between −180° and 180° which satisfy the equation $6\cos\theta-8\sin\theta=9$.

20. Write down the expansion of $\cos\overline{A+B}$ in terms of the sines and cosines of A and B.

If $\sin A=0{\cdot}2$ and $\cos B=0{\cdot}7$, find all the possible values of

$$\cos\overline{A+B}.$$

Shew that $a\cos\overline{\theta+\alpha}+b\sin\theta$ can be expressed in the form $A\cos\overline{\theta+\beta}$, and graph the function from $\theta=0$ to $\theta=\pi$ when $a=2$, $b=1{\cdot}5$ and $\alpha=\frac{\pi}{6}$.

21. Express $a\cos\theta + b\sin\theta$ in the form $r\sin(\theta+\phi)$.
If θ_1, θ_2 are the two smallest positive roots of the equation

$$a\cos\theta + b\sin\theta = c,$$

prove that
$$\tan(\theta_1+\theta_2) = \frac{2ab}{a^2-b^2}.$$

22. Two circular pegs of radii r and R have their centres c apart, and the two arms of a right-angled L piece touch the pegs. Find an expression for the square of the length of the line joining the points of contact in terms of c, r, R, and the angle α which one arm makes with c. Shew that the maximum distance is $\surd\{c^2+r^2+R^2+2c\sqrt{R^2+r^2}\}$.

23. Find the acute angle between the lines

(i) $2x-y+1=0, \quad 3x+4y+2=0$;

(ii) $6x-7y=0, \quad x+y+1=0$;

(iii) $3x^2-4xy+y^2=0$; (iv) $6x^2+7xy-y^2=0$.

24. Find the angle between the lines $y=mx+c$ and $y=m'x+c'$, and use the result to find the condition that these lines should be (1) parallel, (2) perpendicular.

25. Find the tangent of the angle between the straight lines $y=mx+c$, $y=m'x+c'$; and state carefully the meaning of your result.

Find completely the locus of a point such that the angle between the lines joining it to the points $(a, 0)$, $(-a, 0)$ is constant and equal to α; sketch the figure. [S.W.]

26. Shew that one of the angles, α, between the lines

$$y=mx+c, \quad y=m'x+c'$$

is given by $\tan\alpha(1+mm')=m-m'$; and state carefully which of the angles is thus found.

Lines through the points $(3, 4)$, $(-4, 3)$ respectively are drawn at an angle of 45°: find the locus of their intersection and give a diagram. [S.W.]

27. Find an expression for the angle between the two lines

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0.$$

From the vertices of a triangle ABC, perpendiculars AD, BE are drawn to the opposite sides. If the equations of AB, BC and CA are respectively

$$x-3y+4=0, \quad x+y-2=0, \quad 3x-4y+7=0,$$

find the equations of AD and BE. [Man.]

28. $OABC$ is a rectangle in which $OA = 5$ in. and $OC = 3$ in.: D and E are points in CB such that $CB = 2CE = 3CD$. Taking OA, OC as axes of X and Y respectively, find the equations of OB, AC, AE, the coordinates of the point of intersection P of OD and AE, and the angle OPA. [Durham.]

29. Find the angle between the straight lines whose equations are $ax + by + c = 0$, $a'x + b'y + c' = 0$.

Prove that the centres and the points of intersection of the circles whose equations are

$$x^2 + y^2 - 10y + 15 = 0, \quad x^2 + y^2 - 8x - 6y + 15 = 0$$

form the vertices of a square. [B.U.]

*30. Find the equations of the lines which pass through the point (h, k) and make angles ϕ with the line $ax + by + c = 0$.

*31. Prove that the equation $ax^2 + 2hxy + by^2 = 0$ represents two straight lines through the origin; find under what conditions the lines are real, and obtain a formula for the angle between them.

Prove that the lines $x^2 - y^2 = 2kxy$ are at right angles for all values of k, and find their inclinations to OX when $k = 2$. [S.W.]

Multiple and Submultiple Angles.

Putting $A = B$ in Formula (1), p. 304, we have,

$$\left.\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A, \\ \text{and since } \cos^2 A + \sin^2 A = 1, \quad \cos 2A &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A. \end{aligned}\right\} \quad \text{.........(1)}$$

$$\left.\begin{aligned} \text{Also} \quad 2\sin^2 A &= 1 - \cos 2A \\ \text{and} \quad 2\cos^2 A &= 1 + \cos 2A. \end{aligned}\right\} \quad \text{.........................(2)}$$

Similarly, putting $A = B$ in Formula (2), p. 304, we have

$$\sin 2A = 2\sin A \cos A. \quad \text{........................(3)}$$

Evidently $$\tan 2A = \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}.$$

Dividing top and bottom by $\cos^2 A$,

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \quad \text{...........................(4)}$$

These formulae should be learnt and memorised in words. *E.g.* (2) gives "Twice the square of the sine of any angle equals one minus the cosine of twice the angle" and "Twice the square of the cosine of an angle equals one plus the cosine of twice the angle."

Alternatively, "One minus the cosine of any angle is twice the square of the sine of half the angle" and "One plus the cosine of any angle is twice the square of the cosine of half the angle."

Ex. 1. *Find* $\cos 3A$ *and* $\sin 3A$.

$$\begin{aligned}\cos 3A &= \cos(2A+A)\\ &= \cos 2A\cos A - \sin 2A \sin A\\ &= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A\\ &= 2\cos^3 A - \cos A - 2\cos A(1-\cos^2 A)\\ &= 4\cos^3 A - 3\cos A.\end{aligned}$$

$$\begin{aligned}\sin 3A &= \sin(2A+A)\\ &= \sin 2A\cos A + \cos 2A\sin A\\ &= 2\sin A\cos^2 A + (1-2\sin^2 A)\sin A\\ &= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A\\ &= 3\sin A - 4\sin^3 A.\end{aligned}$$

Ex. 2. *Given* $\theta = 18°$, *shew that* $\sin 3\theta = \cos 2\theta$, *and hence find* $\sin 18°$.

As $\theta = 18°$, $\quad 3\theta = 54°$ and $2\theta = 36°$.

$\therefore$ 3θ and 2θ are complementary and $\sin 3\theta = \cos 2\theta$;

$$\therefore\ 3\sin\theta - 4\sin^3\theta = 1 - 2\sin^2\theta;$$

$$\therefore\ 4\sin^3\theta - 2\sin^2\theta - 3\sin\theta + 1 = 0.$$

By inspection, $\sin\theta = 1$ satisfies this equation.

By division, we find the left-hand side is

$$(\sin\theta - 1)(4\sin^2\theta + 2\sin\theta - 1).$$

As $\sin 18° \neq 1$, $\quad \therefore\ 4\sin^2\theta + 2\sin\theta - 1 = 0$.

$$\therefore\ \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4};$$

$$\therefore\ \sin 18° = \frac{-1+\sqrt{5}}{4},\text{ since } \sin 18° \text{ is positive.}$$

Ex. 3. *Prove that* $2\sin^2(\frac{1}{4}\pi - \theta) = 1 - \sin 2\theta$.

$$2\sin^2(\tfrac{1}{4}\pi - \theta) = 1 - \cos 2(\tfrac{1}{4}\pi - \theta) = 1 - \cos(\tfrac{1}{2}\pi - 2\theta) = 1 - \sin 2\theta.$$

Ex. 4. *Prove that* $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$.

We have $\quad \tan^2\theta = \dfrac{2\sin^2\theta}{2\cos^2\theta} = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$.

Ex. 5. *Prove that*

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \quad \text{and} \quad \sin 2A = \frac{2\tan A}{1+\tan^2 A}.$$

We have $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$,

and dividing top and bottom by $\cos^2 A$, this becomes

$$\frac{1-\tan^2 A}{1+\tan^2 A}.$$

Also $\quad \sin 2A = 2\sin A \cos A = \dfrac{2\sin A\cos A}{\cos^2 A + \sin^2 A}$

and dividing top and bottom by $\cos^2 A$, this becomes

$$\frac{2\tan A}{1+\tan^2 A}.$$

EXERCISES XXXVII.

1. Find $\sin 15°$, $\cos 15°$, $\tan 15°$; $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, $\tan 22\frac{1}{2}°$.

2. Prove (i) $\dfrac{1-\cos 2A}{\sin 2A} = \tan A$;

   (ii) $\cot\theta - \tan\theta = 2\cos 2\theta \operatorname{cosec} 2\theta$.

3. Prove $\cos 36° - \sin 18° = 0{\cdot}5$.

4. Prove (i) $2\cos^2\left(\frac{\pi}{4} - A\right) = 1 + \sin 2A$;

   (ii) $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

5. Prove $\cos^2\phi + \cos^2\left(\phi + \frac{\pi}{3}\right) + \cos^2\left(\phi - \frac{\pi}{3}\right) = \frac{3}{2}$.

6. Prove $$\sin 2\theta = \frac{1-\tan^2\left(\frac{\pi}{4}-\theta\right)}{1+\tan^2\left(\frac{\pi}{4}-\theta\right)}.$$

7. Prove that
   $$\sin^2(x+y) - \sin^2(x-y) = \sin 2x \sin 2y.$$

8. Express $\cos 4\theta$ in terms of $\cos\theta$.

9. Express $\cos 5\theta$ in terms of $\cos\theta$ and $\sin 5\theta$ in terms of $\sin\theta$.

10. Find $\cos^4\theta$ as a linear function of $\cos 4\theta$ and $\cos 2\theta$.

Sums and Products.

We have $\sin(A+B)=\sin A\cos B+\cos A\sin B$;

$\sin(A-B)=\sin A\cos B-\cos A\sin B$.

By addition, $2\sin A\cos B=\sin(A+B)+\sin(A-B)$.

In words, **2 sin (any angle) cos (another angle)**

= sin (sum) + sin (difference), ...(1)

where "difference" is the result of taking the second angle from the first.

By subtraction, $2\cos A\sin B=\sin(A+B)-\sin(A-B)$.

In words, **2 cos (any angle) sin (another angle)**

= sin (sum) − sin (difference). ......(2)

Again, $\cos(A+B)=\cos A\cos B-\sin A\sin B$;

$\cos(A-B)=\cos A\cos B+\sin A\sin B$.

By addition, $2\cos A\cos B=\cos(A+B)+\cos(A-B)$.

In words, **2 cos (any angle) cos (another angle)**

= cos (sum) + cos (difference). ......(3)

By subtraction, $2\sin A\sin B=\cos(A-B)-\cos(A+B)$.

In words, **2 sin (any angle) sin (another angle)**

= cos (difference) − cos (sum). ......(4)

Ex. 1. $$2\sin 6A\cos 5A=\sin(6A+5A)+\sin(6A-5A)$$
$$=\sin 11A+\sin A.$$

Ex. 2. $$2\cos 7\theta\sin 2\theta=\sin(7\theta+2\theta)-\sin(7\theta-2\theta)$$
$$=\sin 9\theta-\sin 5\theta.$$

Ex. 3. $$2\cos\theta\sin 4\theta=\sin(\theta+4\theta)-\sin(\theta-4\theta)$$
$$=\sin 5\theta-\sin(-3\theta)$$
$$=\sin 5\theta+\sin 3\theta.$$

Ex. 4. $$2\sin(45^\circ-x)\sin(45^\circ+x)=\cos(-2x)-\cos(90^\circ)$$
$$=\cos 2x.$$

EXERCISES XXXVIII.

Express as sums or differences the following:

1. $2\sin 2\theta\cos\theta$.
2. $2\cos 5A\sin 2A$.
3. $2\cos 6\phi\cos 3\phi$.
4. $2\sin 2x\sin 5x$.
5. $\sin 3\alpha\cos 7\alpha$.
6. $\sin 3y\sin 2y$.

7. $\sin\frac{5x}{2}\cos\frac{9x}{2}$. 8. $\cos 2x\cos(x-y)$.

9. $\sin 4\theta\sin(\theta+\phi)$. 10. $\cos\left(\frac{\pi}{6}+\theta\right)\sin\left(\frac{\pi}{6}-\theta\right)$.

11. $\cos\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{3\pi}{4}-\alpha\right)$. 12. $\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{\pi}{3}\right)$.

13. $\sin(3\theta+\phi)\cos(\theta-2\phi)$. 14. $\cos(n\pi-\alpha)\cos(n\pi+\alpha)$.

It has been shewn that

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

Writing θ for $(A+B)$ and ϕ for $(A-B)$,

$$A=\frac{\theta+\phi}{2},\quad B=\frac{\theta-\phi}{2};$$

$$\therefore\ \sin\theta+\sin\phi=2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}.$$

In words,

the sum of two sines = 2 sin (half sum) cos (half difference). (5)

Again, $\sin(A+B)-\sin(A-B)=2\cos A\sin B$;

i.e. $$\sin\theta-\sin\phi=2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$$

In words, **the difference of two sines
= 2 cos (half sum) sin (half difference). ......(6)**

Again, $\cos(A+B)+\cos(A-B)=2\cos A\cos B$;

i.e. $$\cos\theta+\cos\phi=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}.$$

In words, **the sum of two cosines
= 2 cos (half sum) cos (half difference). ......(7)**

Finally, $$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$
$$=2\sin A\sin(-B);$$

i.e. $$\cos\theta-\cos\phi=2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}.$$

In words, **the differences of two cosines
= 2 sin (half sum) sin (half difference reversed). ......(8)**

Ex. 1. $\sin 8x + \sin 4x = 2 \sin \frac{8x + 4x}{2} \cos \frac{8x - 4x}{2}$

$= 2 \sin 6x \cos 2x.$

Ex. 2. $\sin 6\theta - \sin 2\theta = 2 \cos \frac{6\theta + 2\theta}{2} \sin \frac{6\theta - 2\theta}{2}$

$= 2 \cos 4\theta \sin 2\theta.$

Ex. 3. $\cos 3A + \cos 11A = 2 \cos \frac{3A + 11A}{2} \cos \frac{3A - 11A}{2}$

$= 2 \cos 7A \cos(-4A)$

$= 2 \cos 7A \cos 4A.$

Ex. 4. $\cos 20^\circ - \cos 60^\circ = 2 \sin \frac{20^\circ + 60^\circ}{2} \sin \frac{60^\circ - 20^\circ}{2}$

$= 2 \sin 40^\circ \sin 20^\circ.$

EXERCISES XXXIX.

Express the following as products:

1. $\sin 14x + \sin 6x.$
2. $\sin 11\theta - \sin 7\theta.$
3. $\cos 7x + \cos 5x.$
4. $\cos 2\alpha - \cos 6\alpha.$
5. $\sin 80^\circ + \sin 20^\circ.$
6. $\cos 7x - \cos 3x.$
7. $\cos 90^\circ - \cos 20^\circ.$
8. $\sin\left(\frac{\pi}{6} - A\right) + \sin\left(\frac{\pi}{6} + A\right).$
9. $\sin\left(\frac{\pi}{3} + x\right) - \sin\left(\frac{\pi}{3} - x\right).$
10. $\cos\left(\frac{\pi}{4} - \theta\right) - \cos\left(\frac{\pi}{4} + \theta\right).$
11. $\cos(n\pi + \alpha) - \cos(n\pi - \alpha).$
12. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta).$
13. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta).$
14. $\sin 20^\circ + \sin 100^\circ + \sin 60^\circ.$
15. Shew that $\frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \tan\frac{\alpha - \beta}{2}.$
16. Shew that $\frac{\sin(\theta + 3\phi) + \sin(3\theta + \phi)}{\sin 2\theta + \sin 2\phi} = 2\cos(\theta + \phi).$
17. Shew that $\frac{\sin 2x + \sin 3x}{\cos 2x - \cos 3x} = \cot\frac{x}{2}.$

18. Let P, Q be points whose polar coordinates are (r, θ) and (r, ϕ) respectively referred to OX as initial line and O as origin. (Take $\phi > \theta$ and each angle $< \pi$). Complete the rhombus $OPSQ$ and let the diagonals intersect at R.

Shew that (1) the sum of the projections of $\overline{OP}$ and $\overline{OQ}$ in any direction is twice the projection of $\overline{OR}$ in that direction; (2) the projection of $\overline{OP}$ in any direction minus the projection of $\overline{OQ}$ in that direction is twice the projection of $\overline{QR}$ in the same direction.

From (1) and (2), by projecting in the directions $\overline{OX}$ and $\overline{OY}$ (which makes + 90° with OX), prove that

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\phi-\theta}{2},$$

$$\cos\theta - \cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2},$$

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\phi-\theta}{2},$$

$$\sin\phi - \sin\theta = 2\cos\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}.$$

Miscellaneous Examples.

Ex. 1. *Shew that* $4\cos x\cos\left(x+\frac{2\pi}{3}\right)\cos\left(x+\frac{4\pi}{3}\right)=\cos 3x$.

$$2\cos\left(x+\frac{2\pi}{3}\right)\cos\left(x+\frac{4\pi}{3}\right)=\cos(2x+2\pi)+\cos\left(\frac{2\pi}{3}\right)$$
$$=\cos 2x-\tfrac{1}{2}.$$

$$\therefore\ 4\cos x\cos\left(x+\frac{2\pi}{3}\right)\cos\left(x+\frac{4\pi}{3}\right)=2\cos x\cos 2x-\cos x$$
$$=(\cos 3x+\cos x)-\cos x$$
$$=\cos 3x.$$

Ex. 2. *Shew that* $\sin^2 7\theta-\sin^2 3\theta=\sin 10\theta\sin 4\theta$.

$$\sin^2 7\theta-\sin^2 3\theta=\tfrac{1}{2}(1-\cos 14\theta)-\tfrac{1}{2}(1-\cos 6\theta)$$
$$=\tfrac{1}{2}(\cos 6\theta-\cos 14\theta)$$
$$=\sin\frac{6\theta+14\theta}{2}\sin\frac{14\theta-6\theta}{2}$$
$$=\sin 10\theta\sin 4\theta.$$

Ex. 3. *Find all the values of x which satisfy the equation*

$$\sin 3x - \sin x = \sin 5x.$$

The equation may be written $\sin 5x + \sin x = \sin 3x$;

$$\therefore\ 2\sin 3x \cos 2x = \sin 3x;$$

$$\therefore\ \sin 3x(2\cos 2x - 1) = 0;$$

$$\therefore\ x \text{ must satisfy either } \sin 3x = 0 \quad\text{or}\quad \cos 2x = \tfrac{1}{2};$$

$$\therefore\ 3x = n\pi \quad\text{or}\quad 2x = 2r\pi \pm \tfrac{1}{3}\pi.$$

These formulae give all the values of x required.

Ex. 4. *Simplify* $\cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) + \cos^2\left(\frac{\pi}{3} - x\right)$.

The expression

$$= \frac{1}{2}\left[(1 + \cos 2x) + \left\{1 + \cos\left(\frac{2\pi}{3} + 2x\right)\right\} + \left\{1 + \cos\left(\frac{2\pi}{3} - 2x\right)\right\}\right]$$

$$= \frac{1}{2}\left[3 + \cos 2x + \cos\left(\frac{2\pi}{3} + 2x\right) + \cos\left(\frac{2\pi}{3} - 2x\right)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2x + 2\left\{\cos\frac{2\pi}{3}\cos 2x\right\}\right]$$

$$= \tfrac{1}{2}[3 + \cos 2x - \cos 2x]$$

$$= \tfrac{3}{2}.$$

(See also example 21, p. 303.)

Ex. 5. *If $A + B + C = \pi$, shew that*

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$= 2\cos\frac{C}{2}\cos\frac{A-B}{2}, \quad \text{since } \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2},$$

and $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$;

$$\therefore\ \sin A + \sin B + \sin C = 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right)$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$

$$= 4\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}.$$

Ex. 6. *If* $A+B+C=\pi$, *shew that*

$$1+\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C.$$

$$\cos 2B+\cos 2C=2\cos(B+C)\cos(B-C)$$
$$=-2\cos A\cos(B-C),\quad \text{since } B+C=\pi-A.$$

And $\quad 1+\cos 2A=2\cos^2 A$;

$$\therefore\ 1+\cos 2A+\cos 2B+\cos 2C$$
$$=2\cos A\{\cos A-\cos(B-C)\}$$
$$=2\cos A\{-\cos(B+C)-\cos(B-C)\}$$
$$=-4\cos A\cos B\cos C.$$

Ex. 7. *Express* $18\cos\theta+13\sin\theta=22$ *as a quadratic equation in* t, *where* $t=\tan\frac{1}{2}\theta$, *and thus solve the equation.*

Using Ex. 5, p. 314,

$$\cos\theta=\frac{1-t^2}{1+t^2}\quad\text{and}\quad\sin\theta=\frac{2t}{1+t^2}.$$

Substitute in the given equation, rearrange, and we have

$$18(1-t^2)+26t=22(1+t^2);$$
$$\therefore\ 20t^2-13t+2=0,$$
$$i.e.\qquad t=\tfrac{1}{4}\ \text{ or }\ \tfrac{2}{5};$$
$$\therefore\ \tfrac{1}{2}\theta=n\,.\,180^\circ+14^\circ 2'\ \text{ or }\ k\,.\,180^\circ+21^\circ 48',$$
$$i.e.\qquad \theta=n\,.\,360^\circ+28^\circ 4'\ \text{ or }\ k\,.\,360^\circ+43^\circ 36'.$$

This method of solution should be compared with that of Ex. 4, p. 307.

EXERCISES XL.

1. (i) Given $\tan\theta=x$, express $\sin\theta$, $\cos\theta$, $\sin 2\theta$, $\cos 2\theta$, in terms of x only.

(ii) Write down the expressions for $\sin(A+B)$ and $\sin(A-B)$ in terms of the sines and the cosines of A and B; and deduce the expression for the difference of two sines in terms of the trigonometrical functions of the semi-sum and semi-difference of the angles. [I.C.E.]

2. Prove that

$$\tan\left(\frac{\pi}{4}+x\right)-\tan\left(\frac{\pi}{4}-x\right)=2\tan 2x.$$

3. Express $\sin 3A$ in terms of $\sin A$, and $\cos 3A$ in terms of $\cos A$.

Shew that

$$\tan 3A = \tan(60^\circ - A)\tan A \tan(60^\circ + A). \qquad \text{[L.U.]}$$

4. Prove that

$$\tan\frac{45^\circ + \theta}{2}\tan\frac{45^\circ - \theta}{2} = \frac{\sqrt{2}\cos\theta - 1}{\sqrt{2}\cos\theta + 1}.$$

Deduce the values,

$$\tan 7\tfrac{1}{2}^\circ = (\sqrt{2} - 1)(\sqrt{3} - \sqrt{2}),$$
$$\tan 37\tfrac{1}{2}^\circ = (\sqrt{2} + 1)(\sqrt{3} - \sqrt{2}). \qquad \text{[L.U.]}$$

5. Shew that if $\tan\theta$ be of given magnitude, θ being less than 2π radians, then $\tan\frac{1}{2}\theta$ may have either of two values whose product is -1.

Having given that $\tan\theta = (x^2 - y^2)/2xy$, express $\tan\frac{1}{2}\theta$ in terms of x and y. [L.U.]

6. Shew that

$$\sin\theta - \sin 2\theta + \sin 3\theta = 4\sin\frac{\theta}{2}\cos\theta\cos\frac{3\theta}{2}.$$

7. Shew that

$$\frac{\sin 7x - \sin 3x - \sin 5x + \sin x}{\cos 7x + \cos 3x - \cos 5x - \cos x} = \tan 2x.$$

8. Solve the equations:

(i) $\sin^2 x + \cos 2x = \cos x$; (ii) $\sin\theta + \sin 2\theta = \cos\theta + \cos 2\theta$;
(iii) $\cos 3x \cos x = \cos 4x \cos x$; (iv) $\tan 2\phi + 3\cot\phi = 0$;
(v) $\cot 3\theta + \cot\theta = 0$.

9. Between what limits must θ lie if $\sin^2\theta > \cos^2\theta$?

10. Shew that $a\cos\theta + b\sin\theta$ lies between

$$+\sqrt{a^2 + b^2} \quad \text{and} \quad -\sqrt{a^2 + b^2}.$$

11. Prove that $\sin 40^\circ + \cos 10^\circ = \sqrt{3}\cos 20^\circ$.

12. Prove that $8\sin 10^\circ \sin 50^\circ \sin 70^\circ = 1$.

13. If $\cot\theta = 2{\cdot}4$, find the value of $\sin\theta$ and of $\tan 2\theta$. Explain why $\sin\theta$ has two values and $\tan 2\theta$ only one value.

14. If $t = \tan\dfrac{\theta}{2}$, shew that

$$\cos\theta = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad \sin\theta = \frac{2t}{1 + t^2}.$$

Hence *and* otherwise find all the angles between 0 and 360° that satisfy the equation $52\cos\theta + 39\sin\theta = 60$. [L.U.]

15. If $\cos\theta = x$ and $\sin\theta = y$, express $\cos 3\theta + \sin 2\theta - \sin 4\theta$ in terms of x and y, and when θ lies between π and $3\pi/2$ express the result in terms of x only. Find possible values of x that will make the expression vanish. [L.U.]

16. Solve the equations:

(i) $\sin 2x + \sin 4x = \cos x$; (ii) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$;

(iii) $\sin x + \cos x = \sin 2x + \cos 2x$.

17. Prove that

$$\sin 2\theta + \sin 2\phi + \sin 2\psi - \sin 2(\theta+\phi+\psi) = 4\sin(\theta+\phi)\sin(\phi+\psi)\sin(\psi+\theta).$$

18. If $A+B+C=\pi$, prove the following identities:

(i) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

(ii) $1 + \cos A + \cos B - \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

(iii) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1 = 0$.

(iv) $\dfrac{\sin 2A - \sin 2B + \sin 2C}{\sin 2B + \sin 2C - \sin 2A} = \cot A\tan B$.

(v) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

(vi) $\cos^2 2A + \cos^2 2B + \cos^2 2C - 1 = 2\cos 2A\cos 2B\cos 2C$.

19. Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$ and $\tan C$. If $A+B+C=\pi$, shew that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

If $A+B+C=(2n+1)\frac{\pi}{2}$, shew that

$$1 - \tan A\tan B - \tan B\tan C - \tan C\tan A = 0.$$

20. Prove that

(i) $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$;

(ii) $\dfrac{(1+\tan A)^2 - 2\tan^2 A}{1+\tan^2 A} = \sqrt{2}\sin(45^\circ + 2A)$;

(iii) if $A+B+C=90^\circ$, then
$\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$. [L.U.]

***21.** Prove that

$$\sin x\sin(y-z) + \sin y\sin(z-x) + \sin z\sin(x-y) = 0.$$

***22.** Prove that

$$4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{\pi}{3}-\theta\right)=\sin 3\theta.$$

***23.** If $A+B+C=\pi$, prove that

$$\sin^2 A(\cot B-\cot C)+\sin^2 B(\cot C-\cot A)$$
$$+\sin^2 C(\cot A-\cot B)=0.$$

***24.** If

$$\cos\theta+\cos\phi+\cos x=0 \quad \text{and} \quad \sin\theta+\sin\phi+\sin x$$

shew that $\cot\dfrac{\theta+\phi}{2}=\cot x$ and $\cos\dfrac{\theta-\phi}{2}=\pm\dfrac{1}{2}$.

***25.** If $a\cos\alpha+b\sin\alpha=c$ and $a\cos\beta+b\sin\beta=c$,

shew that $\dfrac{\cos\frac{1}{2}(\alpha+\beta)}{a}=\dfrac{\sin\frac{1}{2}(\alpha+\beta)}{b}=\dfrac{\cos\frac{1}{2}(\alpha-\beta)}{c}$.

***26.** Eliminate θ between the equations

$$\tan\theta+\sin\theta=p, \qquad \tan\theta-\sin\theta=q.$$

Simple Properties of the Triangle.

I. In any triangle, the sides are proportional to the sines of the opposite angles.

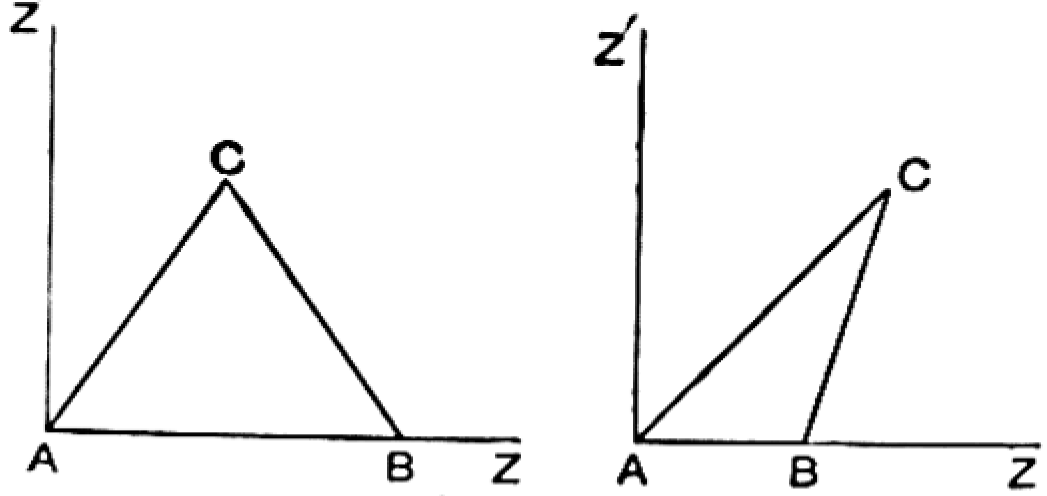


FIG. 140.

In Fig. 140, take the direction of AB as the standard direction, and let $\overline{AZ'}$ make 90° with $\overline{AB}$.

$$\overline{AC}=b_A \quad \text{and} \quad \overline{BC}=a_{180^\circ-B}.$$

Projecting on $\overline{AZ'}$,

the projection of $\overline{AC}$ = the projection of $\overline{BC}$;

$$\therefore\ b\sin A=a\sin(180^\circ-B)=a\sin B;$$

$$\therefore\ \frac{\sin A}{a}=\frac{\sin B}{b}.$$

Similarly $$\frac{\sin A}{a}=\frac{\sin C}{c};$$

$$\therefore \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

Referring to Ex. 6, p. 132, we see that each of these ratios is equal to $1/2R$, where R is the radius of the circumcircle.

II. In any triangle $c=a\cos B+b\cos A$.

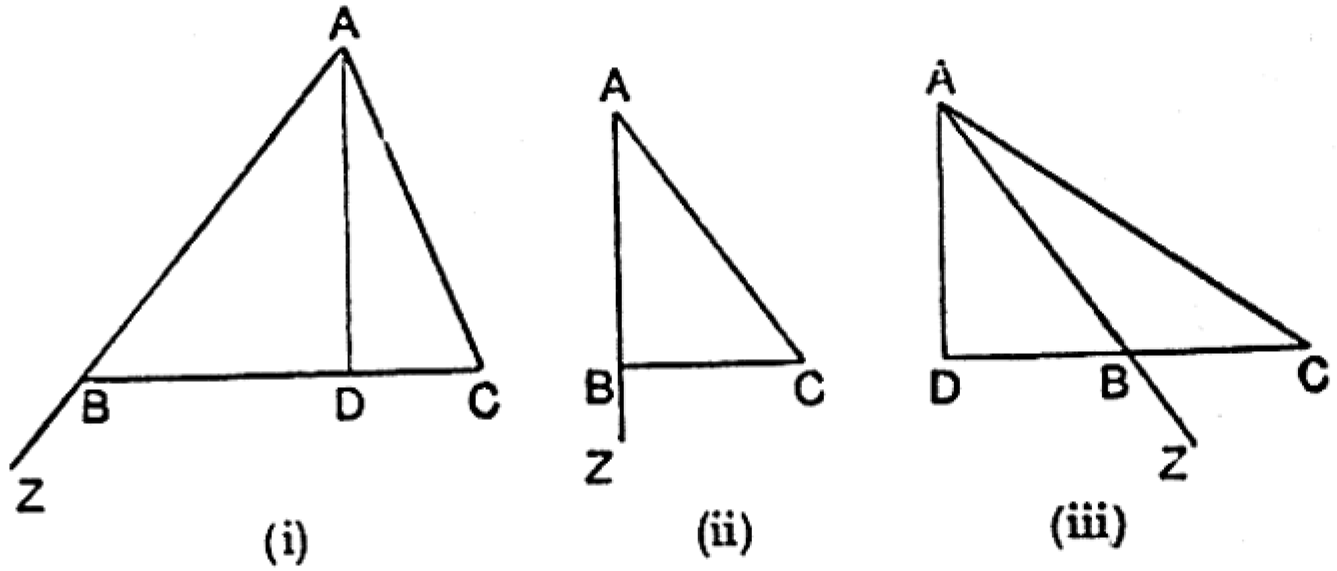


FIG. 141.

In Fig. 141, $$\overline{BC}=\overline{BA}+\overline{AC};$$

$\therefore$ the projection of $\overline{BC}$ in the direction AZ equals the sum of the projections of $\overline{BA}$ and $\overline{AC}$ in that direction,

i.e. $$a\cos(180^\circ-B)=c\cos 180^\circ+b\cos A;$$

$$\therefore \quad -a\cos B=-c+b\cos A;$$

$$\therefore \quad c=a\cos B+b\cos A.$$

Similarly $$a=b\cos C+c\cos B$$

and $$b=c\cos A+a\cos C.$$

III. In any triangle, the square on any side is equal to the sum of the squares on the other two sides minus twice their product multiplied by the cosine of the angle included by them.

Using Fig. 141, in (i)

$$BD=c\cos B \quad \text{and} \quad AD=c\sin B;$$

$$\therefore \quad b^2=AD^2+DC^2=c^2\sin^2 B+(a-c\cos B)^2$$

$$=c^2(\sin^2 B+\cos^2 B)+a^2-2ac\cos B;$$

$$\therefore \quad b^2=c^2+a^2-2ca\cos B.$$

In (ii), $$b^2=c^2+a^2;$$

$$\therefore \quad b^2=c^2+a^2-2ca\cos B, \quad \text{since } \cos B=0.$$

In (iii), $BD = c\cos ABD = c\cos(180° - B) = -c\cos B$
and $AD = c\sin ABD = c\sin(180° - B) = c\sin B$;

$$\therefore\ b^2 = AD^2 + CD^2 = c^2\sin^2 B + (a - c\cos B)^2;$$

$$\therefore\ b^2 = c^2 + a^2 - 2ca\cos B.$$

Alternatively, using II., p. 324,

$$b\cos A = c - a\cos B,$$

and using I., p. 323, $b\sin A = a\sin B$.

Squaring and adding, $b^2 = c^2 + a^2 - 2ac\cos B$.

Similarly, for all triangles,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and $$c^2 = a^2 + b^2 - 2ab\cos C.$$

IV. **The area of any triangle is equal to half the product of two sides multiplied by the sine of the angle included by them.**

Using Fig. 141, and denoting the area by Δ,

in (i), $\Delta = \frac{1}{2}a\,.\,AD = \frac{1}{2}ac\sin B$;
in (ii), $\Delta = \frac{1}{2}a\,.\,AB = \frac{1}{2}ac$, since $\sin B = 1$;
in (iii), $\Delta = \frac{1}{2}a\,.\,AD = \frac{1}{2}ac\sin(180° - B)$
$= \frac{1}{2}ac\sin B$.

Similarly, for all triangles,

$$\Delta = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ab\sin C.$$

(See Ex. 5, p. 132.)

Ex. 1. *In any triangle, prove that*

$$c(\sin^2 A + \sin^2 B) = \sin C(a\sin A + b\sin B).$$

We have $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k},\quad \text{say};$$

$$\therefore\ ka = \sin A,\quad kb = \sin B,\quad kc = \sin C.$$

The left-hand side of the given identity is

$$c(k^2a^2 + k^2b^2)\quad \text{or}\quad k^2c(a^2 + b^2).$$

The right-hand side of the given identity is

$$kc(aka + bkb)\quad \text{or}\quad k^2c(a^2 + b^2);$$

$$\therefore\ c(\sin^2 A + \sin^2 B) = \sin C(a\sin A + b\sin B).$$

Ex. 2. *In any triangle, prove that*

$$a\cos B - b\cos A = \frac{a^2-b^2}{c}.$$

Using formula III., p. 324,

$$\cos B = \frac{c^2+a^2-b^2}{2ac},$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc};$$

$$\therefore\ a\cos B - b\cos A = \frac{c^2+a^2-b^2}{2c} - \frac{b^2+c^2-a^2}{2c}$$

$$= \frac{a^2-b^2}{c}.$$

Ex. 3. *In any triangle, shew that*

$$a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0.$$

Since

$$a = 2R\sin A,\quad b = 2R\sin B \quad\text{and}\quad c = 2R\sin C,$$

the left-hand side of the given identity is

$$2R\sin A\sin(B-C) + 2R\sin B\sin(C-A) + 2R\sin C\sin(A-B). \quad\text{......(1)}$$

Now, since $A+B+C=180°$,

$$2\sin A\sin(B-C) = 2\sin(B+C)\sin(B-C),$$

$$= \cos 2C - \cos 2B.$$

Hence expression (1)

$$= R\{\cos 2C - \cos 2B + \cos 2A - \cos 2C + \cos 2B - \cos 2A\}$$

$$=0.$$

Ex. 4. *X is any point on the side BC of a triangle ABC. If the angle $AXC=\theta$ and $BX/XC=m/n$, shew that*

$$(m+n)\cot\theta = n\cot B - m\cot C.$$

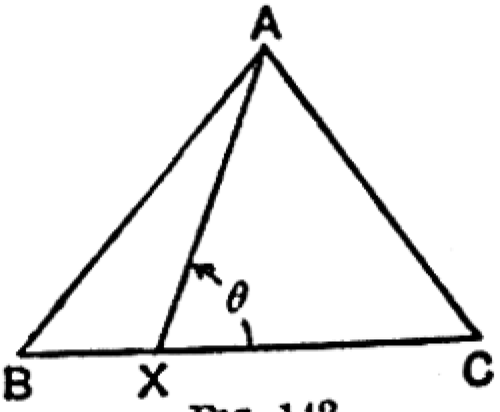


FIG. 142.

In Fig. 142,

$$\frac{AX}{\sin B} = \frac{BX}{\sin BAX};$$

$$\therefore\ \frac{AX}{BX} = \frac{\sin B}{\sin(\theta - B)} \quad\text{.........................(1)}$$

Similarly, $$\frac{AX}{\sin C}=\frac{CX}{\sin XAC};$$

$$\therefore\ \frac{AX}{CX}=\frac{\sin C}{\sin(\theta+C)}, \quad\text{......................}(2)$$

since the angle $XAC=\pi-\theta-C$.

Dividing (1) by (2),

$$\frac{CX}{BX}=\frac{\sin(\theta+C)}{\sin(\theta-B)}\cdot\frac{\sin B}{\sin C},$$

i.e. $$\frac{n}{m}=\frac{\sin\theta\cos C+\cos\theta\sin C}{\sin C}\cdot\frac{\sin B}{\sin\theta\cos B-\cos\theta\sin B},$$

i.e. $$\frac{n}{m}=\frac{\sin\theta\cot C+\cos\theta}{\sin\theta\cot B-\cos\theta}=\frac{\cot C+\cot\theta}{\cot B-\cot\theta};$$

$$\therefore\ n(\cot B-\cot\theta)=m(\cot C+\cot\theta);$$

$$\therefore\ (m+n)\cot\theta=n\cot B-m\cot C.$$

This example may also be solved as follows.

Let P be the foot of the perpendicular from A on BC. Then

$$n\cot B-m\cot C=n\frac{BP}{AP}-m\frac{PC}{AP}$$

$$=\frac{n(BX+XP)-m(XC-XP)}{AP}$$

$$=\frac{(m+n)XC}{AP},\quad\text{since}\quad\frac{BX}{XC}=\frac{m}{n};$$

$$\therefore\ n\cot B-m\cot C=(m+n)\cot\theta.$$

Ex. 5. *Shew that* $(c+a)\sin\frac{1}{2}B=b\cos\frac{1}{2}(C-A)$.

Using formula I., p. 323,

$$c+a=2R(\sin C+\sin A)=4R\sin\tfrac{1}{2}(C+A)\cos\tfrac{1}{2}(C-A),$$

and $$b=2R\sin B=4R\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B.$$

Therefore, since $\sin\frac{1}{2}(C+A)=\cos\frac{1}{2}B$,

$$(c+a)\sin\tfrac{1}{2}B=b\cos\tfrac{1}{2}(C-A).$$

Inverse Notation. The angle whose sine is x is often written $\sin^{-1}x$. But as the number of angles, the sine of each of which is x, is unlimited, it is usual to restrict $\sin^{-1}x$ to mean the acute angle whose sine is x.

Thus $$\sin^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}, \quad \sin^{-1}(-\tfrac{1}{2})=-\frac{\pi}{6}.$$

Similarly, $\tan^{-1}x$ is the acute angle whose tangent is x.

Thus $$\tan^{-1}\sqrt{3}=\frac{\pi}{3}, \quad \tan^{-1}(-1)=-\frac{\pi}{4}.$$

Again, $\cos^{-1}x$ means the smallest positive angle whose cosine is x.

Thus $$\cos^{-1}(\tfrac{1}{2})=\frac{\pi}{3} \quad \text{and} \quad \cos^{-1}(-\tfrac{1}{2})=\frac{2\pi}{3}.$$

Ex. 1. *Shew that*

$$\sin^{-1}\tfrac{3}{5}=\sin^{-1}\tfrac{56}{65}-\cos^{-1}\tfrac{12}{13}.$$

Let $\alpha=\sin^{-1}\frac{3}{5}$, $\beta=\sin^{-1}\frac{56}{65}$, $\gamma=\cos^{-1}\frac{12}{13}$.

Then $\sin\alpha=\frac{3}{5}$, $\sin\beta=\frac{56}{65}$ and $\cos\gamma=\frac{12}{13}$,

and we have to shew that $\alpha=\beta-\gamma$.

Note that α, β and γ are all positive acute angles.

Hence $$\cos\beta=+\sqrt{1-\left(\frac{56}{65}\right)^2}=\frac{33}{65},$$

$$\sin\gamma=+\sqrt{1-\frac{144}{169}}=\frac{5}{13}.$$

Now, $$\begin{aligned}\sin(\beta-\gamma)&=\sin\beta\cos\gamma-\cos\beta\sin\gamma\\&=\frac{56}{65}\cdot\frac{12}{13}-\frac{33}{65}\cdot\frac{5}{13}\\&=\frac{672-165}{65.13}\\&=\frac{3}{5}.\end{aligned}$$

$$\therefore\ \beta-\gamma=\alpha.$$

Hence $\beta-\gamma$ is a positive acute angle whose sine is 3/5.

Ex. 2. *Prove that*

$$\tan^{-1}\tfrac{1}{2}-\tan^{-1}\tfrac{2}{11}=2\tan^{-1}\tfrac{1}{7}.$$

Let $\tan^{-1}\frac{1}{2}=\alpha$, $\tan^{-1}\frac{2}{11}=\beta$ and $\tan^{-1}\frac{1}{7}=\gamma$.

We have to shew that $\alpha-\beta=2\gamma$,
where $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{2}{11}$ and $\tan\gamma=\frac{1}{7}$.

Now
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\frac{1}{2}-\frac{2}{11}}{1+\frac{1}{11}}=\frac{7}{24}$$

and
$$\tan 2\gamma=\frac{2\tan\gamma}{1-\tan^2\gamma}=\frac{\frac{2}{7}}{1-\frac{1}{49}}=\frac{7}{24};$$

$$\therefore\ \alpha-\beta=2\gamma.$$

*Ex. 3. *Find a value of x which satisfies the equation*

$$2\tan^{-1}x+\sec^{-1}x=\frac{\pi}{2}.$$

Let
$$\tan^{-1}x=\alpha \quad\text{and}\quad \sec^{-1}x=\beta,$$

so that
$$2\alpha+\beta=\frac{\pi}{2} \qquad\ldots\ldots\ldots\ldots(1)$$

and
$$\tan\alpha=x=\sec\beta. \qquad\ldots\ldots\ldots\ldots(2)$$

Using (1), $2\alpha=\frac{\pi}{2}-\beta$ and $\tan 2\alpha=\cot\beta$;

$$\therefore\ \frac{2\tan\alpha}{1-\tan^2\alpha}=\cot\beta=\frac{1}{\sqrt{\sec^2\beta-1}},$$

i.e., using (2),
$$\frac{2x}{1-x^2}=\frac{1}{\sqrt{x^2-1}}.$$

Squaring,
$$4x^2(x^2-1)=(1-x^2)^2,$$

i.e.
$$(x^2-1)(4x^2-x^2+1)=0.$$

The first factor gives $x=\pm 1$; the second factor gives unreal solutions.

Testing $x=1$, the left-hand side of the given equation is

$$2\tan^{-1}1+\sec^{-1}1,$$

i.e.
$$2\cdot\frac{\pi}{4}+0 \quad\text{or}\quad \frac{\pi}{2}.$$

Testing $x=-1$, the left-hand side of the given equation is

$$2\tan^{-1}(-1)+\sec^{-1}(-1),$$

i.e.
$$2\left(-\frac{\pi}{4}\right)+\pi \quad\text{or}\quad \frac{\pi}{2};$$

$$\therefore\ x=\pm 1 \text{ are solutions.}$$

EXERCISES XLI.

1. The sides of a triangle are 5, 6 and 7 ft. long. Find the smallest angle.

2. In any triangle, prove that

(i) $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$;

(ii) $\sin A+\sin B>\sin C$;

(iii) $b\sin^2\frac{C}{2}+c\sin^2\frac{B}{2}=\frac{1}{2}(b+c-a)$;

(iv) $\tan A=\dfrac{a\sin C}{b-a\cos C}$.   (v) $\dfrac{a\sin(B-C)}{b\sin(C-A)}=\dfrac{b^2-c^2}{c^2-a^2}$.

(vi) $2bc\cos A+2ca\cos B+2ab\cos C=a^2+b^2+c^2$.

3. Prove, using projection methods, that

$$b\cos(C-\theta)+c\cos(B+\theta)=a\cos\theta.$$

4. The sides of a triangle are x, y, $\sqrt{x^2+y^2+xy}$; find the greatest angle.

5. Find the lengths of the medians and the length of the bisector of the angle A of a triangle in terms of the sides a, b, c.

6. In any triangle, prove that

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$$

7. If X is the middle point of AB in the triangle ABC, and if AD is perpendicular to BC and cuts CX in Y, shew that

$$AY=\frac{ab\sin C}{a+b\cos C}.$$

8. The side BC of a triangle ABC is divided at X so that $q\,.\,BX=r\,.\,XC$. Shew that, if the angles BAX, XAC are β, γ, respectively, then $qc\sin\beta=rb\sin\gamma$.

9. If $ABCD$ is a cyclic quadrilateral, shew that

$$2(AB\,.\,AD+BC\,.\,CD)\cos A=AB^2+AD^2-BC^2-CD^2.$$

10. The sides of a triangle are 4, 5, 6 ft. long; shew that one angle is twice one of the others.

11. Prove that, in any triangle,

(i) $a\cos\dfrac{B-C}{2}=(b+c)\sin\dfrac{A}{2}$;   (ii) $a\sin\dfrac{B-C}{2}=(b-c)\cos\dfrac{A}{2}$

Given $b=56{\cdot}28$, $c=63{\cdot}28$, $A=46°\ 29'$, find the other sides and angles. [B.U.]

12. Shew that the area of a triangle $= \dfrac{a^2}{2(\cot B + \cot C)}$.

13. The bisector of the angle A of a triangle ABC meets BC in D. Shew that the perpendiculars from B and C on AD are given by

$$\frac{AD \tan\theta \tan\frac{A}{2}}{\tan\theta \pm \tan\frac{A}{2}},$$

where θ is the angle ADC. Shew also that the area of the triangle ABC is

$$AD^2 \tan\frac{A}{2}\left\{1 + \frac{\tan^2\frac{A}{2}}{\tan^2\theta - \tan^2\frac{A}{2}}\right\}.$$

14. In any triangle, shew that

$$\cot C = \frac{a}{c}\operatorname{cosec} B - \cot B = \frac{b}{c}\operatorname{cosec} A - \cot A.$$

The elevation of a mountain from a place due E. of it is 45°: on walking $1\frac{1}{4}$ miles towards the S.E. it becomes 40°. Find the height of the mountain.

15. One end P of a rod PQ of length l is made to move round a circle of centre C and radius a, while the other end Q moves along a tangent to the circle at the fixed point O. Prove that the distance x of Q from O is given by

$$x^2 - 2ax\sin\theta = l^2 - 2a^2(1 - \cos\theta),$$

where $O\hat{C}P = \theta$.

[L.U.]

16. In order to fix the position of a point P at sea with reference to the shore the angles APB, BPC, and APC, where A, B, and C are known points on land, are measured with a sextant and found to be α, β, and $\alpha + \beta$. Describe a geometrical method for obtaining the position of P on a plan, and shew that if BP meets AC in E, then

$$AE : EC :: \cot\beta - \cot A : \cot\alpha - \cot C,$$

where A and C are the angles at A and C of the triangle ABC.

[L.U.]

17. Two tangents PT, QT are drawn to a circle at the ends of an arc PQ, which subtends an angle 2θ in radians at the centre. Prove that, if the arc PQ bisects the area of the triangle PTQ,

$$2\theta = \tan\theta + \sin\theta\cos\theta.$$

Shew from the tables that the value of θ lies between 55° and 60°.

[L.U.]

18. A derrick, consisting of three unequal poles, has to be made with its feet at the corners of a horizontal triangle ABC, of which $AB = 14$ ft. and $BC = 16$ ft. The upper ends of the three poles are held together at D by an iron frame. The point E in the plane of ABC vertically under D is 8 ft. from A, 9 ft. from B and 10 ft. from C, while DE is 18 ft. Calculate the lengths of the poles and the angles ADB, ADC. [L.U.]

19. Prove that

(i) $\cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\dfrac{1}{\sqrt{2}} = \dfrac{5\pi}{12}$;

(ii) $\tan^{-1}1 + \cot^{-1}\dfrac{1}{\sqrt{3}} + \sin^{-1}\dfrac{1}{2} = \dfrac{3\pi}{4}$;

(iii) $2\cot^{-1}2 + \sin^{-1}\dfrac{3}{5} = \dfrac{\pi}{2}$;

(iv) $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{11}{10}$;

(v) $\cos^{-1}\frac{4}{5} - \sin^{-1}\frac{16}{55} = 2\tan^{-1}\frac{1}{5}$.

***20.** Find the value of x which satisfies:

(i) $2\tan^{-1}2x - \sin^{-1}x = \cos^{-1}x$;

(ii) $\tan^{-1}2x + \cos^{-1}x = \dfrac{\pi}{2}$;

(iii) $\cos^{-1}x + 2\sin^{-1}x = \dfrac{3\pi}{4}$.

CHAPTER XII.

THE SOLUTION OF TRIANGLES. FURTHER PROPERTIES OF TRIANGLES. POLAR EQUATIONS OF LINE AND CIRCLE.

WHEN certain of the *parts* of a triangle are given, it is possible to find the remaining parts by using the formulae proved on pp. 323-325. But of these, the only one adapted for logarithmic calculation is

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

The remaining formulae are replaced by others, which are proved below.

I. Given the sides, to find the angles.

Since $a^2=b^2+c^2-2bc\cos A$,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}; \qquad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$\therefore\ 1+\cos A=\frac{b^2+2bc+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc};$$

$$\therefore\ 2\cos^2\frac{A}{2}=\frac{(b+c+a)(b+c-a)}{2bc}.$$

It is usual to write $2s$ for $a+b+c$; *i.e.* s is the semi-perimeter of the triangle.

$$\therefore\ 2\cos^2\frac{A}{2}=\frac{2s(2s-2a)}{2bc};$$

$$\therefore\ \cos^2\frac{A}{2}=\frac{s(s-a)}{bc};$$

and since $\frac{A}{2}$ is an acute angle,

$$\cos\frac{\mathbf{A}}{2} = +\sqrt{\frac{\mathbf{s(s-a)}}{\mathbf{bc}}}.$$

Again, from (1),

$$1-\cos A = \frac{a^2-b^2+2bc-c^2}{2bc};$$

$$\therefore\ 2\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}$$

$$= \frac{(2s-2b)(2s-2c)}{2bc};$$

$$\therefore\ \sin\frac{\mathbf{A}}{2} = +\sqrt{\frac{\mathbf{(s-b)(s-c)}}{\mathbf{bc}}}.$$

Finally, $$\tan\frac{\mathbf{A}}{2} = \frac{\sin\frac{\mathbf{A}}{2}}{\cos\frac{\mathbf{A}}{2}} = \sqrt{\frac{\mathbf{(s-b)(s-c)}}{\mathbf{s(s-a)}}}.$$

These formulae are adapted for logarithmic calculation, and enable us to solve any triangle when the sides are given.

Ex. 1. *Express the area of a triangle in terms of its sides.*

On p. 325 the area of a triangle ABC is given by

$$\Delta = \tfrac{1}{2}bc\sin A;$$

$$\therefore\ \Delta = bc\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Ex. 2. *Find the angles of the triangle whose sides are* 17·4, 10·6 *and* 8·1 *inches.*

Let $a = 17{\cdot}4$, $b = 10{\cdot}6$ and $c = 8{\cdot}1$;

$$\therefore\ 2s = 36{\cdot}1, \quad i.e.\ s = 18{\cdot}05;$$

$$\therefore\ s-a = 0{\cdot}65, \quad s-b = 7{\cdot}45 \quad \text{and} \quad s-c = 9{\cdot}95.$$

Now $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{7\cdot45\times9\cdot95}{18\cdot05\times0\cdot65}};$$

$$\therefore\ \log\tan\frac{A}{2}=\tfrac{1}{2}\{0\cdot8722+0\cdot9978-1\cdot2565-1\cdot8129\}$$
$$=\tfrac{1}{2}\{0\cdot8006\}$$
$$=0\cdot4003;$$
$$\therefore\ \tfrac{1}{2}A=68^\circ\,18', \quad \textit{i.e.}\ A=136^\circ\,36'.$$

Again, $$\log\tan\frac{B}{2}=\tfrac{1}{2}\{0\cdot9978+\overline{1}\cdot8129-1\cdot2565-0\cdot8722\}$$
$$=\tfrac{1}{2}\{\overline{2}\cdot6820\}$$
$$=\overline{1}\cdot3410;$$
$$\therefore\ \tfrac{1}{2}B=12^\circ\,22', \quad \textit{i.e.}\ B=24^\circ\,44'.$$

And $$\log\tan\frac{C}{2}=\tfrac{1}{2}\{\overline{1}\cdot8129+0\cdot8722-1\cdot2565-0\cdot9978\}$$
$$=\tfrac{1}{2}\{\overline{2}\cdot4380\}$$
$$=\overline{1}\cdot2154;$$
$$\therefore\ \tfrac{1}{2}C=9^\circ\,20', \quad \textit{i.e.}\ C=18^\circ\,40'.$$

The value of C was found as a check on the accuracy of the work. It is clear that $A+B+C=180^\circ$.

II. Given two sides and the included angle, to solve the triangle.

Since $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$

$$\frac{b-c}{b+c}=\frac{2R\sin B-2R\sin C}{2R\sin B+2R\sin C}$$
$$=\frac{\sin B-\sin C}{\sin B+\sin C}$$
$$=\frac{2\sin\frac{1}{2}(B-C)\cos\frac{1}{2}(B+C)}{2\cos\frac{1}{2}(B-C)\sin\frac{1}{2}(B+C)}$$
$$=\tan\tfrac{1}{2}(B-C)\,.\,\cot\tfrac{1}{2}(B+C);$$

$$\therefore\ \tan\frac{B-C}{2}\tan\frac{A}{2}=\frac{b-c}{b+c}, \quad \text{since}\ \frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2};$$

$$\therefore\ \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

The method of solving a triangle in this case is illustrated below.

Ex. *Solve the triangle when*

$$b = 28{\cdot}7, \quad c = 15{\cdot}6 \quad \textit{and} \quad A = 55^\circ\, 22'.$$

$$\tan\frac{B-C}{2} = \frac{28{\cdot}7 - 15{\cdot}6}{28{\cdot}7 + 15{\cdot}6}\cot 27^\circ\, 41' = \frac{13{\cdot}1}{44{\cdot}3}\cot 27^\circ\, 41';$$

$$\therefore \log\tan\tfrac{1}{2}(B-C) = 1{\cdot}1173 + 0{\cdot}2802 - 1{\cdot}6464 = \bar{1}{\cdot}7511;$$

$$\therefore \tfrac{1}{2}(B-C) = 29^\circ\, 25',$$

and $\tfrac{1}{2}(B+C) = 90^\circ - \tfrac{1}{2}A = 62^\circ\, 19'$;

$\therefore$ adding, $B = 91^\circ\, 44'$ and subtracting, $C = 32^\circ\, 54'$.

Again, $$\frac{a}{\sin 55^\circ\, 22'} = \frac{28{\cdot}7}{\sin 91^\circ\, 44'};$$

$$\therefore \log a = 1{\cdot}4579 + \bar{1}{\cdot}9152 - \bar{1}{\cdot}9998 = 1{\cdot}3733;$$

$$\therefore a = 23{\cdot}6.$$

III. **Given two angles and any side, the triangle can be solved by the formula** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. For, if A, B and c are given, the angle C is known, being $180^\circ - B - A$;

b is given by $\dfrac{c\sin B}{\sin C}$, and a is given by $\dfrac{c\sin A}{\sin C}$.

IV. **Given two sides and an angle not contained by them.** This is known as the *Ambiguous Case*, for there may be two, one or no solutions.

If A, a, and b are given, it can be shewn by drawing figures that there is always one solution if $a > b$.

Fig. 143 shews that

(i) if $a < b$ and $> b\sin A$, there are two triangles, ACB_1 and ACB_2;

(ii) if $a < b$ and $= b\sin A$, there is one triangle;

while (iii) if $a < b\sin A$, no triangle is possible.

Ex. *Solve the triangle, given* $A = 40^\circ$, $a = 6{\cdot}2$, $b = 8{\cdot}5$.

Since $$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{6{\cdot}2}{\sin 40^\circ} = \frac{8{\cdot}5}{\sin B}.$$

$$\therefore \log\sin B = 0{\cdot}9294 + \bar{1}{\cdot}8081 - 0{\cdot}7924 = \bar{1}{\cdot}9451;$$

$$\therefore B = 61^\circ\, 48' \quad \text{or} \quad 180^\circ - 61^\circ\, 48'.$$

Using the notation of Fig. 143, $B_1 = 61° 48'$, $B_2 = 118° 12'$. If $AB_1 = c_1$ and $ACB_1 = C_1$, then
$$C_1 = 180° - 40° - 61° 48' = 78° 12';$$
and
$$\frac{c_1}{\sin 78° 12'} = \frac{6\cdot 2}{\sin 40°}, \quad \text{whence } c_1 = 9\cdot 44.$$
Again
$$C_2 = 180° - 40° - 118° 12' = 21° 48'; \quad \text{whence } c_2 = 3\cdot 58.$$

Thus, the two solutions are

(i) $B_1 = 61° 48'$, $C_1 = 78° 12'$, $c_1 = 9\cdot 44$,

(ii) $B_2 = 118° 12'$, $C_2 = 21° 48'$, $c_2 = 3\cdot 58$.

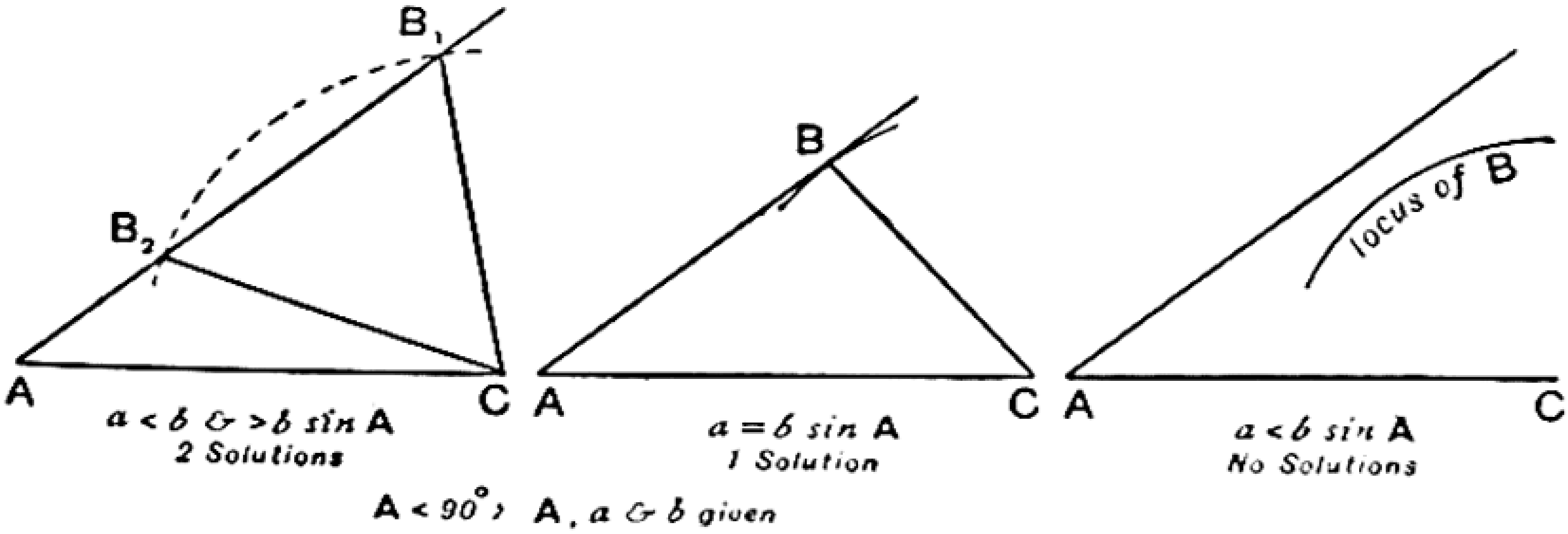


FIG. 143.

From Fig. 143 it should be clear that $c_1 + c_2 = 2b \cos A$ which gives a check on the accuracy of the calculation.

EXERCISES XLII.

Solve the triangles in Examples 1-3.

1. (i) $A = 36° 25'$, $B = 66° 10'$, $c = 8\cdot 3$;
(ii) $B = 135°$, $C = 21°$, $a = 10$;
(iii) $B = 75° 27'$, $C = 45° 9'$, $b = 28$.

2. (i) $a = 34\cdot 5$, $b = 28\cdot 2$, $c = 26\cdot 2$
(ii) $a = 56\cdot 0$, $b = 76\cdot 0$, $c = 108\cdot 0$,
(iii) $a = 5\cdot 09$, $b = 2\cdot 73$, $c = 3\cdot 92$.

3. (i) $b = 387$, $c = 428$, $A = 49° 30'$;
(ii) $a = 76\cdot 3$, $b = 56\cdot 7$, $C = 108°$;
(iii) $a = 34$, $c = 58$, $B = 45° 37'$.

4. Ascertain how many triangles satisfy the following conditions and solve each triangle completely.

(i) $C = 150°$, $c = 9{\cdot}3$, $b = 7{\cdot}5$;
(ii) $C = 50°$, $c = 9{\cdot}3$, $b = 7{\cdot}5$;
(iii) $C = 50°$, $c = 7{\cdot}5$, $b = 9{\cdot}3$;
(iv) $C = 50°$, $c = 7{\cdot}5$, $b = 10{\cdot}3$.

5. Solve the following triangles:

(i) $a = 44$, $b = 72$, $C = 110°$;
(ii) $b = 37{\cdot}4$, $A = 81° 14'$, $C = 35° 17'$;
(iii) $b = 5$, $a = 6$, $c = 4$;
(iv) $C = 107°$, $b = 312$, $c = 456$;
(v) $b = 9{\cdot}52$, $c = 13{\cdot}84$, $a = 9{\cdot}87$;
(vi) $a = 12{\cdot}65$, $b = 9{\cdot}45$, $C = 72° 14'$.

6. By means of tables solve the equation $3 \sin x = 2 \sin (70° - x)$. Shew that the solution may be found by a geometrical construction as follows: describe a triangle ABC in which the ratio of AB to AC is $3 : 2$ and the angle BAC is 70°; bisect BC at D, then the angle BAD is a solution of the equation. Verify by means of the construction the solution found from the tables. [L.U.]

7. Prove that, in any triangle, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, and deduce the value of $\tan \frac{1}{2}A$ in terms of the sides.

The sides of a triangle are 123 yards, 79 yards and 97 yards. Find its angles as accurately as you can. [B.U.]

8. Obtain an expression for the radius of the circumcircle of a triangle in terms of the sides and angles.

Solve the triangle in which $R = a = 5{\cdot}5$ cm., $B = 75°$. [B.U.]

9. Find the area of the quadrilateral field $ABCD$, when AB is parallel to DC, given that AB, BC, CD are 80, 110 and 176 yards respectively, and that the angle ABC is 105° 23′.

Find also the length of the diagonal AC. [B.U.]

10. Prove that, in any triangle, $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

Find the greatest angle, the area and the radius of the inscribed circle of a triangle having the sides 7·3, 6·8 and 9·7. [Durham.]

11. Establish for a triangle the formula

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Given $b = 532{\cdot}4$, $c = 647{\cdot}1$, $A = 75° 14'$, find B, C and a. [S.W.]

12. Explain how to determine the angles of a triangle, whose sides are given, proving the formulae employed.

The sides of a triangle are 777, 673, 400; find the angles. Shew that one angle is accurately 60°. [L.U.]

13. Shew that there may be two triangles, or possibly no triangles at all, having assigned values for a, b, A.

When $a=25$, $b=30$, $A=50°$, determine how many such triangles exist and complete their solution. [L.U.]

14. Prove the formulae for any plane triangle

$$\text{(i)} \quad \tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\ ;$$

$$\text{(ii)} \quad \text{area} = s^2 \tan \tfrac{1}{2}A \,.\, \tan \tfrac{1}{2}B \,.\, \tan \tfrac{1}{2}C.$$

Find the perimeter of a triangle of angles 40°, 65° and 75° whose area is 20 sq. ft. [L.U.]

15. Explain how to solve a triangle when one side and two angles are given, proving the necessary formula.

AB is a base line of length 3000 yards, and C, D are points such that $B\hat{A}C = 32° \, 15'$. $A\hat{B}C = 119° \, 5'$, $D\hat{B}C = 60° \, 10'$, $B\hat{C}D = 78° \, 45'$, A and D being on the same side of BC; prove that the length of CD is 4405 yards approximately. [L.U.]

16. Prove that in a triangle

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

when s is half the sum of the sides a, b, c.

A quadrilateral $ABCD$ has sides $AB=8$, $BC=7$, $CD=9$, $DA=10$ and diagonal $BD=12$, all in the same units. Find its area and the angle at A. [L.U.]

17. Prove that, in any triangle, $\tan \frac{1}{2}(A-B) = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$.

Given $a=3621$, $b=2854$ and $C=39°$, solve the triangle completely. [L.U.]

18. In Fig. 143, a, b, A are given. If $AB_1 = c_1$, $AB_2 = c_2$, $A\hat{B}_1C = B_1$, $A\hat{B}_2C = B_2$, $A\hat{C}B_1 = C_1$ and $A\hat{C}B_2 = C_2$, prove the following relations:

$$(1) \quad c_1 - c_2 = 2a \cos B_1\ ;$$

$$(2) \quad a \cos \frac{C_1 - C_2}{2} = b \sin A\ ;$$

$$(3) \quad c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2 \cos^2 A\ ;$$

$$(4) \quad c_1^2 + c_1c_2 + c_2^2 = a^2 - b^2(1 - 4\cos^2 A).$$

Practical Applications of Solutions of Triangles.

Ex. 1. *A hill-top is due South-west of a certain station at sea-level, from which its elevation is observed to be 23° 30′. A second station whose bearing is N. 69° W. from the first is situated at a height 850 ft. above sea-level, and from this station the hill-top bears due South and has an elevation of 18° 9′. Find the height of the hill-top.*

Fig. 144 shews the plans A, B, C of the first station, the second station and the hill-top respectively.

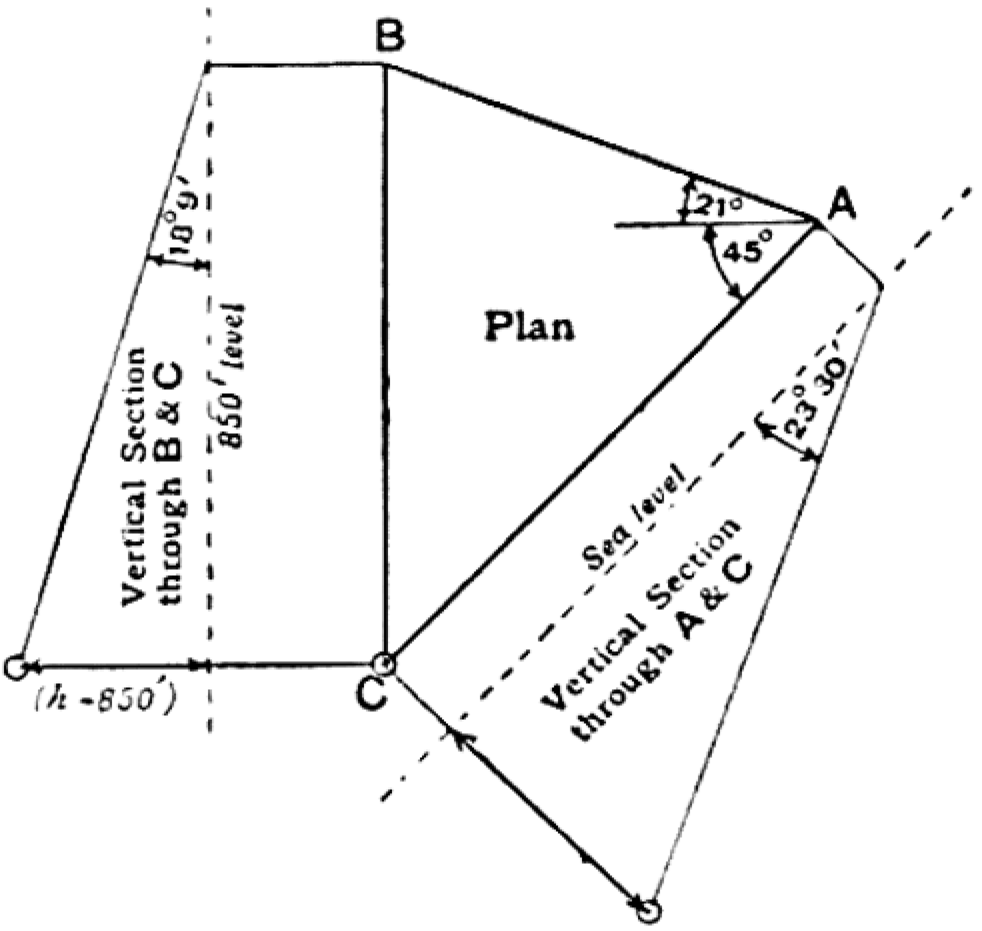

B
A
21°
45°
Plan
18°9′
850′ level
Vertical Section through B & C
(h -850′)
C
23°30′
Sea level
Vertical Section through A & C

FIG 144.

If h ft. be the height of the hill-top above sea-level,

$$AC = h \cot 23^\circ 30',$$

and

$$BC = (h - 850) \cot 18^\circ 9'.$$

These should be clear from the vertical sections shewn in the figure.

The angle

$$CBA = 90^\circ - 21^\circ = 69^\circ.$$

The angle

$$CAB = 21^\circ + 45^\circ = 66^\circ.$$

Using the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$,

we have
$$\frac{(h-850)\cot 18^\circ 9'}{\sin 66^\circ} = \frac{h \cot 23^\circ 30'}{\sin 69^\circ};$$

$$\therefore \log\left(\frac{h-850}{h}\right) = \log\cot 23^\circ 30' + \log\sin 66^\circ - \log\cot 18^\circ 9' - \log\sin 69^\circ$$

$$= 0{\cdot}3617 + \bar{1}{\cdot}9607 - 0{\cdot}4844 - \bar{1}{\cdot}9702$$

$$= \bar{1}{\cdot}8678;$$

$$\therefore h - 850 = 0{\cdot}7375h,$$

whence
$$h = \frac{850}{0{\cdot}2625} \doteqdot 3240.$$

*Ex. 2. *Two planes intersect at right angles in the line XY. Another plane cuts them in lines AP, AQ. If the angle $PAY = \theta$ and the angle $QAY = \phi$, shew that the tangent of the angle which XY makes with the plane PAQ is*

$$(\tan\theta \tan\phi)/\surd(\tan^2\theta + \tan^2\phi).$$

Fig. 145 shews the vertical trace AP and the horizontal trace AQ of the given plane PAQ. $X'Y'$ is the horizontal trace of a new vertical plane perpendicular to AQ. BC is perpendicular to $X'Y'$ and equal to BP, so that QC is the edge view of the plane PAQ.

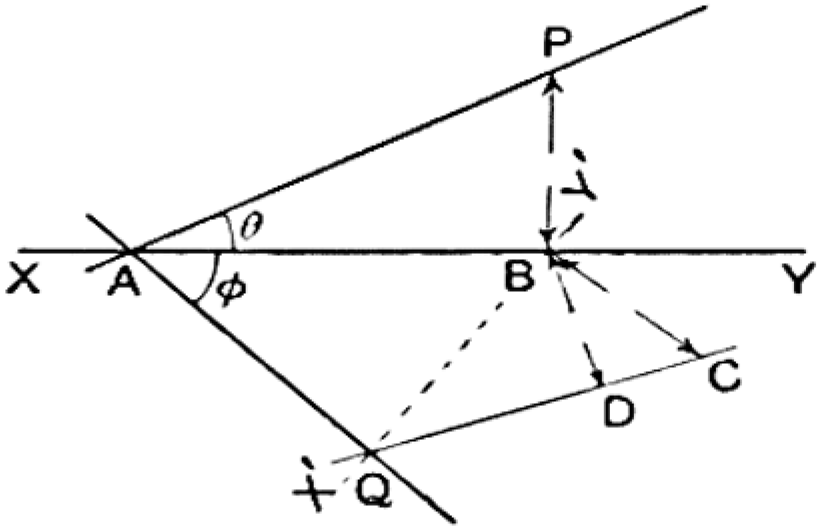


FIG. 145.

If now BD be drawn perpendicular to QC, BD is the length of the perpendicular from B to the plane PAQ.

If α be the angle between XY and the plane PAQ,

$$\sin\alpha = BD/AB. \quad \text{.........................(1)}$$

Let $AB = l$, then $PB = l\tan\theta = BC$, and $BQ = l\sin\phi$.

Also $\qquad BD \times QC = QB \times BC$;

$$\therefore \quad BD = \frac{l\sin\phi \times l\tan\theta}{\surd l^2\sin^2\phi + l^2\tan^2\theta} = \frac{l\sin\phi\tan\theta}{\surd\sin^2\phi + \tan^2\theta}.$$

Using (1), $\quad \sin\alpha = \dfrac{\sin\phi\tan\theta}{\surd\sin^2\phi + \tan^2\theta}$;

$$\therefore \quad \tan\alpha = \frac{\sin\phi\tan\theta}{\surd\sin^2\phi + \tan^2\theta - \sin^2\phi\tan^2\theta}$$

$$= \frac{\sin\phi\tan\theta}{\surd\sin^2\phi + \tan^2\theta\cos^2\phi}$$

$$= \frac{\tan\phi\tan\theta}{\surd\tan^2\phi + \tan^2\theta}.$$

Alternative Solution.

Fig. 146 shews a perspective view of the plane PAQ. B is any point in XY, BP is in the vertical plane and perpendicular to XY, BQ is in the horizontal plane and perpendicular to XY, BD is perpendicular to the plane PAQ.

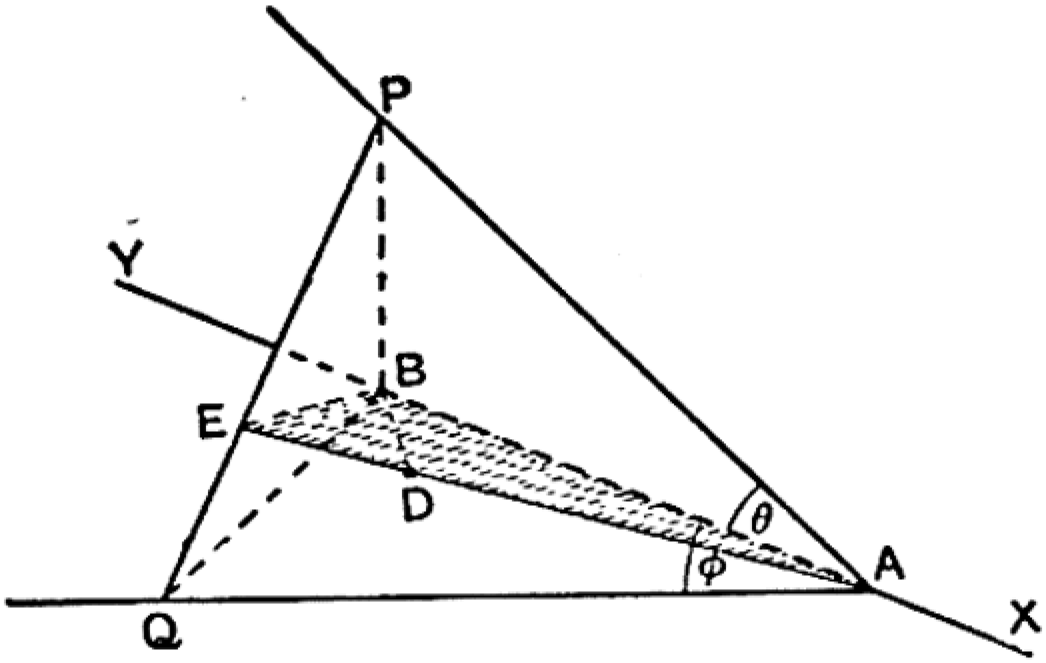


FIG. 146.

Clearly AB is perpendicular to the plane PBQ, and therefore PQ is perpendicular to AB.

Also PQ is perpendicular to BD, since BD is perpendicular to the plane PAQ.

Hence PQ, being perpendicular to both AB and BD, is perpendicular to the plane ABD, and therefore PQ is perpendicular to BE.

Hence $BE \times PQ = BP \times BQ$. ....(2)

If $AB = l$ and $B\hat{A}E = a$, $BE = l \tan a$,

$$BP = l \tan \theta,$$
$$BQ = l \tan \phi,$$
$$PQ = \sqrt{l^2 \tan^2 \theta + l^2 \tan^2 \phi}.$$

Using (2), $\tan a \sqrt{\tan^2 \theta + \tan^2 \phi} = \tan \theta \tan \phi$.

$$\therefore \tan a = \frac{\tan \theta \tan \phi}{\sqrt{\tan^2 \theta + \tan^2 \phi}}.$$

*Ex. 3. *The edges of a parallelepiped are x, y, z, and the angles at a corner are 2θ, a and a. Shew that its volume is*

$$2xyz \sin \theta \sqrt{\{\sin^2 a - \sin^2 \theta\}}.$$

Fig. 147 shews one face $OCDB$ and one inclined edge OA of the parallelepiped. AE is perpendicular to the plane OCD and AF is perpendicular to OB.

By symmetry, E lies in OD.

Since AE is perpendicular to the plane OCD, AE and OB are at right angles. Also AF and OB are at right angles.

$\therefore$ OB is perpendicular to the plane AFE;

$\therefore$ FE is perpendicular to OB.

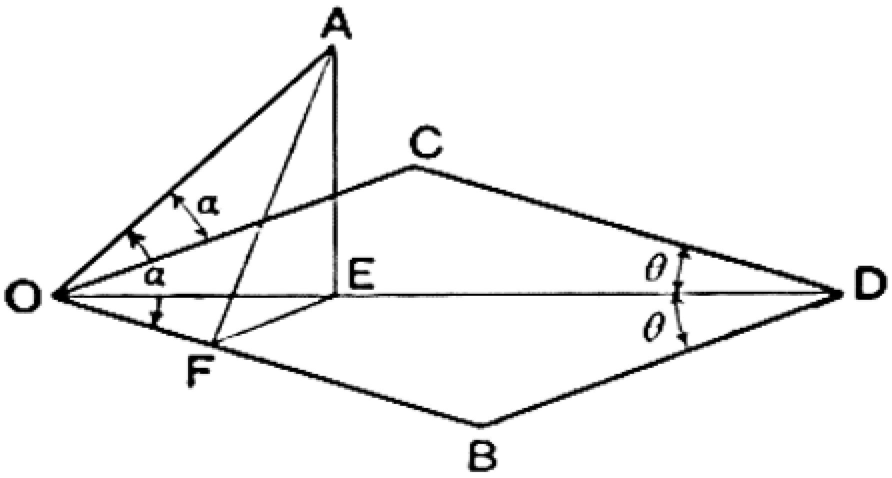


FIG. 147.

The volume of the parallelepiped

$$= \text{area } OCDB \times \text{height } AE. \quad \dots\dots\dots\dots\dots (1)$$

Now, area $OCDB = yz \sin 2\theta$.

Also $OF = x \cos a$;

$$\therefore \quad EF = x \cos a \tan \theta,$$

$$\text{and} \quad AF = x \sin a,$$

$$\therefore \quad AE = \surd\{x^2 \sin^2 a - x^2 \cos^2 a \tan^2 \theta\}$$

$$= \frac{x}{\cos \theta} \surd\{\sin^2 a \cos^2 \theta - \cos^2 a \sin^2 \theta\}$$

$$= \frac{x}{\cos \theta} \surd\{\sin^2 a - \sin^2 \theta\}.$$

Using (1), volume $= yz \sin 2\theta \times \dfrac{x}{\cos \theta} \surd\{\sin^2 a - \sin^2 \theta\}$,

$$= 2xyz \sin \theta \surd\{\sin^2 a - \sin^2 \theta\}.$$

EXERCISES XLIII.

1. From a place A due North of a tower, the angular elevation of its top is 30°: from B, due East of and 150 yards away from A, the angular elevation of the top of the tower is 18°. Find the height of the tower.

2. $PQRS$ is the plan of a rectangular hall and $PQ = 48$ ft. The corner of the ceiling over R subtends an angle of 18° at P and an angle of 30° at Q. Find the height of the room.

3. In a tetrahedron $ABCD$, $AB = AC = AD = a$ and these lines make equal angles θ with each other; prove that the distance of A from the face BCD is

$$a\left(\frac{1+2\cos\theta}{3}\right)^{\frac{1}{2}}.$$

[B.U.]

4. Define the angle between two planes.

A pyramid on a square base, whose side is 6 inches, has its sloping edges all of the same length, viz. 8 inches.

Find (1) the volume of the pyramid,

(2) the angle between a sloping face and the base,

(3) the angle between two adjacent sloping faces. [B.U.]

5. Three points, A, B, C are in a straight line; the distance AB is 5 miles and BC is 4 miles. A point O not in the line is $6\frac{1}{2}$ miles from A, and the distance OC exceeds the distance OB by 3 miles. Calculate the distances OB and OC, and the angle OBA. [B.U.]

6. The triangle ABC has a right angle at A, and AB is 70 while AC is 100. The plane of the triangle makes an angle of 56° with a horizontal plane through BC; find the height of A above this plane, and the inclination of AB to the plane. [S.W.]

7. A post AB is observed from two points P and Q in the same horizontal plane as B, the foot of the post. P and Q are 200 ft. apart. The angles BPQ, BQP, APB are 65°, 60°, 20° respectively. Find the height of the post to the nearest quarter of a foot. [Man.]

8. A balloon is observed from two stations A and B at the same level, A being 1000 ft. North of B. At a given instant, the balloon appears from A to be in a direction N. 33° 12′ E. and to have an elevation 53° 25′, while from B it appears in a direction N. 21° 27′ E. Obtain the height of the balloon, and find the angle of elevation which should be observed at B. [Man.]

9. Two observers 500 yd. apart take the bearing and elevation of a balloon at the same instant. One finds that the bearing is N. 41° E. and the elevation 24°; the other finds that the bearing is N. 32° E. and the elevation 16°. Calculate the height of the balloon. [B.U.]

10. An aeroplane is observed by two men A and B stationed at consecutive milestones on a level road. When it is over the road between them A observes it to have an elevation of 60° and B one of 45°. Shortly afterwards the elevations have become 45° and 30° respectively. Assuming that the aeroplane is flying horizontally at 40 miles per hour, find the time which has elapsed between the observations. [B.U.]

11. If a straight line a is perpendicular to each of two intersecting straight lines, b and c, prove that it is perpendicular to the plane through b and c.

Two skew straight lines AC, BD are both at right angles to AB. From a point P of AC a perpendicular PM is drawn to BD. Prove that $PM^2 - AB^2 = AP^2 \sin^2 \alpha$, where α is the angle between the lines AC and BD. [Man.]

12. A ship is steaming at 13 knots in a direction E. 57° 48′ N. and a lighthouse is observed to bear E. 26° 37′ N. Ten minutes later the lighthouse bears E. 18° 53′ N. How near will the ship approach the lighthouse ?

13. From a ship two lighthouses X, Y, are observed to bear E 19° 37′ S. and E. 59° 43′ S. If the bearing of Y from X is W. 63° S. and the distance XY is 5 miles, find the distances of the ship from X and Y.

14. Three points A, B, C, are in a line inclined at an angle θ to the horizon. Of these points, A is the lowest and C is the highest; and D is a point vertically above C. If $AB=a$, $CD=h$, the angle $DAB=\alpha$, and the angle $DBC=\beta$, prove that:

$$\cos\theta = \frac{a}{h}\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

15. The faces of a pyramid are isosceles triangles of vertical angle θ, and each face is inclined to the base of the pyramid at an angle ϕ. Find the angle each sloping edge makes with the base.

16. The angle of elevation of the top of a mountain is observed from a point A to be 24° and it bears due East. From a point B at the same level as A and bearing S. 5° W. from A, the mountain appears in a direction N. 59° E. If AB is 1400 yards, find the height of the mountain.

17. X is the foot of a vertical pole ; A and B are due East of X and Y is due South of B. $XA = 10$ ft., $AB = 20$ ft. and $BY = 30$ ft. If the elevation of the pole at A is double that at B, find the height of the pole and its angle of elevation at Y. Find also the angle which XA subtends at Y.

***18.** ABC is a triangle having the angle A 90° and AP is perpendicular to the plane of the triangle. If $PB=b$, $PC=c$, and the angle $BPC=\alpha$, shew that the sine of the angle between the planes BPC and BAC is

$$\sqrt{\left\{\frac{b^2+c^2-2bc\cos\alpha}{bc\sin\alpha\tan\alpha}\right\}}.$$

*19. The angle of elevation of a tower from a place, A, due South of it is θ, and from a place, B, due East of A, the elevation is ϕ. If $AB=a$, shew that the height of the tower is

$$\frac{a\sin\theta\sin\phi}{\sqrt{\{\sin(\theta-\phi)\sin(\theta+\phi)\}}}.$$

*20. A, B are the feet of two vertical poles. From a point C midway between A and B the elevation of the first pole is observed to be α. CD is at right angles to AB and in the horizontal plane. At D the elevation of the first pole is β and of the second γ. Shew that the heights of the poles are respectively

$$\frac{a\sin\alpha\sin\beta}{\sqrt{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}}} \quad\text{and}\quad \frac{a\sin\alpha\cos\beta\tan\gamma}{\sqrt{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}}}.$$

MISCELLANEOUS EXERCISES.

1. In any triangle prove that

$$\text{(i)}\quad a^2=b^2+c^2-2bc\cos A\ ;$$

$$\text{(ii)}\quad 4AD^2=b^2+c^2+2bc\cos A,$$

where D is the middle point of the side BC. [I.C.E.]

2. The sides of a triangle are in the proportion 11 : 9 : 4. Calculate the angles to the nearest minute, and make a rough check of your results by drawing a diagram to scale.

3. (1) With the usual notation, prove that the radius, R, of the circumscribed circle of a triangle is equal to $abc/4\triangle$.

(2) AD, BE, CF are the perpendiculars from the vertices of a triangle upon the opposite sides. If O is the intersection of these lines, prove that

$$\frac{OD}{\sec A}=\frac{OE}{\sec B}=\frac{OF}{\sec C}=2R\cos A\cos B\cos C.$$

[I.C.E.]

4. Two landmarks A and B are observed by a man to be at the same instant in a line due East. After he has walked $4\frac{1}{2}$ miles in a direction 30° North of East, A is observed to be due South, while B is 38° South of East. Find the distance between A and B. [Durham.]

5. Express $3\cos\theta-7\sin\theta$ in the form $A\cos(\theta+\alpha)$, and hence, or otherwise, solve the equation

$$3\cos\theta-7\sin\theta=5.$$

[B.U.]

6. Prove that

$$2\sin x + 3\cos x = r\sin(x+\alpha)$$

where (r, α) are polar coordinates of the point whose rectangular coordinates are 2, 3. Find the value of x which makes the given expression equal to 1 : find also the greatest value of the expression. [S.W.]

7. Prove the formula

$$\frac{1-\cos x}{1+\cos x} = \tan^2 \tfrac{1}{2}x.$$

Find what values of x (if any) lying between 0 and 180° satisfy respectively the equations

$$\cos x = -\cos \tfrac{1}{2}x, \qquad \cos x = -\sin \tfrac{1}{2}x.$$ [S.W.]

8. The axis of a right cone whose vertical angle is 2α, passes through the centre of a sphere of radius r, the distance of the vertex of the cone from the centre of the sphere being c. Show that the cone cuts the sphere in two circles, whose radii x_1 and x_2 are the roots of the quadratic

$$x^2 - cx\sin 2\alpha + (c^2 - r^2)\sin^2\alpha = 0.$$ [L.U.]

9. Prove that $\cos 2\alpha - \cos 2\beta = 2\sin(\alpha+\beta)\sin(\beta-\alpha)$.

Resolve the expression

$$\sin(\theta+\phi)\sin(\theta-\phi) + \sin\alpha\sin(\alpha+2\theta)$$

into two factors by applying the above formula or otherwise.

10. Prove the formula giving the cosine of any angle in a triangle in terms of its sides.

If $a = 5{\cdot}6$, $b = 6{\cdot}1$, $c = 7{\cdot}5$, find the greatest angle, and deduce the area, the unit of length being half an inch.

Verify your result approximately by constructing the triangle and measuring the shortest distance of a vertex from the opposite side. [S.W.]

11. Prove :

(1) $2\cot 2A = \cot A - \tan A$;

(2) $\left(\cot\dfrac{A}{2} - \tan\dfrac{A}{2}\right)^2 (1 - 2\tan A\cot 2A) = 4.$ [Leeds.]

12. Find the angles between 0° and 360° which satisfy the equations :

(i) $2\sin\theta + 3\cos\theta = 4$;

(ii) $\cos\theta \,.\, \cot\theta = 1 + \sin\theta.$ [S.W.]

13. Establish the identities:

$$\text{(i)}\quad \frac{1-(\sec\theta-\tan\theta)^2}{(\sec\theta+\tan\theta)^2-1}=\frac{1-\sin\theta}{1+\sin\theta},$$

$$\text{(ii)}\quad \cos 3A=\cos A\,.\,\frac{1-\tan A\tan 2A}{1+\tan A\tan 2A}.$$

Prove $\tan 7\tfrac{1}{2}^\circ=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$. [S.W.]

14. Prove the formula

$$\triangle^2=s(s-a)(s-b)(s-a).$$

The sides of a triangle are 95, 100, 105; find to three significant figures the ratio of its area to that of an equilateral triangle of the same perimeter. [S.W.]

15. Prove the formula $\cos(A+B)=\cos A\cos B-\sin A\sin B$ by a method applicable to angles of any magnitude and sign; and deduce the value of $\cos 3A$ in terms of $\cos A$.

Prove that

$$\frac{\cos A+\cos 2A+\cos 3A+\cos 4A}{\sin A+\sin 2A+\sin 3A+\sin 4A}=\cot\frac{5A}{2}\qquad\text{[B.U.]}$$

16. Write down the formula for $\tan(\theta+\phi)$ in terms of $\tan\theta$ and $\tan\phi$. Deduce that

$$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}.$$

From the equation $\tan 3\theta=\cot 2\theta$, or otherwise, prove that $\tan 18^\circ$ is a root of the equation in x

$$5x^4-10x^2+1=0,$$

and deduce that, to four decimal places, $\tan^2 18^\circ$ is equal to 0·1056. [S.W.]

17. Prove:

(1) $\sin 3A-\sin 2A-\sin A+4\sin A\sin\frac{A}{2}\sin\frac{3A}{2}=0.$

(2) $\cot(A-15^\circ)-\tan(A+15^\circ)=\dfrac{4\cos 2A}{2\sin 2A-1}.$

Deduce that $\tan 15^\circ=2-\sqrt{3}$.

(3) Prove that in any triangle

$$b\cos\theta=c\cos(A-\theta)+a\cos(C+\theta),$$

where θ is any angle. [B.U.]

18. If $r\cos(\theta+\gamma)=a\cos(\theta+\alpha)+b\cos(\theta+\beta)$, for all values of θ, shew that $r^2=a^2+b^2+2ab\cos(\alpha-\beta)$, and

$$\tan\gamma=\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}.$$

Solve the equation

$$2\cos(\theta+10°)+3\cos(\theta-20°)=0,$$

giving all values of θ between 0° and 360°. (B.U.)

19. Find the area of a regular pentagon the length of whose side is a, and find the value of a in order that the area may be equal to that of a circle of 1 foot radius.

Prove that

$$\cos A+\cos(A+72°)+\cos(A+144°)+\cos(A-144°)$$
$$+\cos(A-72°)=0. \quad \text{(B.U.)}$$

20. (i) Prove that, for angles of any magnitude,

$$\cos(A+B)=\cos A\,.\,\cos B-\sin A\,.\,\sin B,$$

and deduce the formula $\cos\phi-\cos\theta=2\sin\tfrac{1}{2}(\theta+\phi)\,.\,\sin\tfrac{1}{2}(\theta-\phi)$.

(ii) Shew that $\dfrac{\sin 3\phi}{\sin\phi}-\dfrac{\cos 3\phi}{\cos\phi}=2.$ (Man.)

21. Prove that

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B};$$

and apply the formula to the following problem.

D is the middle point of AB, and $DXYZ$ is drawn at right angles to AB, so that

$$2XD=AB, \quad 2YD=2AB \quad \text{and} \quad 2ZD=3AB;$$

shew that the sum of the angles AXB, AYB, AZB is two right angles. (Man.)

22. Prove that $\cos\theta=(1-t^2)/(1+t^2)$ and $\sin\theta=2t/(1+t^2)$, where t stands for $\tan\tfrac{1}{2}\theta$.

Hence, or otherwise, by the help of tables, find the general solution of the equation $5\cos\theta+6\sin\theta=7$. (Man.)

23. Prove the following identities:

(i) $\cos(A-B)=\cos A\cos B+\sin A\sin B$;

(ii) $\cos^2\theta+\cos^2(\alpha+\theta)-\sin^2\alpha=2\cos\alpha\cos\theta\cos(\alpha+\theta)$,

and find the complete solution of the equation

$$\sin 3\theta-\sin\theta=1-\cos 2\theta. \quad \text{(Durham.)}$$

24. State and prove the formulae for

$$\cos\theta+\cos\phi \quad \text{and} \quad \sin\theta-\sin\phi$$

as products.

A ladder makes an angle α with the horizontal when resting against a wall. The foot is pulled outwards a distance x and the top falls a distance y along the wall, the inclination then becoming β; shew that

$$y=x\cot\tfrac{1}{2}(\alpha+\beta).$$

25. Express $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of $\tan\tfrac{1}{2}\theta$.

If $\tan\tfrac{1}{2}\theta=\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta$, prove that $\cos\theta=\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$.

[B.U.]

26. Prove that

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A},$$

$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

Segments AB, BC, CD of a horizontal line subtend equal angles at a point O vertically above A. If the length of OA is a and that of AB is b, prove that

$$CD=\frac{b(b^2+a^2)^2}{(a^2-3b^2)(a^2-b^2)}.$$

[Man.]

27. Prove geometrically that

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

If $\tan\theta\tan\phi=k$, shew that

$$(1+k)\cos(\theta+\phi)=(1-k)\cos(\theta-\phi).$$

Hence solve the equation $\tan\theta\tan(\theta-60^\circ)=\dfrac{1}{3}$. [Durham.]

28. Establish geometrically the equations $\cos 2\alpha=1-2\sin^2\alpha$, $\sin 2\alpha=2\sin\alpha\cos\alpha$, where 2α is an acute angle.

If $\tan^2\alpha=\cos 2\beta$, express $\cos 2\alpha$ in terms of $\tan^2\beta$. [S.W.]

29. Prove the formulae

$$\sin\tfrac{1}{2}(A-B)=\frac{a-b}{c}\cos\tfrac{1}{2}C;$$

$$\cos\tfrac{1}{2}(A-B)=\frac{a+b}{c}\sin\tfrac{1}{2}C.$$

A triangle has a given base and a given perimeter. Prove that the vertical angle is greatest when the triangle is isosceles.

[S.W.]

30. The sides of a quadrilateral taken in order are of lengths a, b, c, d, and its diagonals, of lengths x, y, make an angle θ with one another. Prove that the area of the quadrilateral is $\frac{1}{2}xy\sin\theta$.

Prove also that, if the quadrilateral is cyclic, then $xy = ac + bd$, and that in this case the sine of the angle between the sides a, b is equal to $\dfrac{ac+bd}{ab+cd}\sin\theta$. [Durham.]

31. Prove that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy};$$

and deduce the value of

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z.$$

Shew that the sum of the three acute angles

$$\tan^{-1}\tfrac{3}{11},\quad \tan^{-1}\tfrac{19}{82},\quad \tan^{-1}\tfrac{3}{10} \text{ is } 45^\circ.$$ [S.W.]

32. Explain what is meant by $\sin^{-1}0{\cdot}25$. Prove that

$$\tan^{-1}\frac{2}{3} = \frac{\pi}{2} - \sec^{-1}\frac{\sqrt{13}}{2},$$

and find a value of x satisfying the equation

$$\operatorname{cosec}^{-1}x = 2\cot^{-1}2 + \cos^{-1}\tfrac{3}{5}.$$

How would you solve the equation $\tan^{-1}2x = x$? [L.U.]

33. Obtain expressions for $\cos\frac{1}{2}A$ and $\sin A$ in terms of the sides of the triangle ABC.

Hence, or otherwise, prove that if A, B, C are the angles of a triangle, then

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

34. In a triangle ABC, $a = 389{\cdot}1$ ft., $b = 247{\cdot}8$ ft., $C = 53^\circ\,42'$; calculate A, B, and c. [Leeds.]

35. If $\tan\dfrac{x}{2} = t$, shew that

$$\frac{3\cos x + 4\sin x}{3 - 5\sin x} = \frac{1+3t}{1-3t}.$$ [L.U.]

36. Solve the equations:

$$\text{(i)}\quad \cos\theta\cos 2\theta = \cos 3\theta,$$

$$\text{(ii)}\quad \begin{cases} x - y = 60^\circ, \\ \cos x - \cos y = \dfrac{1}{\sqrt{2}}. \end{cases}$$ [L.U.]

Further Properties of Triangles. If the angles of a triangle ABC be bisected internally and externally, four points I, I_1, I_2 and I_3 are obtained which are the centres of the inscribed and the three escribed circles (Fig. 148).

To find r, the radius of the inscribed circle.

From Fig. 148,

$$\text{area of triangle } BIC = \tfrac{1}{2}ID \,.\, BC = \tfrac{1}{2}r \,.\, a.$$

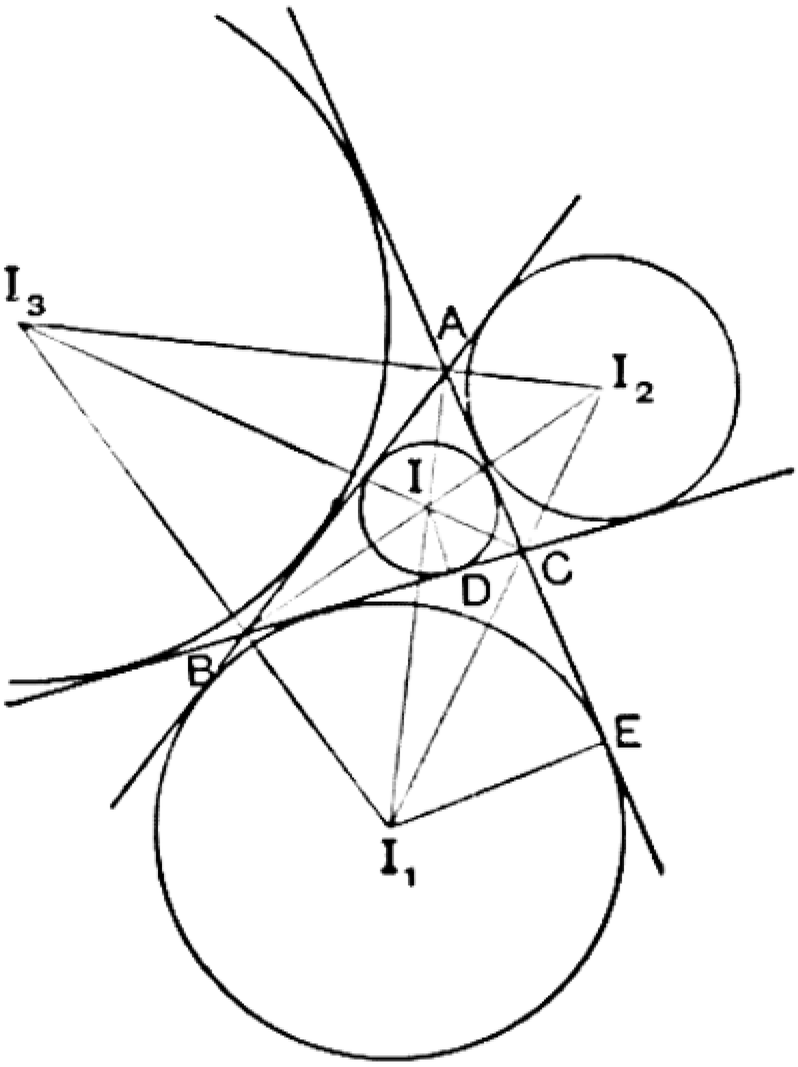


FIG. 148.

Similarly, area of triangle

$$CIA = \tfrac{1}{2}rb,$$

area of triangle

$$AIB = \tfrac{1}{2}rc.$$

Adding, area of triangle

$$ABC = \tfrac{1}{2}r(a+b+c),$$

$$i.e. \qquad \Delta = rs\,;$$

$$\therefore\ r = \Delta/s.$$

To find r_1, the radius of the escribed circle touching BC.

From Fig. 148,

area of triangle

$$BI_1A = \tfrac{1}{2}r_1c. \quad \ldots\ldots(1)$$

Similarly, area of triangle

$$I_1CA = \tfrac{1}{2}r_1b \quad \ldots\ldots..(2)$$

and area of triangle

$$BI_1C = \tfrac{1}{2}r_1a. \quad \ldots\ldots.(3)$$

Adding (1) and (2) and subtracting (3),

$$\text{area of triangle } ABC = \tfrac{1}{2}r_1(c+b-a),$$

$$i.e. \qquad \Delta = r_1(s-a)\,;$$

$$\therefore\ r_1 = \Delta/(s-a).$$

Similarly, $r_2 = \Delta/(s-b)$ and $r_3 = \Delta/(s-c)$.

Other formulae for r, r_1, etc., can be obtained from Fig. 148. It can be proved geometrically that $AE = s$, and $CD = s - c$. The triangle AI_1E, therefore, gives $r_1 = s \tan \tfrac{1}{2}A$; and the triangle IDC gives $r = (s-c) \tan \tfrac{1}{2}C$. See Ex. 1 of Exercises XLIV. Another important formula is proved in the following example.

Ex. 1. *Prove that* $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

From Fig. 148, $BD = ID \cot IBD$,

i.e. $BD = r \cot \frac{1}{2}B$. Similarly, $CD = r \cot \frac{1}{2}C$.

Adding, we have $a = r(\cot \frac{1}{2}B + \cot \frac{1}{2}C)$;

$$\therefore\ a = \frac{r(\cos \frac{1}{2}B \sin \frac{1}{2}C + \cos \frac{1}{2}C \sin \frac{1}{2}B)}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = \frac{r \sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C};$$

$$\therefore\ r = \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} = \frac{2R \sin A \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A}$$

$$\therefore\ r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$$

Ex. 2. *Shew that the sides of the pedal triangle are* $a \cos A$ $b \cos B$ *and* $c \cos C$.

From Fig. 149, the angles BFC and BEC are each 90°;

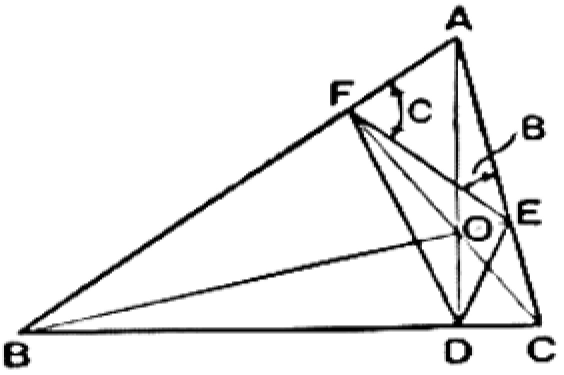


FIG. 149.

$\therefore$ $BFEC$ is a cyclic quadrilateral.

Hence the angle $AEF = B$ and the angle $AFE = C$.

From the triangle AFE,

$$\frac{EF}{\sin A} = \frac{AE}{\sin C};$$

$$\therefore\ EF = \frac{\sin A}{\sin C}(AB \cos A)$$

$$= \frac{c \sin A \cos A}{\sin C}$$

$$= a \cos A.$$

Similarly the other sides of the pedal triangle DEF are $b \cos B$ and $c \cos C$.

Ex. 3. *Shew that* $AO = 2R \cos A$, *where* O *is the orthocentre.*

From the triangle OAE (Fig. 149), $OA = AE \sec OAE$;

$\therefore$ $OA = AE \sec(90° - C) = c \cos A \operatorname{cosec} C = 2R \cos A$.

EXERCISES XLIV.

1. Using Fig. 148, shew that $CD = (s-c)$, $AE = s$. From these and similar relations, deduce

(i) $r = (s-a)\tan\frac{1}{2}A = (s-b)\tan\frac{1}{2}B = (s-c)\tan\frac{1}{2}C$;

(ii) $r_1 = s\tan\frac{1}{2}A$; (iii) $II_1 = a\cos\frac{1}{2}A$.

2. Prove the following formulae:

(i) $r_1 = r\cot\frac{1}{2}B\cot\frac{1}{2}C$; (ii) $1/r_1 + 1/r_2 = 1/r - 1/r_3$;

(iii) $\Delta = r_2 r_3 \tan\frac{1}{2}A$;

(iv) $r_2 r_3 + r_3 r_1 + r_1 r_2 = (r_2+r_3)(r_3+r_1)(r_1+r_2)/4R$;

(v) $r_1 = (s-b)\cot\frac{1}{2}C = (s-c)\cot\frac{1}{2}B$;

(vi) $\Delta = r^2 \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$;

(vii) $r_1 + r_2 + r_3 = r + 4R$; (viii) $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0.$

3. If R is the radius of the circumcircle of a triangle ABC, shew that $a = 2R\sin A$; hence, if s is the semiperimeter of the triangle, prove that

$$s = R(\sin A + \sin B + \sin C) = 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

[Man.]

4. If r is the radius of the inscribed circle of a triangle ABC, prove that $r = a\sin\frac{1}{2}B\sin\frac{1}{2}C\sec\frac{1}{2}A$, and also find r in terms of the sides.

Find the radii of the inscribed and circumscribed circles of the triangle whose sides are 777, 673 and 400. [L.U.]

5. Obtain the length of the radius (R) of the circumcircle of the triangle ABC in the forms

(i) $abc/4S$, (ii) $\frac{1}{2}(b+c)\sec\frac{1}{2}A\sec\frac{1}{2}(B-C)$.

If $b = 43$ in., $c = 52$ in., $A = 47°$, find B, C and R. [L.U.]

6. Express in terms of the sides of the triangle the distances of its vertices from the centre of the inscribed circle, and calculate the distance from A, when $a = 50$, $b = 37$, $c = 33$. [L.U.]

7. If r_1, r_2 denote the radii of the escribed circles touching respectively the sides a, b of a triangle, shew that

$$r_1 = \frac{\Delta}{s-a} \quad \text{and} \quad \frac{r_1 r_2}{r_1 + r_2} = \frac{\Delta}{c},$$

where Δ = the area of the triangle and s the half sum of the sides.

If $\Delta = 100$ sq. in., $r_1 = 10$ in., $r_2 = 50$ in., find c and the altitude of the triangle.

Prove that $b - a = 8$, and shew how to find b and a. [L.U.]

8. Prove that in a plane triangle

(i) the area $= \frac{1}{2}a^2 \sin B \sin C \operatorname{cosec}(B+C)$;

(ii) the radius of the inscribed circle

$$= a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \operatorname{cosec} \tfrac{1}{2}(B+C).$$

Calculate these for the triangle in which $a = 59$ in., $B = 35°$, $C = 72°$. [L.U.]

9. Prove that $r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$ and write down similar expressions for r_2 and r_3.

10. Prove that, if I_1, I_2, I_3 are the centres of the escribed circles of the triangle ABC and R is the radius of the circumscribing circle, then

$$AI_1 = 4R \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C, \quad AI_2 = 4R \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C,$$
$$\text{and} \quad AI_3 = 4R \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

Also prove that the triangles I_1BC, I_2CA, I_3AB are similar triangles whose areas are as $\sin^2 \frac{1}{2}A : \sin^2 \frac{1}{2}B : \sin^2 \frac{1}{2}C$. [L.U.]

11. Find the sides and angles of the pedal triangle of a given triangle ABC, and deduce that its area is $2\Delta \cos A \cos B \cos C$, where Δ is the area of the latter triangle.

Find the areas of the triangles BOC, COA, AOB, where O is the orthocentre. Hence, or otherwise, prove that

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1. \quad \text{[B.U.]}$$

12. Prove that in a triangle

$$1 + \cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C.$$

If perpendiculars AD, BE and CF be drawn from the vertices to the opposite sides of a triangle ABC of area Δ, shew that the area of the triangle AEF is $\Delta \cos^2 A$, and that the area of the triangle DEF is $2\Delta \cos A \cos B \cos C$. [Man.]

13. Shew that the radius of the circle circumscribing the pedal triangle is $R/2$.

14. In a triangle ABC, prove that $\dfrac{a}{\sin A}$ = the diameter of the circumcircle.

If I is the incentre of the triangle, and J the centre of the circle escribed to the side BC, prove that IJ is the diameter of a circle which passes through B and C, and that $IJ = a \sec \frac{1}{2}A$. [I.M.E.]

The Polar Equation of a Straight Line. Let PQ be the given straight line (Fig. 150), O the pole and OX the initial line.

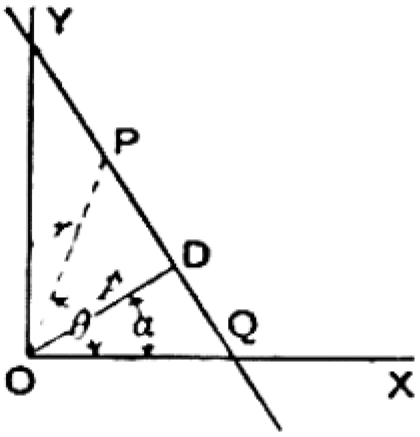


FIG. 150.

Let p be the length of the perpendicular OD from the pole to the given line, and let the angle DOX be α; so that (p, α) are the polar coordinates of D.

Let $P, (r, \theta)$ be any point on the given line.

$$\text{Then} \qquad OP \cos P\hat{O}D = OD,$$

$$\text{i.e.} \qquad r\cos(\theta - \alpha) = p. \quad \dots\dots\dots\dots\dots(1)$$

This is the polar equation of the given line; that is, it is the relation which holds between the polar coordinates of any point on the line and the constants p, α which determine the line.

If OY is drawn so that the angle XOY is 90°, the rectangular coordinates (x, y) of any point, referred to the axes OX, OY, are connected with its polar coordinates (r, θ) by the relations

$$r\cos\theta = x \quad \text{and} \quad r\sin\theta = y. \quad \dots\dots\dots\dots\dots(2)$$

$$\text{Now (1) is} \qquad r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p,$$

$$\text{i.e.} \qquad x\cos\alpha + y\sin\alpha = p,$$

which is the equation of the straight line in rectangular coordinates, already obtained on page 185.

Transforming a polar equation to the corresponding equation in rectangular coordinates is frequently useful, and is easily carried out with the aid of the transformation formulae (2).

Ex. 1. *What is the polar equation of the line POQ* (Fig. 151), *for which the angle POX equals α?*

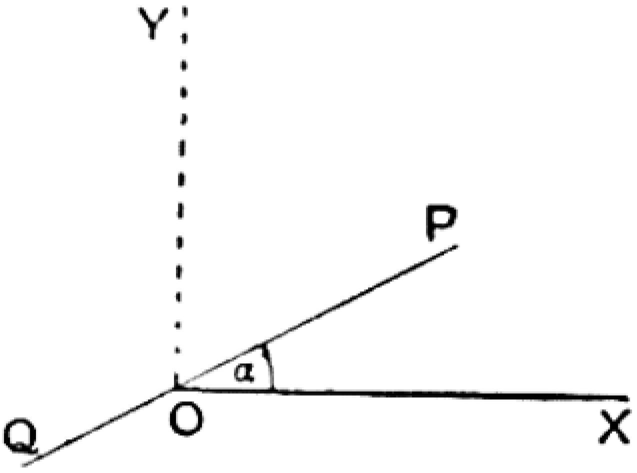


FIG. 151.

For any point (r, θ) in OP, it is clear that $\theta = \alpha$.
For any point (r, θ) in OQ, $\theta = \pi + \alpha$.

Thus $\theta = \alpha$ is the equation of OP

and $\theta = \pi + \alpha$ is the equation of OQ.

Using rectangular coordinates the whole of the line POQ is represented by $y = x \tan \alpha$,
or transforming to polars,

$$\tan \theta = \tan \alpha \quad (\text{since } y/x = r \sin \theta / r \cos \theta).$$

Note that $\tan \theta = \tan \alpha$ is the true polar equation of the line POQ, and that $\theta = \alpha$, $\theta = \pi + \alpha$ are the polar equations of the separate "half-lines" OP, OQ respectively.

Ex. 2. *Find the distance between the points* (r_1, θ_1) *and* (r_2, θ_2).

Let P, Q be the points (Fig. 152).

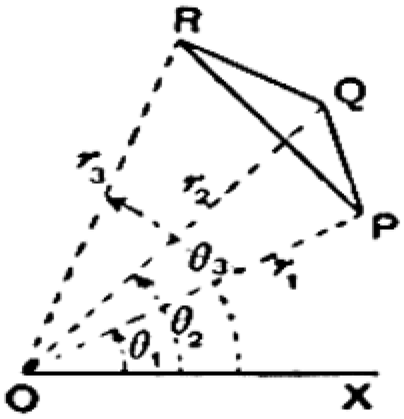


FIG. 152.

Then
$$PQ^2 = OP^2 + OQ^2 - 2OP \,.\, OQ \cos POQ$$
$$= {r_1}^2 + {r_2}^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1).$$

Ex. 3. *Find the area of the triangle formed by the points*

$$(r_1, \theta_1), \quad (r_2, \theta_2), \quad (r_3, \theta_3).$$

The area of the triangle PQR (Fig. 152) is

$$\triangle OPQ + \triangle OQR - \triangle OPR;$$

i.e. $\frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) - \frac{1}{2} r_1 r_3 \sin(\theta_3 - \theta_1)$,

i.e. $\frac{1}{2} r_1 r_2 r_3 \left\{ \frac{\sin(\theta_2 - \theta_1)}{r_3} + \frac{\sin(\theta_3 - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta_3)}{r_2} \right\}$.

N.B.—As in the case of rectangular coordinates (p. 240), this formula gives a positive result for the area, if it is on the left hand as the boundary is described from P through Q to R.

Ex. 4. *What does the equation* $\cos(\theta - \alpha) = \cos\beta$ *represent?*

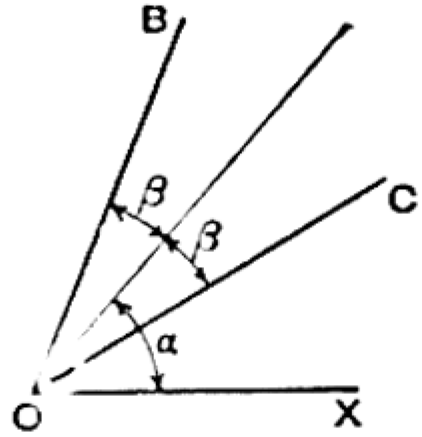


FIG. 153.

If $$\cos(\theta - \alpha) = \cos\beta,$$
then $$\theta - \alpha = 2n\pi \pm \beta. \qquad \text{.........(1)}$$

In Fig. 153, the angle $BOX = \alpha + \beta$, the angle $COX = \alpha - \beta$.

The equation of OB is
$$\theta = \alpha + \beta \quad \text{or} \quad \theta = 2n\pi + \alpha + \beta.$$

The equation of OC is
$$\theta = \alpha - \beta \quad \text{or} \quad \theta = 2n\pi + \alpha - \beta.$$

Thus, from (1), the given equation represents both arms of the angle BOC.

Ex. 5. *O is a fixed point. A variable straight line through O cuts two fixed lines in P and Q. On OPQ a point R is taken so that*
$$\frac{a}{OP} + \frac{b}{OQ} = \frac{c}{OR}.$$
Find the locus of R.

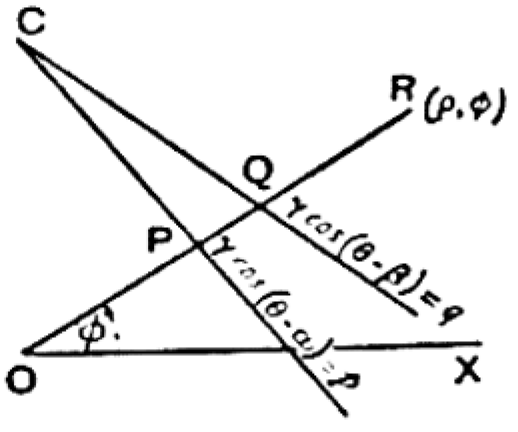


FIG. 154.

Using Fig. 154, let the equations of the fixed lines CP, CQ be $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \beta) = q$ respectively.

Let the coordinates of R be ρ, ϕ; then $X\hat{O}P = \phi$ and $OR = \rho$.

Also $\quad OP\cos(\phi - \alpha) = p \quad$ and $\quad OQ\cos(\phi - \beta) = q$.

Since $$\frac{a}{OP} + \frac{b}{OQ} = \frac{c}{OR},$$

we have $$\frac{a\cos(\phi - \alpha)}{p} + \frac{b\cos(\phi - \beta)}{q} = \frac{c}{\rho}, \qquad \text{.................(1)}$$

which is the required locus in polar coordinates, since it is a relation connecting ρ and ϕ.

It is easy to see that (1) represents a straight line by transforming to rectangular coordinates as follows:

Multiplying through by ρ and writing $\rho\cos\phi = x$ and $\rho\sin\phi = y$, (1) becomes
$$\frac{a}{p}(x\cos\alpha + y\sin\alpha) + \frac{b}{q}(x\cos\beta + y\sin\beta) = c,$$

which is of the first degree in x and y, and therefore represents a straight line.

EXERCISES XLV.

1. What is represented by the equations $\sin\theta=0$, $\sin\theta=\sin\alpha$, $a\cos\theta+b\sin\theta=0$, $3\cos\theta+4\sin\theta=2{\cdot}5$?

2. Plot the points $\left(1, \frac{\pi}{3}\right)$, $\left(2, \frac{\pi}{6}\right)$, $\left(3, -\frac{\pi}{6}\right)$, and find the sides and area of the triangle defined by these points.

3. Find the sides and area of the triangle whose vertices are

$$\left(2, \frac{\pi}{8}\right), \quad \left(4, \frac{3\pi}{8}\right) \quad \text{and} \quad \left(-1, \frac{7\pi}{8}\right).$$

4. What does the equation $\frac{l}{r}=a\cos\theta+b\cos(\theta+\alpha)$ represent ?

5. Shew that the angle between the lines $r\cos(\theta-\alpha)=p$ and $r\cos(\theta-\alpha')=p'$ is $\alpha-\alpha'$.

6. Prove that the perpendicular from (r_1, θ_1) to the line

$$r\cos(\theta-\alpha)=p \quad \text{is} \quad \pm\{r_1\cos(\theta_1-\alpha)-p\}.$$

Shew that the plus sign is taken if r_1, θ_1 and the origin are on opposite sides of the line ; the minus sign, if they are on the same side.

7. Find the polar equation of the line joining the points (r_1, θ_1), (r_2, θ_2). (*Hint.*—If P, Q are these points and R is *any* point on the line, consider the areas of the triangles OPQ, OQR, ORP.)

8. Shew that the lines $r\cos(\theta-\alpha)=p$ and $r\sin(\theta-\alpha)=q$ are perpendicular.

9. Find the equation of the straight line through the point (r_1, θ_1) perpendicular to the line $r\cos(\theta-\alpha)=p$.

10. Two fixed straight lines intersect at C. A variable straight line passing through a fixed point O cuts the two given lines in P and Q. On OPQ a point R is taken so that $\frac{l}{OP}+\frac{m}{OQ}=\frac{l+m}{OR}$.

Shew that the locus of R is a straight line passing through C.

11. Investigate the polar equation of a straight line.

A straight line $OPQR$ is drawn from the origin O to meet the given straight lines $r\cos(\theta-\alpha)=a$, $r\cos(\theta-\beta)=b$, in P, Q. Find the locus of R when it is so taken that

$$\frac{1}{OR}=\frac{1}{OP}-\frac{1}{OQ}.$$

[S.W.]

12. Find the coordinates of the point of intersection of the lines

$$r\sin\theta=2a, \quad r\cos\left(\theta-\frac{\pi}{6}\right)=a.$$

*The Polar Equation of a Circle.

(i) When the pole does not lie on the circle.

Let (c, α), Fig. 155, be the polar coordinates of the centre C. Let the radius be a and let P, (r, θ) be *any* point on the circumference.

Then
$$a^2 = r^2 + c^2 - 2rc \cos POC,$$

i.e.
$$r^2 - 2rc \cos(\theta - \alpha) + c^2 - a^2 = 0,$$

which is the required equation.

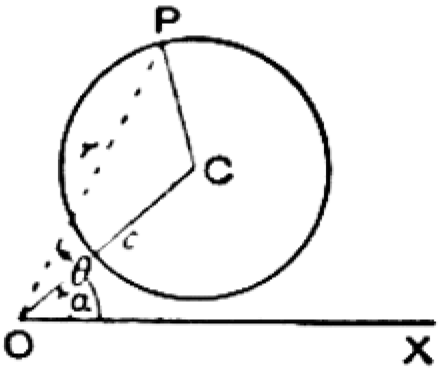


FIG. 155.

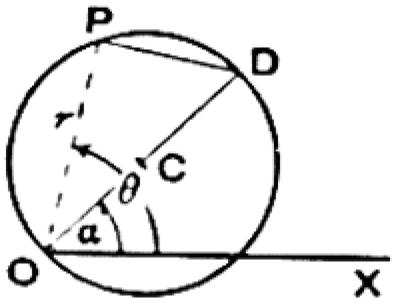


FIG. 156.

(ii) When the pole lies on the circle.

Let the centre be (a, α), so that the radius is a. Let OD be the diameter through the pole and P, (r, θ), any point on the circumference. (Fig. 156.)

Then
$$OP = OD \cos(\theta - \alpha),$$

i.e.
$$r = 2a \cos(\theta - \alpha),$$

which is the required equation.

Ex. 1. *If O is a fixed point and any straight line through O meets a circle in the points P and Q, the rectangle $OP \,.\, OQ$ is constant.*

Take O as pole, and let the equation of the circle be

$$r^2 - 2cr \cos(\theta - \alpha) + c^2 - a^2 = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let any straight line $\theta = \beta$ cut the circle in the points P, Q. Substituting $\theta = \beta$ in (1), we have

$$r^2 - 2cr \cos(\beta - \alpha) + c^2 - a^2 = 0$$

as the equation whose roots are OP, OQ;

$\therefore$ $OP \,.\, OQ = c^2 - a^2$, which is independent of β.

Ex. 2. *Shew that* $r\cos(\theta-\alpha)=2a\cos^2\frac{\alpha}{2}$ *is a tangent to the circle* $r=2a\cos\theta$.

Let O be the pole and C the centre of the given circle (Fig. 157).

Let CQ be a radius making an angle α with the initial line OX.

If p is the perpendicular from O on the tangent at Q, the figure shews that

$$p=a+OD=a(1+\cos\alpha)=2a\cos^2\frac{\alpha}{2}.$$

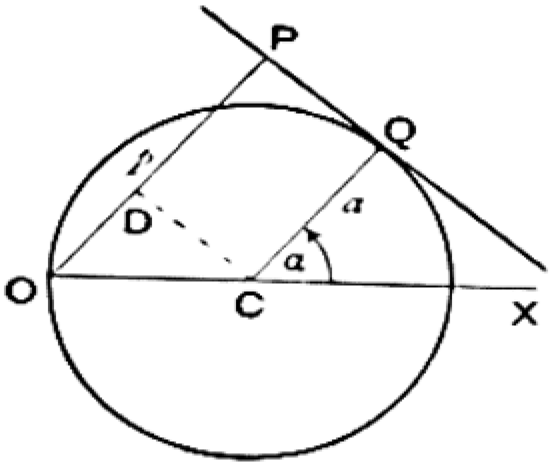


FIG. 157.

But the equation of the tangent at Q is

$$r\cos(\theta-\alpha)=p,$$

i.e. $$r\cos(\theta-\alpha)=2a\cos^2\frac{\alpha}{2}.$$

Ex. 3. *O is a fixed point. Any line through O cuts a fixed circle in P and P′. In OPP′ a point Q is taken so that*

$$\frac{2}{OQ}=\frac{1}{OP}+\frac{1}{OP'}.$$

Find the locus of Q.

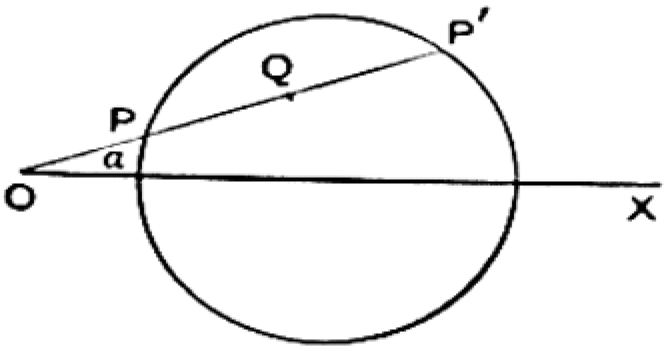


FIG. 158.

Let O be the pole (Fig. 158) and let

$$r^2-2cr\cos\theta+c^2-a^2=0$$

be the circle, so that the initial line goes through the centre.

Let the line OPP' be $\theta=\alpha$, and ρ, α the coordinates of Q. Then OP, OP' are the roots of

$$r^2-2cr\cos\alpha+c^2-a^2=0;$$

$$\therefore\ \frac{1}{OP}+\frac{1}{OP'}=\frac{OP+OP'}{OP\,.\,OP'}=\frac{2c\cos\alpha}{c^2-a^2};$$

$$\therefore\ \frac{2}{\rho}=\frac{2c\cos\alpha}{c^2-a^2},$$

which is the condition that Q, *i.e.* $(\rho,\ \alpha)$, should lie on the straight line

$$r\cos\theta=\frac{c^2-a^2}{c}.$$

Ex. 4. *Find the polar coordinates of the centres and the radii of the circles*

$$r=4\cos(\theta-\tfrac{2}{3}\pi) \quad \textit{and} \quad r^2-2r\cos\theta-2=0,$$

and shew that they cut orthogonally.

From the form of its equation, the circle $r=4\cos(\theta-\frac{2}{3}\pi)$ passes through the origin (Fig. 159) and has a diameter 4 which makes an angle $\frac{2}{3}\pi$ with the initial line.

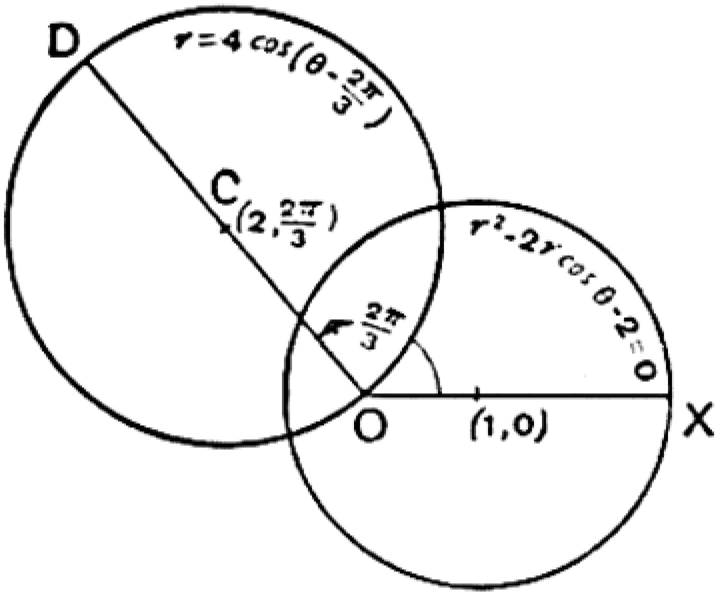


FIG. 159.

Its centre C is therefore the point $2, \frac{2}{3}\pi$.

The circle $r^2-2r\cos\theta-2=0$ can be written in rectangular coordinates as

$$x^2+y^2-2x-2=0,$$

$$\textit{i.e.} \qquad (x-1)^2+y^2=3.$$

Therefore its centre is at the point $(1, 0)$ and its radius is $\sqrt{3}$.

The polar coordinates of the centre are $(1, 0)$ (see figure).

Circles cut orthogonally if the square of the distance between the centres equals the sum of the squares of the radii.

For the given circles the square of the distance between the centres $(2, \frac{2}{3}\pi)$ and $(1, 0)$ is $2^2+1^2-2\,.\,2\cos\frac{2}{3}\pi$, *i.e.* 7.

The sum of the squares of the radii is $4+3$, *i.e.* 7. Therefore the circles cut orthogonally.

EXERCISES XLVI.

1. Shew that the equation of a circle whose circumference passes through the pole can be deduced from the general polar equation of a circle by putting $c=a$.

2. What is represented by the equations

(i) $r=a$, (ii) $r=4\cos\theta$, (iii) $r=4\sec\theta$,

(iv) $r^2-3r+2=0$, (v) $r^2-3r\cos\theta+2=0$?

3. Find the polar coordinates of the points of intersection of $r \sin \theta = a$ and $r \operatorname{cosec} \theta = 2a$. Plot both curves.

4. The polar coordinates of the ends of a diameter of a circle are (r_1, θ_1), (r_2, θ_2). What is the equation of the circle ?

5. If (x_1, y_1), (x_2, y_2) are the rectangular coordinates of the points (r_1, θ_1), (r_2, θ_2) in Example 4, deduce the equation of the circle in rectangular coordinates.

6. Find the polar equation of a circle, the centre being on the initial line.

What loci are represented by the equations

$$r \sin \theta = a, \quad r = a \sin \theta, \quad r = a \sin \theta + b \cos \theta,$$

respectively ? Give figures. [S.W.]

7. Shew that $r \cos (\theta - \gamma) = a + c \cos (\alpha - \gamma)$ is a tangent to the circle $r^2 - 2cr \cos (\theta - \alpha) + c^2 - a^2 = 0$.

8. Shew that the circles

(i) $r = 2a \cos \theta$ and $r = 2b \sin \theta$.

(ii) $r = 4 \cos (\theta - \frac{1}{3}\pi)$ and $r^2 - 6r \cos \theta + 6 = 0$,

cut orthogonally.

9. Draw graphs of the loci

(i) $r = 2 \operatorname{cosec} \theta$, (ii) $r = 4 \sin \theta$.

Prove that they cut at right angles ; and find the coordinates of the points of intersection. [S.W.]

10. Draw the polar loci

$$r = 2 \cos \theta + 4 \sin \theta \quad \text{and} \quad \frac{1}{r} = \frac{\cos \theta}{2} + \frac{\sin \theta}{4},$$

and prove that they cut each other at right angles. [S.W.]

11. Shew that $r = a \cos \theta + b \sin \theta$ and $\dfrac{l+m}{r} = \dfrac{2m}{a} \cos \theta + \dfrac{2l}{b} \sin \theta$ cut orthogonally. (*Hint.*—Shew that the centre of the circle lies on the straight line.)

12. P is any point on a fixed circle and O is a fixed point on its circumference. On OP a point Q is taken so that

(i) $\dfrac{OP}{OQ} = \text{constant}$, (ii) $OP \, . \, OQ = \text{constant}$.

Find the locus of Q in each case. [Take O as pole, and the diameter through O of the fixed circle as the initial line.]

13. P is any point on the circle whose equation is

$$r^2 - 2rc\cos(\theta - \alpha) + c^2 - a^2 = 0.$$

If O is the pole and Q is a point on OP such that

$$\text{(i)}\ \frac{OP}{OQ} = k, \qquad \text{(ii)}\ OP\,.\,OQ = l^2,$$

find the locus of Q in each case.

14. O is the pole and OPQ is a triangle of given species (*i.e.* its angles are given). If P moves in the circle whose equation is

$$r^2 - 2rc\cos\theta + c^2 - a^2 = 0,$$

show that the locus of Q is the circle whose equation is

$$k^2r^2 - 2ckr\cos(\theta - \alpha) + c^2 - a^2 = 0,$$

where k is the constant ratio OP/OQ and α is the angle POQ.

15. Find the condition that the circles

$$r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0$$

$$\text{and} \qquad r^2 + 2c'r\cos(\theta - \alpha') + c'^2 - a'^2 = 0$$

cut orthogonally.

16. $ABCD$ is a rhombus formed of smoothly jointed rods of length a. The corners B, D are connected by equal rods BO, DO of length b ($> a$) to a fixed point O about which the whole can turn. Prove that, if C moves along a straight line at a distance c from O, the locus of A is the circle

$$rc = (b^2 - a^2)\cos\theta,$$

r being the distance from O, and the initial line being that along which c is measured. [L.U.]

CHAPTER XIII.

THE CALCULUS. SIMPLE APPLICATIONS.

Measurement of Rates, Speed, etc. A train is said to be moving at the rate of 20 miles per hour at a certain instant. Does this mean that it has travelled 20 miles in the last hour, or that it will travel 20 miles in the next hour? Evidently not, for it may only have left the station 5 minutes ago, and it may be stopped within the next minute. Similar considerations will shew that the statement does not necessarily mean that the train will travel

1 mile in the next 3 minutes,
or 52·8 ft. (0·01 mile) in the next 1·8 seconds (0·03 minute),
or 5·28 ft. in the next 0·18 second,
or 0·528 ft. in the next 0·018 second.

But the distances in the above table would be those actually covered by the train in the times stated, **if the train moved steadily;** yet it cannot be inferred that any definite distance would be travelled in any definite time interval, when the speed **at a particular instant only** is given. One can say, however, that if the speed of the train is 20 miles per hour at a certain moment, then in the succeeding n^{th} part of 1 hour the train will travel one n^{th} part of 20 miles, provided n is large enough, *i.e.* **provided that the time interval is small enough,** or, roughly speaking, is so short that the speed of the train does not alter appreciably.

Suppose a body to have moved through s ft. in t seconds, and that it moves through a further Δs ft. in the next Δt seconds. What is its average speed during the interval of Δt seconds? Since in this interval the body moves through Δs ft., it travels Δs ft. in Δt seconds. Now, if it moved *uniformly* so as to pass over the same distance in the same time interval, it would pass over $\Delta s/\Delta t$ ft. in 1 second at this uniform speed. This uniform speed, $\Delta s/\Delta t$ ft. per second, is called the **average speed** during the interval.

In words, the average speed during any time interval is measured by the quotient

$$\frac{\text{distance covered in the interval}}{\text{time occupied}}.$$

Three trains A, B and C travel on parallel lines. At a certain instant they are all opposite a point P, and 5 seconds later they are all opposite another point Q. The **maps of the paths** (Fig. 160) shew the positions occupied by the trains at the ends of successive seconds.

Time in seconds.	Distance in feet travelled by		
	A	B	C
1	18·0	10·0	1·50
2	29·0	20·0	5·0
3	38·0	30·0	14·0
4	45·0	40·0	27·0
5	50·0	50·0	50·0

The table gives the same information as the maps, and is represented by the displacement-time graphs in the same figure.

Since each train travels 50 ft. in 5 seconds, the **average speed** of each during this interval is 10 ft. per second, and this is also the speed of B at any instant, since B is moving uniformly.

Now, during the first second A moves 18 ft., B 10 ft. and C 1·5 ft., so that, as far as the first second is concerned, A moves faster than B and C slower than B.

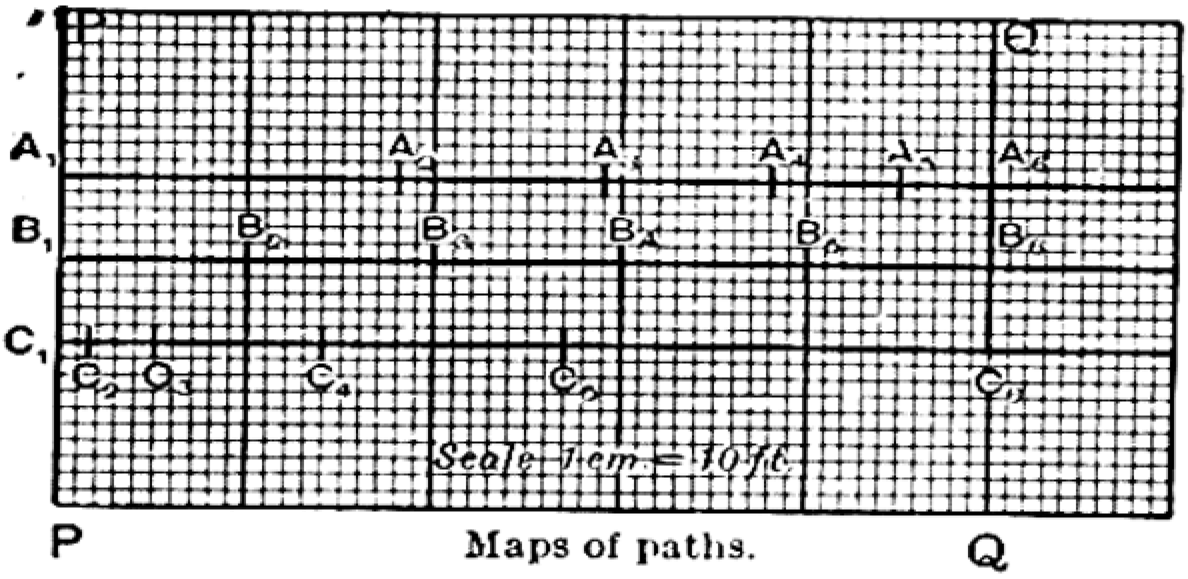
A glance at the displacement-time graph makes this clear; for, near the point O, the graph for A is steeper and the graph for C is less steep than that for B.

Again, during the last second A moves 5 ft., B 10 ft. and C 23 ft., so that for this second A moves slower than B and C faster; while, near the point R, the graph for A is less steep than that for B, and that for C is steeper.

From the above it may be inferred that there is a connection between the slope of the graph and the speed.

Considering the motion of *A* in more detail, it is clear that

During the 1st second, the average speed is 18 ft. per sec.
,, 2nd ,, ,, ,, 11 ,,
,, 3rd ,, ,, ,, 9·0 ,,
,, 4th ,, ,, ,, 7·0 ,,
,, 5th ,, ,, ,, 5·0 ,,



Maps of paths.

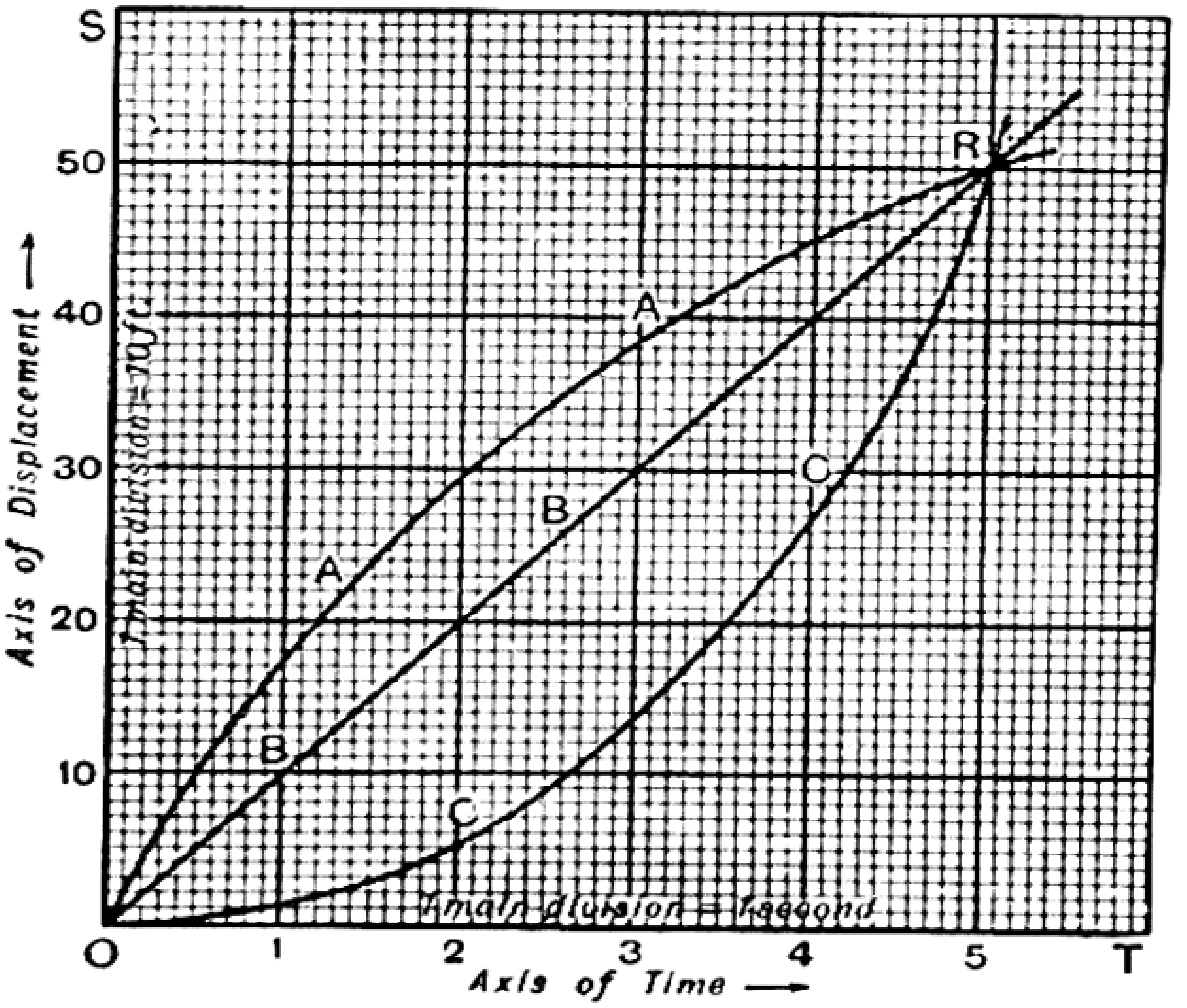


FIG. 160.—Displacement-time graphs.

So that *A*'s speed varies. How is the speed at any instant, say after 1 second, to be found ?

The graph (Fig. 161) gives the displacement of the train A from the point P on a larger scale, the ordinates at H, L, M, N, R giving the displacements of the train from P at the end of successive seconds.

Now, the average speed of this train over the distance A_2A_6 (Fig. 160) is 8 ft. per second, for it covers 32 ft. in 4 seconds. The slope of the line HR (Fig. 161) is $(50-18)/4$, *i.e.* 8 (see p. 36). Hence, the average speed over the distance A_2A_6 is measured by the slope of the line HR. But the speed of the train at A_2 is greater than this average speed, since the graph at H is steeper than the line HR at H.

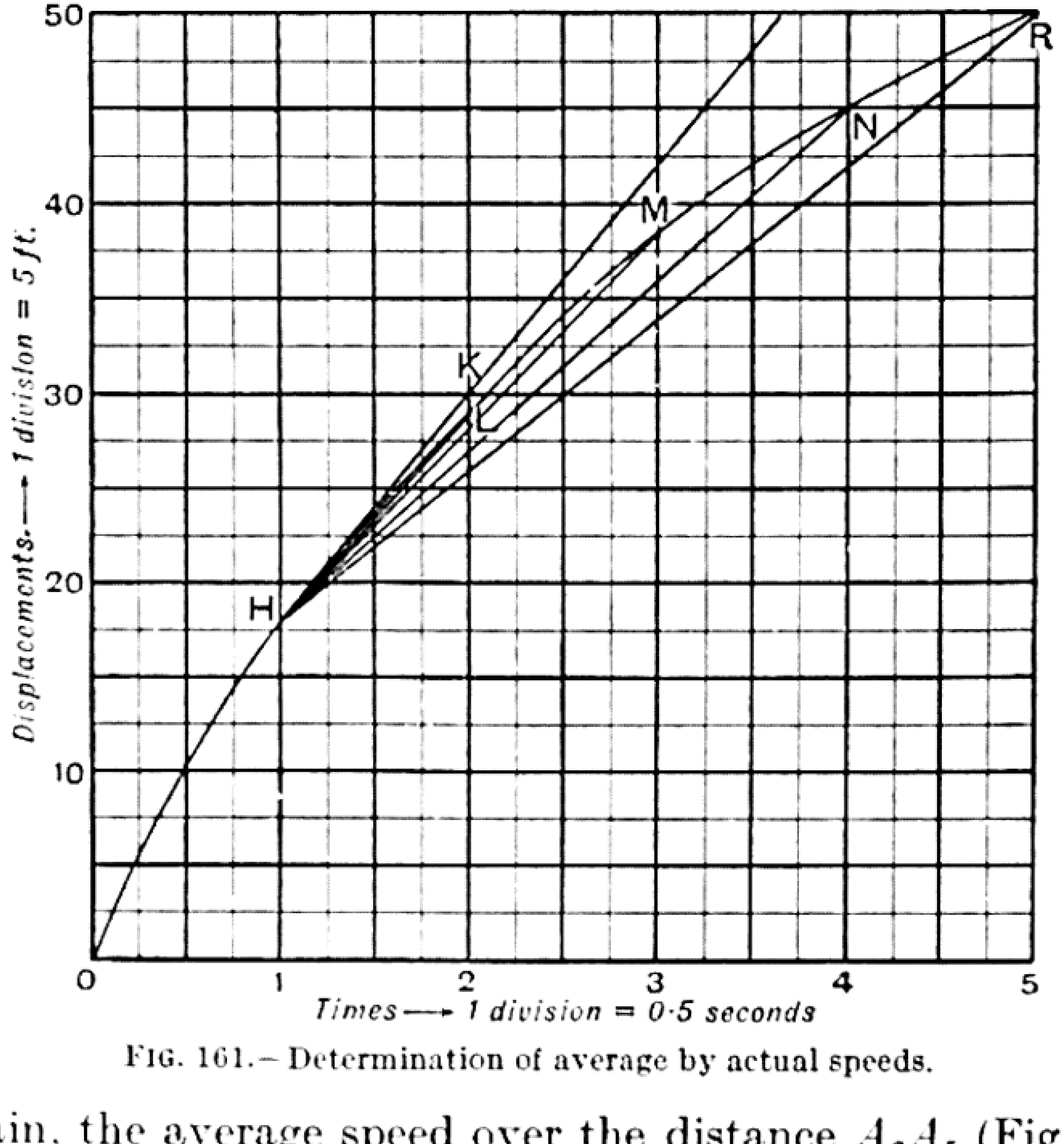


FIG. 161.—Determination of average by actual speeds.

Again, the average speed over the distance A_2A_5 (Fig. 160) is 9 ft. per second, for it covers 27 ft. in 3 seconds, and the slope of the line HN (Fig. 161) is $(45-18)/3$, *i.e.* 9. Hence the average speed over the distance A_2A_5 is measured by the slope of the line HN. But the speed of the train at A_2 is greater than this average speed, since the graph at H is steeper than the straight line HN at H.

In the same way the average speed over A_2A_4 is 10·0 ft. per sec., and this is the slope of the line HM; and still the speed at A_2 is greater than this average speed. Inspection shews that lines like HR, HN, etc., get steeper and steeper, approaching a limiting position HK, as the moving end of the line approaches H. That is, as the time interval is diminished, the line through H tends towards a definite position HK, which is **the tangent to the graph at H.** The slope of the line through H always measures the average speed of the train during the interval, and **the slope of the tangent at H measures the speed of the train at the instant when it is at A_2.**

Recapitulation in Tabular Form. Readings taken from graph (Fig. 160).

Time Interval Δt seconds.	Distance Travelled Δs feet.	Average Speed $\Delta s/\Delta t$ ft. per sec.	Slope of Corresponding Chord.
4	$50 - 18 = 32$	8	8
3	$45 - 18 = 27$	9	9
2	$38{\cdot}0 - 18 = 20$	10·0	10·0
1	$29{\cdot}0 - 18 = 11{\cdot}0$	11·0	11·0
0·5	$23{\cdot}7 - 18 = 5{\cdot}7$	11·4	11·4

Now, the slope of the line HK (Fig. 161)

$$= \frac{\text{ordinate of } K - \text{ordinate of } H}{\text{abscissa of } K - \text{abscissa of } H} = \frac{30 - 18}{1} = 12,$$

and this is the limit of $\Delta s/\Delta t$ as Δt tends to zero. That is, the speed of A after 1 second is 12 ft. per second.

Slope of any Curve at a Point. Let P, Q be any two points whose coordinates are (x, y), $(x + \Delta x, y + \Delta y)$. (Fig. 162.)

The slope of the chord PQ is $\frac{\Delta y}{\Delta x}$.

The slope of the curve at P is the slope of the tangent PT, and this is the limit as Δx approaches zero of $\Delta y/\Delta x$, *i.e.*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

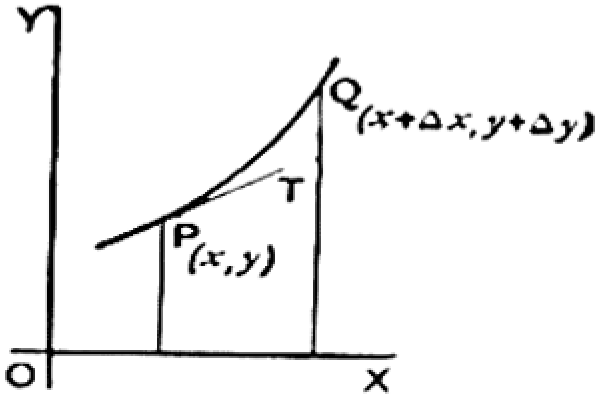


FIG. 162.

By the preceding paragraph, it should be clear that the slope of the curve at the point P *measures the rate of increase of* y *with respect to* x. (See pp. 36-39.)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{is written} \quad \frac{dy}{dx}.$$

It must be clearly understood that $\frac{dy}{dx}$ is merely an abbreviated form of $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. As here defined, $\frac{dy}{dx}$ cannot be regarded as a fraction.

If the slope of the curve at a point is positive, as in the figure, $\frac{dy}{dx}$ is positive and y increases with x. Similarly, if the slope is negative, $\frac{dy}{dx}$ is negative and y decreases as x increases.

N.B.—The graph of $y=c$ is a straight line parallel to the axis of x, and the slope at all points is zero;

$$\therefore \text{ if } y=c, \quad \frac{dy}{dx}=0.$$

The graph of $y=mx+c$ is a straight line, the slope of which at all points is m;

$$\therefore \text{ if } y=mx+c, \quad \frac{dy}{dx}=m.$$

Ex. 1. *Given* $y=2{\cdot}4x^{3\cdot8}$, *calculate the values of* Δy *and* $\Delta y/\Delta x$ *when* $x=0{\cdot}4$ *for the following values of* Δx: 0·1, 0·05, 0·01.

x	$\log x$	$3{\cdot}8 \log x$	$\log y$	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
0·4	$\bar{1}{\cdot}6021$	$\bar{2}{\cdot}4880$	$\bar{2}{\cdot}8682$	0·07382			
0·5	$\bar{1}{\cdot}6990$	$\bar{2}{\cdot}8562$	$\bar{1}{\cdot}2364$	0·1724	0·1	0·0986	0·986
0·45	$\bar{1}{\cdot}6532$	$\bar{2}{\cdot}6822$	$\bar{1}{\cdot}0624$	0·1156	0·05	0·0418	0·836
0·41	$\bar{1}{\cdot}6128$	$\bar{2}{\cdot}5286$	$\bar{2}{\cdot}9088$	0·08106	0·01	0·00724	0·724

The table shews that as Δx diminishes, the values of $\Delta y/\Delta x$ for this curve also diminish. Now $\Delta y/\Delta x$ is the slope of the chord joining two points on the curve. In Fig. 163 let P, A, B, C be the points whose abscissae are 0·4, 0·41, 0·45 and 0·5 respectively. The slopes of the lines PC, PB, PA clearly diminish, but tend towards a definite limit, viz. the slope of the tangent at P. From Fig. 163 the slope of the tangent at P is $DN/NP = 0{\cdot}70$.

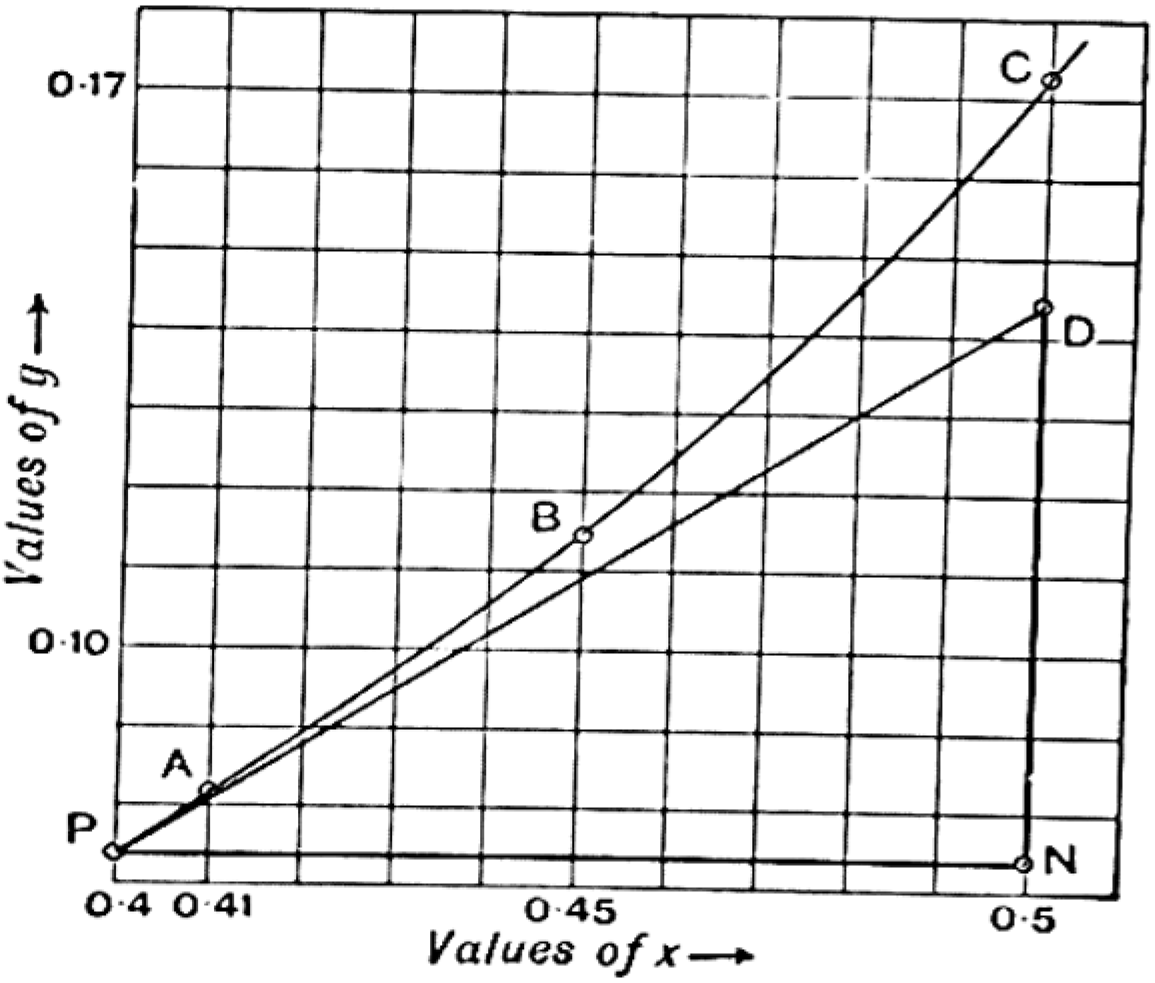


FIG. 163.

The slope of the line joining the points whose abscissae are 0·4 and 0·4001 can be calculated, if seven-figure logarithms are used, as follows :

x	$\log x$	$3{\cdot}8 \log x$	$\log y$	y	Δx	Δy	$\dfrac{\Delta y}{\Delta x}$
0·4	$\bar{1}{\cdot}6020600$	$\bar{2}{\cdot}4878280$	$\bar{2}{\cdot}8680392$	0·0737971	0·0001	0·0000701	0·701
0·4001	$\bar{1}{\cdot}6021686$	$\bar{2}{\cdot}4882407$	$\bar{2}{\cdot}8684519$	0·0738672			

Hence, there is a close agreement between the values obtained arithmetically and graphically for the slope of the curve $y = 2{\cdot}4x^{3{\cdot}8}$ at the point $x = 0{\cdot}4$.

Ex. 2. *Given* $y = 5{\cdot}3e^{-2{\cdot}7x}$, *calculate the values of* Δy *and* $\Delta y/\Delta x$ *when* $x = 1$ *for the following values of* Δx: 0·1, 0·05, 0·01.

x	$\log y$	y	Δx	Δy	$\dfrac{\Delta y}{\Delta x}$
1	$\bar{1}{\cdot}5517$	0·3562			
			0·1	− 0·0843	− 0·843
1·1	$\bar{1}{\cdot}4344$	0·2719			
			0·05	− 0·0449	− 0·898
1·05	$\bar{1}{\cdot}4931$	0·3113			
			0·01	− 0·0095	− 0·95
1·01	$\bar{1}{\cdot}5400$	0·3467			

As in the last example, $\Delta y/\Delta x$ tends to a definite limit as Δx diminishes. From Fig. 164 the limit is $-PN/ND$ ($\doteqdot -0{\cdot}96$); therefore at P the slope of the curve is $-0{\cdot}96$. Using seven-figure logarithms and taking $\Delta x = 0{\cdot}001$, $\Delta y/\Delta x$ is found to be $-0{\cdot}96$.

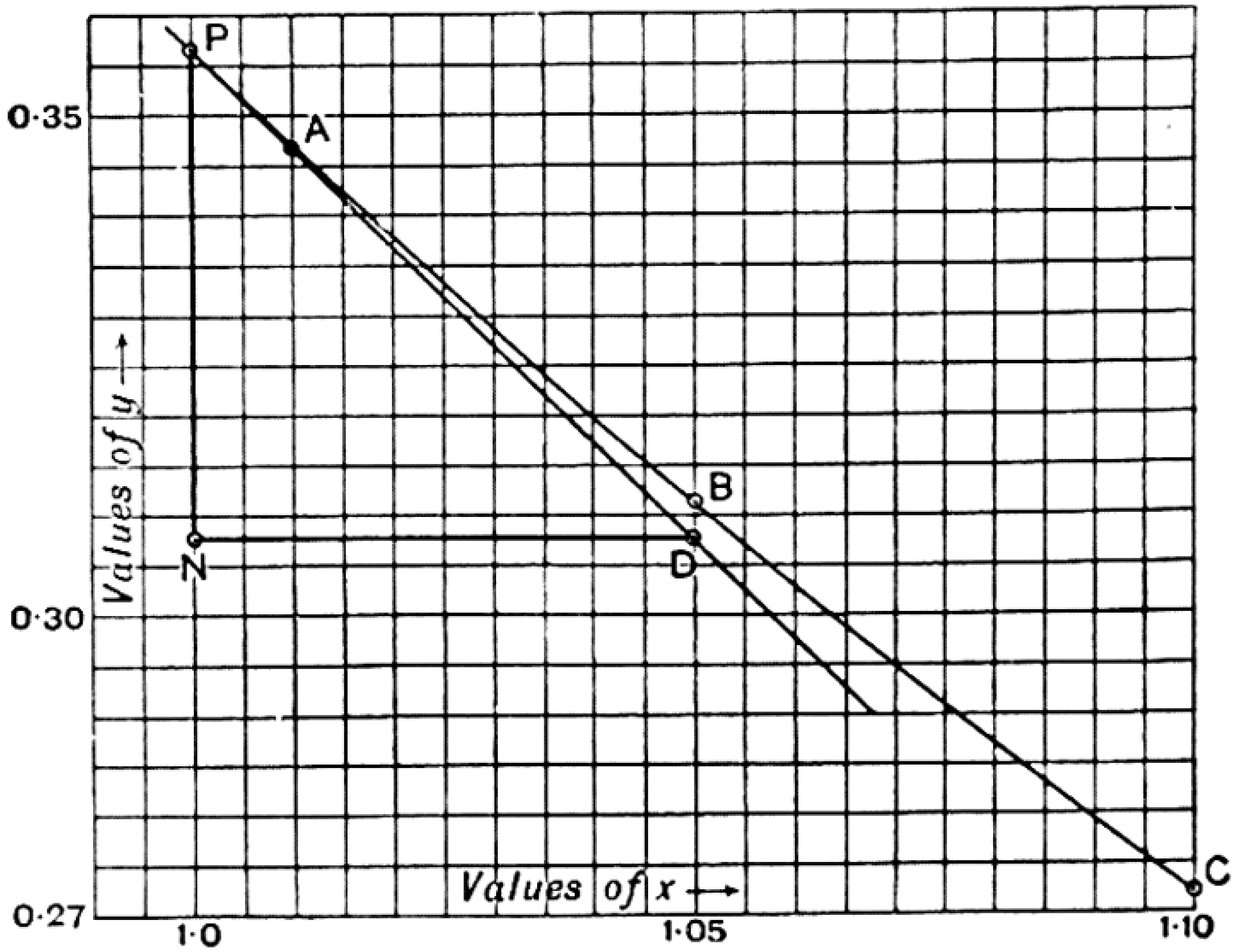


FIG. 164.

Illustrations. (i) **Coefficient of Expansion.**—To determine experimentally the mean coefficient of linear expansion of a rod, its lengths L, L' are measured at the temperatures T, T'. The coefficient α is then

$$\frac{1}{L}\cdot\frac{L'-L}{T'-T}, \quad \textit{i.e.}\ \frac{1}{OA}\times\text{slope of chord } AC \text{ (Fig. 165)},$$

where OA and BC are the lengths at temperatures T and T' respectively.

Now the graph connecting the length of a rod with its temperature is approximately linear, so that the slope of the chord AC is approximately the slope of the tangent at A (Fig. 165), and α can therefore be written as $\frac{1}{L}\cdot\frac{dL}{dT}$. This last expression is the actual coefficient of linear expansion of the rod at the temperature T when its length is L. $\frac{dL}{dT}$ is the rate of increase of length with temperature; $\frac{1}{L}\cdot\frac{dL}{dT}$ is the rate of increase of length with temperature per unit length.

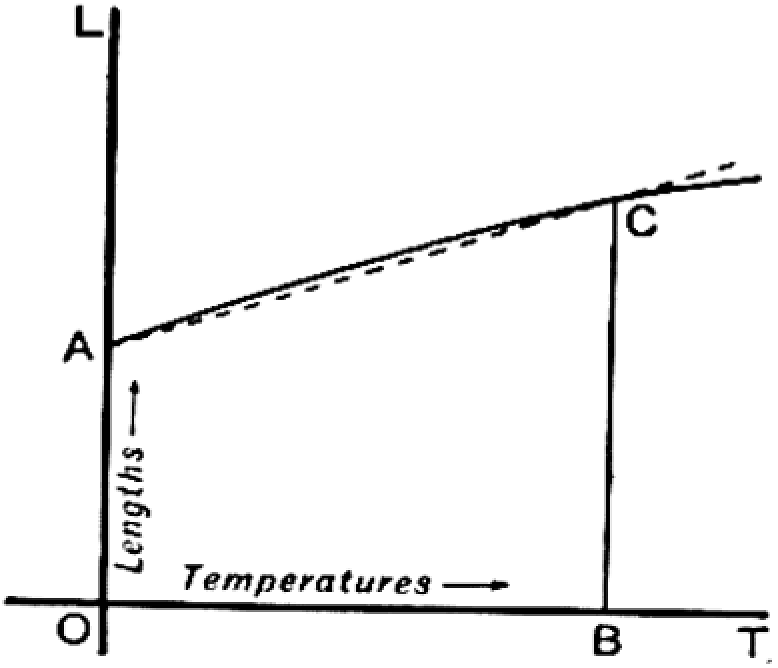


FIG. 165.

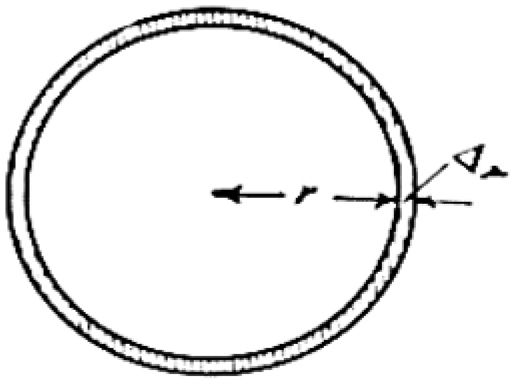


FIG. 166.

(ii) **Expanding Circle.**—Imagine a circle to expand (Fig. 166). Let A be the area when the radius is r and $A+\Delta A$ the area when the radius is $r+\Delta r$.

Then $\qquad A=\pi r^2 \quad\text{and}\quad A+\Delta A=\pi(r+\Delta r)^2;$

$$\therefore\ \Delta A=2\pi r\,\Delta r+\pi(\Delta r)^2;$$

$$\therefore\ \frac{\Delta A}{\Delta r}=2\pi r+\pi\,\Delta r.$$

Taking the limit as $\Delta r\to 0$, we have $\frac{dA}{dr}=2\pi r$, *i.e.* the rate of increase of area with the radius is equal to the circumference of the circle.

(iii) **Expanding Sphere**—Imagine a sphere to expand.

Let V and $V+\Delta V$ be the volumes corresponding to the radii r and $r+\Delta r$.

Then $\quad V=\frac{4}{3}\pi r^3 \quad$ and $\quad V+\Delta V=\frac{4}{3}\pi(r+\Delta r)^3$;

$$\therefore\ \Delta V=\tfrac{4}{3}\pi\{3r^2\Delta r+3r(\Delta r)^2+(\Delta r)^3\};$$

$$\therefore\ \frac{\Delta V}{\Delta r}=\tfrac{4}{3}\pi\{3r^2+3r(\Delta r)+(\Delta r)^2\}.$$

Taking the limit as $\Delta r\to 0$, we have $\dfrac{dV}{dr}=4\pi r^2$,

i.e. the rate of increase of volume with radius is 4π times the square of the radius. Note that $4\pi r^2$ is the area of the surface of a sphere of radius r.

(iv) **Acceleration.**—If the velocity of a body moving in a straight line is v ft. per second, t seconds after the start, $\dfrac{dv}{dt}$ is the rate of increase of v with respect to t, *i.e.* measures the acceleration.

(v) **Elasticity of Gas.**—Suppose a volume of gas v to be at pressure p. If the pressure be made $p+\Delta p$, the volume becomes $v+\Delta v$ (Fig. 167). Obviously, if Δp is positive, Δv will be negative. The **strain** is defined as the change of

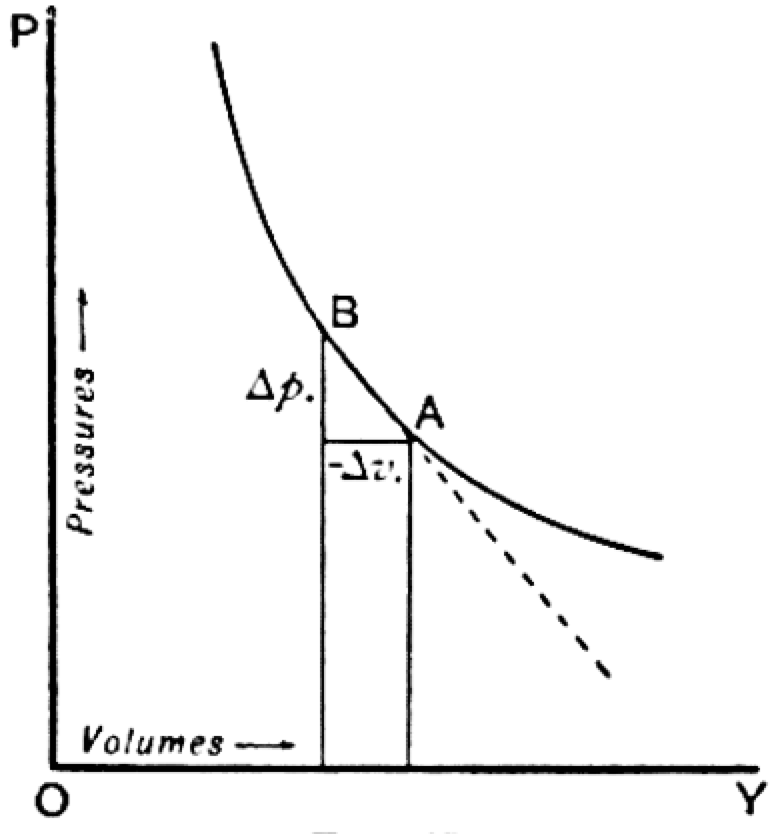


FIG. 167.

volume per unit of volume, and is therefore $-\Delta v/v$. The **elasticity** at pressure p is defined as the ratio of the change in **stress**, Δp, to the strain produced, $-\Delta v/v$, when Δp is indefinitely small, *i.e.* the elasticity is

$$\lim_{\Delta p \to 0} -\Delta p \Big/ \frac{\Delta v}{v}, \quad i.e. \quad \lim_{\Delta p \to 0} -v\frac{\Delta p}{\Delta v} \quad \text{or} \quad -v\frac{dp}{dv}.$$

(vi) **Back E.M.F.**—If a current of electricity is altering in any way, there is a reaction or back electromotive force due to self-induction which is proportional to the rate of change of current. Thus, if i denotes current and t time,

$$\text{the back electromotive force} \propto \frac{di}{dt}.$$

(vii) **Measurement of Stress.**—If the total pressure on an area S is P, the intensity of stress, or stress per unit area, p, is P/S. If the distribution of stress is not uniform, and ΔP be the small stress on a very small element of area ΔS surrounding a point A, then

$$\text{the intensity of stress at } A \text{ is } \frac{dP}{dS}.$$

(viii) **Bending of Beams.**—*Find the connection between the bending moment and the shearing force at any point of a loaded beam.*

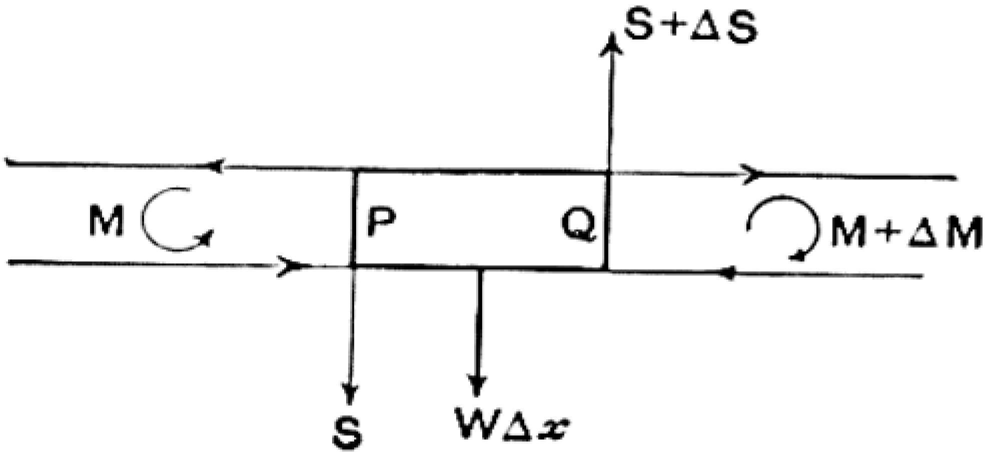


FIG. 168.

Consider the equilibrium of an element PQ of length Δx of a beam loaded and supported in any manner whatever.

Let S, M be the shearing force and bending moment respectively at the section P (Fig. 168). Then the S.F. and B.M. at section Q are $S+\Delta S$ and $M+\Delta M$. If the load per unit length is W, we have, resolving vertically,

$$S+\Delta S = S + W\Delta x, \quad \text{or} \quad \Delta S = W\Delta x;$$

$$\therefore \; W = \frac{\Delta S}{\Delta x}, \quad \text{and, in the limit, } W = \frac{dS}{dx} \; \dots\dots\dots\dots(1)$$

Again, taking moments about Q:

$$S\Delta x+\frac{W(\Delta x)^2}{2}+M=M+\Delta M, \quad \text{or } S+\frac{W\Delta x}{2}=\frac{\Delta M}{\Delta x}.$$

And, since $W\Delta x/2 \to 0$ as $\Delta x \to 0$, we have

$$S=\frac{dM}{dx}. \quad \text{..............................(2)}$$

In words, the shearing force is the rate of increase of bending moment with length; the load per unit length is the rate of increase of shear with length.

The Differential Coefficient. Illustrations (ii) and (iii) given in the preceding paragraph are really examples of the method of finding the differential coefficient of one variable with respect to another.

$\dfrac{dy}{dx}$ is called the **differential coefficient** or **derivative** of y with respect to x.

If $y=f(x)$,

$$y+\Delta y=f(x+\Delta x);$$

$$\therefore \ \Delta y=f(x+\Delta x)-f(x)$$

and

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The derivative of $f(x)$ is sometimes written $f'(x)$.

Ex. 1. *Find the differential coefficient of* $1/x$ *with respect to* x.

Let $y=\dfrac{1}{x}$, then $y+\Delta y=\dfrac{1}{x+\Delta x}$:

$$\therefore \ \Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{-\Delta x}{x(x+\Delta x)};$$

$$\therefore \ \frac{dy}{dx}=\lim_{\Delta x\to 0}-\frac{1}{x(x+\Delta x)}=-\frac{1}{x^2}.$$

Ex. 2. *Find the derivative of* $\sqrt{x}$.

Let $y=\sqrt{x}$, then $y+\Delta y=\sqrt{x+\Delta x}$;

$$\therefore \ \Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\frac{\sqrt{x+\Delta x}-\sqrt{x}}{1}\times\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}};$$

i.e.

$$\Delta y=\frac{x+\Delta x-x}{\sqrt{x+\Delta x}+\sqrt{x}}, \quad \text{or} \quad \frac{\Delta y}{\Delta x}=\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}};$$

$$\therefore \ \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Ex. 3. *If $f(x)=3ax^2+5bx+c$, find $f'(x)$.*

We have $f(x+\Delta x)=3a(x+\Delta x)^2+5b(x+\Delta x)+c$

$$=3ax^2+6ax\,.\,\Delta x+3a(\Delta x)^2+5bx+5b\,.\,\Delta x+c$$

$$=f(x)+(\Delta x)(6ax+5b)+3a(\Delta x)^2\,;$$

$$\therefore\ \frac{f(x+\Delta x)-f(x)}{\Delta x}=6ax+5b+3a\,.\,\Delta x\,;$$

$$\therefore\ f'(x)=\lim_{\Delta x\to 0}(6ax+5b+3a\,.\,\Delta x)$$

$$=6ax+5b.$$

At this stage it is convenient to assume the truth of the following results. Proofs for the cases when n has rational values will be found in Ch. XIV.

If $y=ax^n$, $\qquad \dfrac{dy}{dx}=anx^{n-1}$.

If $y=(ax+b)^n$, $\qquad \dfrac{dy}{dx}=na(ax+b)^{n-1}$.

If y is the sum of a number of functions of x, $\dfrac{dy}{dx}$ is the sum of their derivatives ; *e.g.* if $y=ax^3+bx^2+cx+d$, then

$$\frac{dy}{dx}=3ax^2+2bx+c.$$

Ex. 1. *Find the equations of the tangent and normal to the parabola $y=4x^2$ at the point* (1, 4).

Since $y=4x^2$, $\qquad \dfrac{dy}{dx}=8x$.

Therefore, the slope of the tangent to this parabola at any point is 8 times the abscissa at that point.

The slope at (1, 4) is 8 × 1, *i.e.* 8.

The tangent is therefore the straight line passing through the point (1, 4), whose slope is 8.

Its equation is $y-4=8(x-1)$, *i.e.* $y-8x+4=0$.

The **normal** is a straight line through the point of contact at right angles to the tangent. Its equation is

$$(y-4)=-\tfrac{1}{8}(x-1),\ \text{i.e. } 8y+x=33.$$

Ex. 2. *The distance s ft., traversed by a point in t seconds from the start, is given by* $s = at^{2\cdot5} + bt^2$. *Find its velocity and acceleration after t seconds and at the start.*

We have $$s = at^{2\cdot5} + bt^2;$$

$$\therefore \frac{ds}{dt} = 2\cdot5\,at^{1\cdot5} + 2bt,$$

so that if v ft. per sec. is the velocity after t seconds,

$$v = 2\cdot5\,at^{1\cdot5} + 2bt \quad\ldots\ldots\ldots\ldots(1)$$

Evidently when $t = 0$, $\quad v = 0$.

Again, $$\frac{dv}{dt} = 2\cdot5 \times 1\cdot5\,at^{0\cdot5} + 2b,$$

so that if f ft. per sec. per sec. is the acceleration after t seconds,

$$f = 3\cdot75\,at^{0\cdot5} + 2b.$$

Evidently when $t = 0$, $\quad f = 2b$.

EXERCISES XLVII.

1. The space s ft. passed through by a body in a time t seconds is given by $s = 16t^2$. Find the values of s corresponding to the following values of t: 1, 1·1, 1·01, 1·001. What is the average speed in the intervals 0·1, 0·01 and 0·001 seconds after the first second? What is the speed when $t = 1$?

2. In t seconds a body travels a distance s ft. reckoned from its starting point. It is known that $s = 33 + 68t - 7t^2$. What is the speed at the time $t = 4$? At what time is the speed zero?

3. Find the slope of the tangent to the curve $y = (4 - 5x)^2$ at the point (1, 1).

4. Assume $y = 0\cdot5 + 2\cdot3x^2$. When x becomes $x + \Delta x$, y becomes $y + \Delta y$. Write down the value of $y + \Delta y$, and calculate the value of $\Delta y/\Delta x$. To what limit does this tend as Δx is diminished indefinitely?

5. If $y = 2\cdot4 + 1\cdot6x^2$, calculate y when $x = 2\cdot3$. Now calculate Δy when Δx equals 0·1, 0·01 and 0·001. Write down the corresponding values of $\frac{\Delta y}{\Delta x}$ in each case. What is $\frac{dy}{dx}$ when $x = 2\cdot3$?

6. The side of a square is increasing in length at the rate of 0·01 inch per second. At what rate is the area of the square increasing when the side is 6 inches?

7. Water is being poured into a cylinder of radius r ft. at the rate of U cubic ft. per second. At what rate is the depth of water increasing ?

8. Find the value of $f'(x)$ if $f(x)$ equals (i) $2x+1$, (ii) $3x^3+4x$, (iii) $2x^{1/3}+x^{2/5}$, (iv) $1/x^2$, (v) $3\sqrt{x}+5/x$.

9. Write down the derivatives with respect to x, y, z, t, etc., of

(i) $\frac{1}{3}x^3+2x^2+5x+1$, (ii) $(2y+1)^2$, (iii) $1/\sqrt{z}$,
(iv) $(at+b)^3$, (v) $av^3+b\sqrt{v}-c/v$.

10. Find the equations of the tangents and normals to

(i) $y=5x^2-4x+3$ at the point $(1, 4)$,
(ii) $y=2x^3+3/x$ at the point $(3, 55)$,
(iii) $y=(4ax)^{\frac{1}{2}}$ at the point $(at^2, 2at)$,
(iv) $y=ax^3$ at the point (h, ah^3).

11. Find $\dfrac{dy}{dx}$ when $y=2x^3-6x^2+6x-4$.

At what point is $\dfrac{dy}{dx}$ zero ? Explain this geometrically.

Find the slope at $x=1{\cdot}1$ and $x=0{\cdot}9$, and deduce the shape of the curve near the point $x=1$. [B.U.]

12. Differentiate x^3-2x+3.

Find the equation of the tangent at the point $(2, 7)$ to the curve $y=x^3-2x+3$. [B.U.]

13. Find the differential coefficient of x^2, explaining your procedure.

Write down the differential coefficient of

$$7x^3+2x+3.$$ [B.U.]

14. Find the limits to which the following expressions tend as Δx tends to zero :

(i) $\left\{\dfrac{1}{2(x+\Delta x)^2}-\dfrac{1}{2x^2}\right\}\Big/\Delta x$; (ii) $\left\{\dfrac{1}{\sqrt{x+\Delta x}}-\dfrac{1}{\sqrt{x}}\right\}\Big/\Delta x$;

(iii) $\{(x+\Delta x)^{\frac{1}{3}}-x^{\frac{1}{3}}\}/\Delta x$.

15. Write down the differential coefficients of the following expressions as limits, and evaluate each limit.

(i) x^3, (ii) $\dfrac{1}{x^2}$, (iii) $\dfrac{x}{x+1}$, (iv) $\dfrac{ax+b}{cx+d}$,
(v) $(x+1)^2$, (vi) $(ax+b)^2$, (vii) $\sqrt{x+1}$, (viii) $\sqrt[3]{x}$.

Derived Curves. At every point of a smooth continuous curve there is a definite slope, and hence another curve can be plotted connecting the abscissae of the first curve with the corresponding slopes. The **first derived curve** of any primitive curve is such that its ordinates measure the slopes of the primitive curve at the corresponding points. In Fig. 169 the curve $P'Q'R'S'T' \ldots$ is the first derived curve of $PQRST \ldots$, and was obtained by drawing tangents to the latter and measuring slopes; thus $Q'L'$ measures the slope of the line Qq, $R'M'$ that of the line Rr, $T'N'$ that of the line Tt (which is negative), and so on.

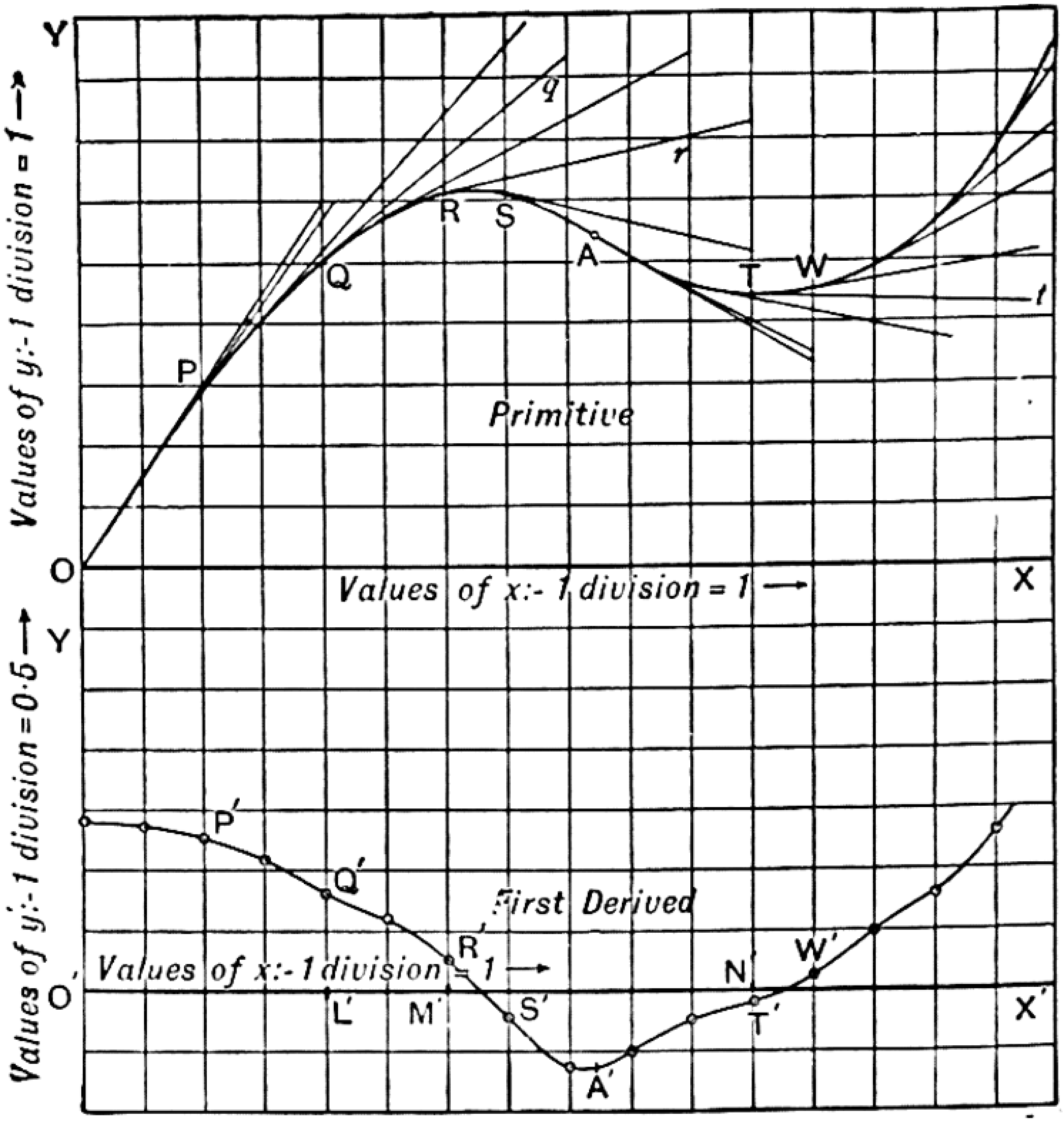


FIG. 169.

While errors are likely to arise in measuring the slope of a curve at a point (by this tangent method), it should be clear that these errors are not cumulative, and that by taking a series of points a fair idea of the variation in the slope can be obtained.

Graphic Determination of Slope. In obtaining the slope of a curve at any point by the tangent method, it is unnecessary actually to draw the tangents; the slope is measured as follows: Place a ruler tangentially to the curve (obtaining the exact reading of the point of contact *after* placing the ruler in position); then note the positions at which the ruler cuts two convenient ordinates. Divide the difference of these two readings by the distance between the ordinates.

To find the slope of a curve at a particular point P, a good method is to choose two points Q, R on the curve equidistant from, but fairly close to, P on opposite sides of it. The slope of the line QR can be taken to be the slope of the curve at P to a close degree of approximation. This method gives better results than attempting to draw the actual tangent at P.

Properties of Derived Curves. If all the ordinates of the primitive curve be increased by any, and the same, amount, it should be clear that the derived curve remains the same. For increasing each ordinate by the amount 'a' is equivalent to lowering the axis OX by this amount, and clearly the slope at every point is unaltered.

If the coordinates of P, a point on the primitive curve, be x, y, then the coordinates of P', the corresponding point on the derived curve, may be written x, y'; so that y' is the slope of the tangent to the primitive curve at P.

Between R and S, Fig. 169, the ordinate y has a **maximum** value, and the corresponding point on the derived curve is the point between R' and S', where the derived curve cuts its axis of x. It is clear that $y'=0$ for any maximum value of y, since at such maxima the tangents to the primitive curve are parallel to the axis of x. Similarly when y is a **minimum**, $y'=0$; *e.g.* y is a minimum at a point between T and W, and $y'=0$ at the corresponding point on the derived curve, viz. where the latter cuts the axis of x between T' and W'.

Further, if we pass along the derived curve from left to right, y' changes from positive to negative where y is a maximum, from negative to positive where y is a minimum.

The point A on the primitive* is called a **point of inflexion,** that is, it is a point at which the tangent to the curve, after turning in one direction as we pass along the curve, stops and turns in the other direction. It is a point at which the *slope* is either a maximum or a minimum. Note the turning point near A' on the first derived.*

---

* For brevity, 'primitive' is written for 'primitive curve.'

The points at which a curve has a maximum or minimum ordinate are called **turning points.**

Determination of Maxima and Minima. If the graph of $f(x)$ has a maximum or minimum value when $x=a$, then $f'(x)=0$ when $x=a$; *i.e.* $f'(a)=0$.

The method of determining whether the ordinate is a maximum or a minimum at a turning point is illustrated in the following examples.

Ex. 1. *Find the turning points on the curve*

$$y=4x^3+15x^2-18x+3.$$

We have $\dfrac{dy}{dx}=12x^2+30x-18=6(2x-1)(x+3)$;

$$\therefore\ \frac{dy}{dx}=0, \quad \text{where } x=\tfrac{1}{2} \text{ or } -3.$$

To determine whether $x=\frac{1}{2}$ gives a maximum or a minimum ordinate, the slope of the curve at points where x is just less than and just greater than $\frac{1}{2}$ must be examined, say, at the points where x is $\frac{1}{2}-h$ and $\frac{1}{2}+h$ where h is very small.

x	$\frac{dy}{dx}$	
$\frac{1}{2}-h$	$6(-2h)(3\frac{1}{2}-h)$,	which is negative, since h is small.
$\frac{1}{2}$	0	
$\frac{1}{2}+h$	$6(2h)(3\frac{1}{2}+h)$,	which is positive.

Therefore in the neighbourhood of the point where $x=\frac{1}{2}$, the tangents must look like .

Therefore y is a minimum where $x=\frac{1}{2}$ and the minimum value of $y=-1\frac{3}{4}$.

Examining the curve in the region $x=-3$, we have

x	$\frac{dy}{dx}$	
$-3-h$	$6(-7-2h)(-h)$,	which is positive.
-3	0	
$-3+h$	$6(-7+2h)(h)$,	which is negative, since h is small.

---

* For brevity, 'derived' is written for 'derived curve.'

Therefore in the neighbourhood of the point where $x = -3$ the tangents must look like / ‾ \\.

Therefore y is a maximum where $x = -3$ and the maximum value of y is 84.

A rough graph of the curve is shewn in Fig. 170.

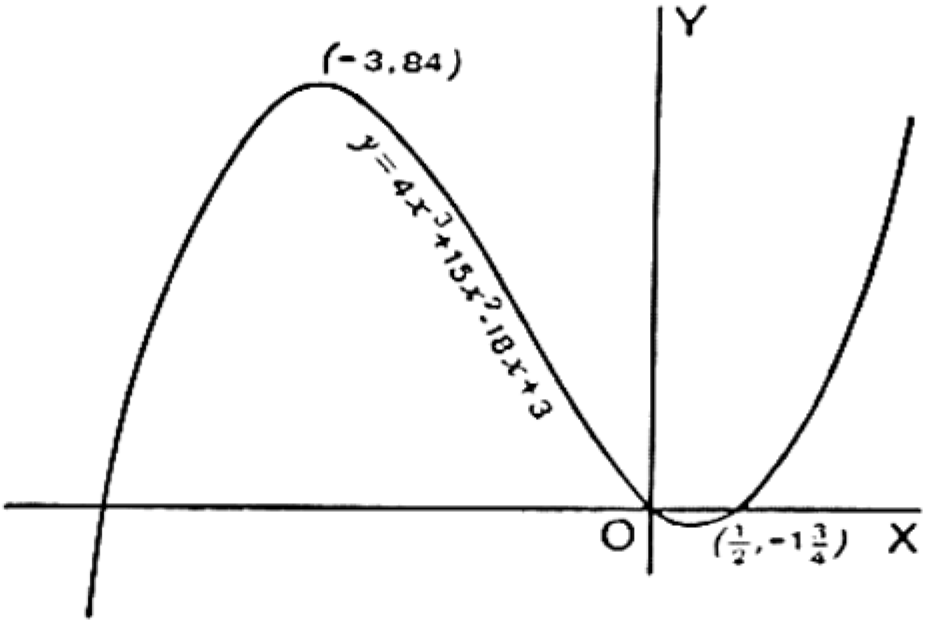


FIG. 170.

Ex. 2. *Find the turning points on the graph of* $\frac{1}{8}x^3 - \frac{3}{4}x^2$.

We have
$$f(x) = \tfrac{1}{8}x^3 - \tfrac{3}{4}x^2;$$
$$\therefore\ f'(x) = \tfrac{3}{8}x^2 - \tfrac{3}{2}x = \tfrac{3}{8}x(x-4);$$
$$\therefore\ f'(x) = 0, \quad \text{where } x = 0 \text{ and } x = 4.$$

x	$f'(x)$	
$-h$	$-\frac{3}{8}h(-h-4)$,	which is positive.
0	0	
h	$\frac{3}{8}h(h-4)$,	which is negative.

Near $x = 0$, the tangents are arranged / ‾ \\ ;

$$\therefore\ x = 0 \text{ gives a maximum, viz. zero.}$$

x	$f'(x)$	
$4-h$	$\frac{3}{8}(4-h)(-h)$,	which is negative.
4	0	
$4+h$	$\frac{3}{8}(4+h)(h)$,	which is positive.

Near $x = 4$, the tangents are arranged \\ __ / ;

$$\therefore\ x = 4 \text{ gives a minimum, viz. } -4.$$

The curve is shewn in Fig. 171.

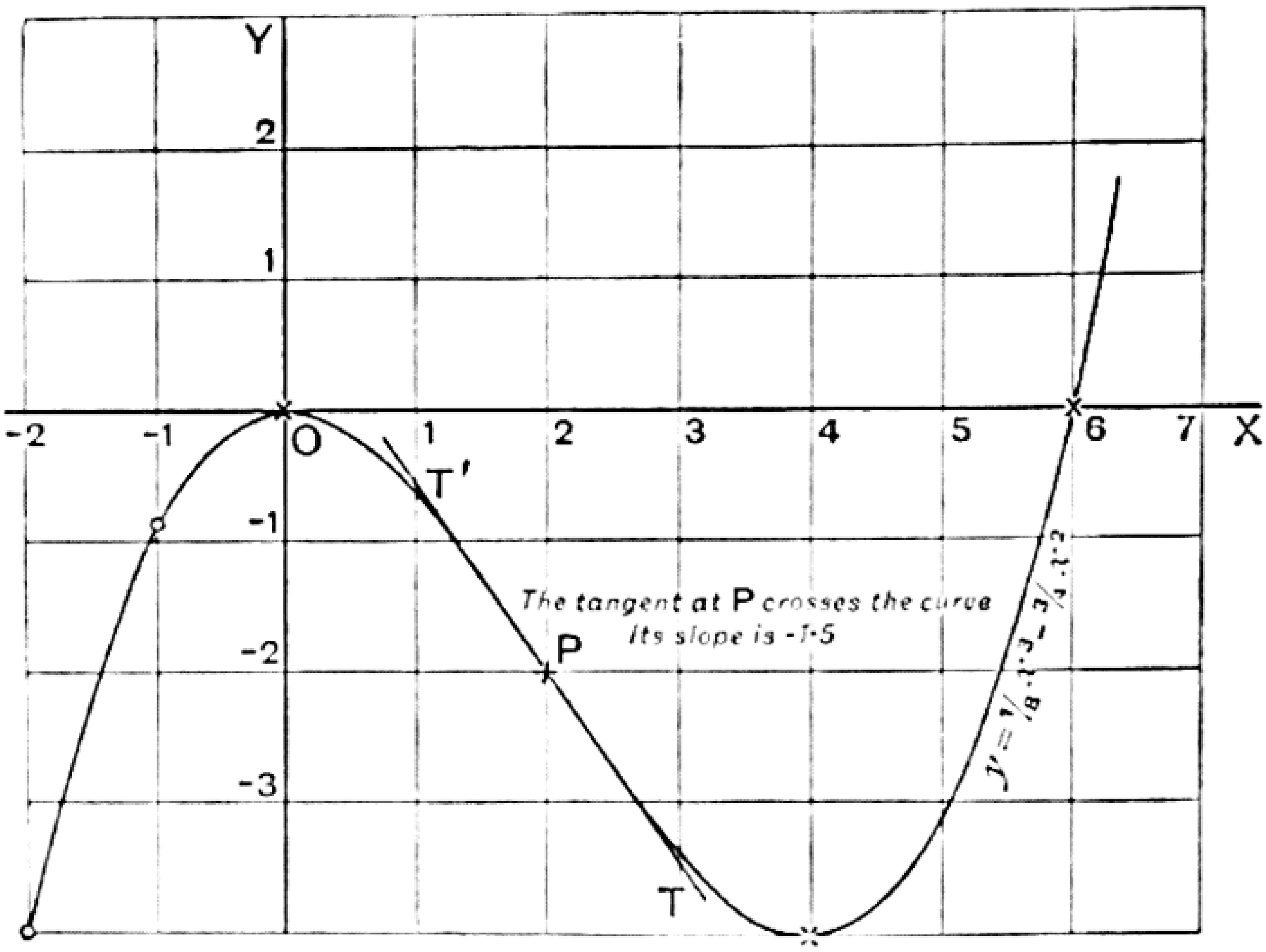


FIG. 171.

Ex. 3. *Find the point of inflexion on the graph of* $\frac{x^3}{8} - \frac{3x^2}{4}$.

From the last example, $f'(x) = \frac{3}{8}(x^2 - 4x)$.

At a point of inflexion, **the slope f′(x)** is either a maximum or a minimum (see pp. 381 and 382).

$\therefore$ the derivative of $f'(x)$, conveniently denoted by $f''(x)$, is zero.

We have
$$f''(x) = \tfrac{3}{8}(2x - 4);$$
$$\therefore f''(x) = 0, \quad \text{where } x = 2.$$

$\therefore$ the curve has a point of inflexion when $x = 2$, and the slope of the tangent TT' at this point is $\frac{3}{8}(4 - 8)$, *i.e.* $-\frac{3}{2}$.

The point is shewn at P on Fig. 171, and the curve *crosses* the tangent there.

Ex. 4. *Find the turning point on the parabola* $y = ax^2 + bx + c$.

We have
$$\frac{dy}{dx} = 2ax + b;$$
$$\therefore \frac{dy}{dx} = 0, \quad \text{where} \quad x = -\frac{b}{2a}.$$

Then, proceeding as before,

x	$\frac{dy}{dx}$
$-b/2a-h$	$-2ah$
$-b/2a$	0
$-b/2a+h$	$+2ah$

As h is positive, therefore, in the neighbourhood of $x=-b/2a$, the tangents must look like /‾‾\ if a is negative and like \__/ if a is positive.

Thus the point $-b/2a$, $(4ac-b^2)/4a$ is the turning point. y is a maximum at this point when a is negative and a minimum when a is positive. See p. 74 and Fig. 43.

Ex. 5. *Examine the nature of the curve at the point* (1, 1) *on the curve* $y=3x^4-8x^3+6x^2$.

We have $\frac{dy}{dx}=12x^3-24x^2+12x=12x\{x-1\}^2$.

When $x=1$, $\frac{dy}{dx}=0$.

When $x=1-h$, $\frac{dy}{dx}=12(1-h)(-h)^2$, which is positive.

When $x=1+h$, $\frac{dy}{dx}=12(1+h)(h)^2$, which is also positive.

Near $x=1$, the tangents appear like /‾‾/, and there is a point of inflexion at this point (1, 1).

Ex. 6. *The strength of a rectangular beam of given length is proportional to the breadth of the section multiplied by the square of the depth. Find the strongest beam which can be cut from a cylindrical log, 12 inches in diameter.*

Let x in. be the breadth of the beam: then the depth is $\sqrt{12^2-x^2}$ in. (see Fig. 172).

Let S stand for the strength of the beam.

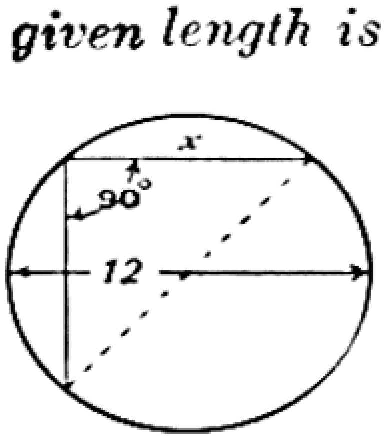


FIG. 172.

$$\therefore\ S=kx(144-x^2)=k(144x-x^3);$$

$$\therefore\ \frac{dS}{dx}=k(144-3x^2)=3k(48-x^2)=3k(4\sqrt{3}-x)(4\sqrt{3}+x).$$

x	$\frac{dS}{dx}$	
$4\sqrt{3}-h$	$3k(h) \times$ (a positive factor),	which is positive.
$4\sqrt{3}$	0	
$4\sqrt{3}+h$	$3k(-h) \times$ (a positive factor),	which is negative.

Near $x=4\sqrt{3}$, the tangents look like / ‾ \\.

$\therefore$ $x=4\sqrt{3}$ gives a maximum value to S, and the strongest beam has dimensions $4\sqrt{3}$ in. by $4\sqrt{6}$ in.

EXERCISES XLVIII.

1. Find the turning points on the parabolas:

(a) $y=5+6x-3x^2$, (b) $y=(x-1)(x+1)$,

(c) $y=2x^2-x+1$.

Shew whether each is a maximum or a minimum.

2. What value of z makes $z(1-z^2)$ a maximum?

3. Divide a number a into two parts, so that twice the square of one part plus three times the square of the other shall be a minimum.

How do you know that you have found a minimum value?

4. Divide a number p into two parts, such that the sum of their squares is a minimum.

5. A line AB, 12 in. long, is divided at P. Find the length of AP when AP^3+PB^3 is a minimum.

6. Find the turning points on the curve whose equation is

$$y=x^3-3x.$$

Shew whether each is a maximum or a minimum, and draw a rough graph of the curve.

7. What number exceeds its fourth power by the greatest number possible?

8. Shew that the dimensions of the greatest open box of square base which can be made from a sheet of tin plate, a inches square, are $a/6$ in., $2a/3$ in., and $2a/3$ in.

9. When sending a parcel through the post its combined length and girth must not exceed 6 ft. Find the maximum volume of a rectangular parcel with square ends.

10. The total cost of a ship per hour (including interest, depreciation, wages, coal, etc.) is, in pounds,

$$c = 3{\cdot}2 + \frac{s^3}{2200},$$

where s is the speed of the ship in knots.

Express the total cost of a passage of 3000 miles in terms of s. What value of s will make this total cost a minimum? At speeds 10 per cent. less and greater than this, compare the total cost with its minimum value. [B.E.]

11. The total cost of a certain ship per hour (including interest, depreciation, wages, coal, etc.) is, in pounds,

$$c = 8{\cdot}21 + \frac{s^3}{1200},$$

where s is the speed in knots.

Express the total cost of a passage of 3400 nautical miles in terms of s. What value of s will make this total cost a minimum? Show that at speeds 10 per cent. greater or less than this, the total cost is not very much greater than what it is for the best speed. [B.E.]

12. A tank with open top and square section is to hold 100 cubic feet. Find the cost of lining it with lead at 6*d*. per sq. ft., when that cost is a minimum.

13. There is a machine consisting of two parts, whose weights are x and y. The cost of the machine in pounds is $12x + 5y$. The power of the machine is proportional to xy. Find x and y if the cost is £100 and if we desire to have the greatest power possible.

14. The work, w, done by a motor per second when the current is C is given by $w = VC - C^2R$, where V and R are constants. Find the value of C in order that w may be a maximum. Hence shew that to get the maximum work per second from a motor, it must run at such a speed as will bring the current to half the value it would have if the motor were at rest.

15. The length and circumference of a cylinder are together equal to 6 ft. Find its maximum volume.

16. Find the turning points on the following curves, and state their nature in each case. Find also the points of inflexion, if any.

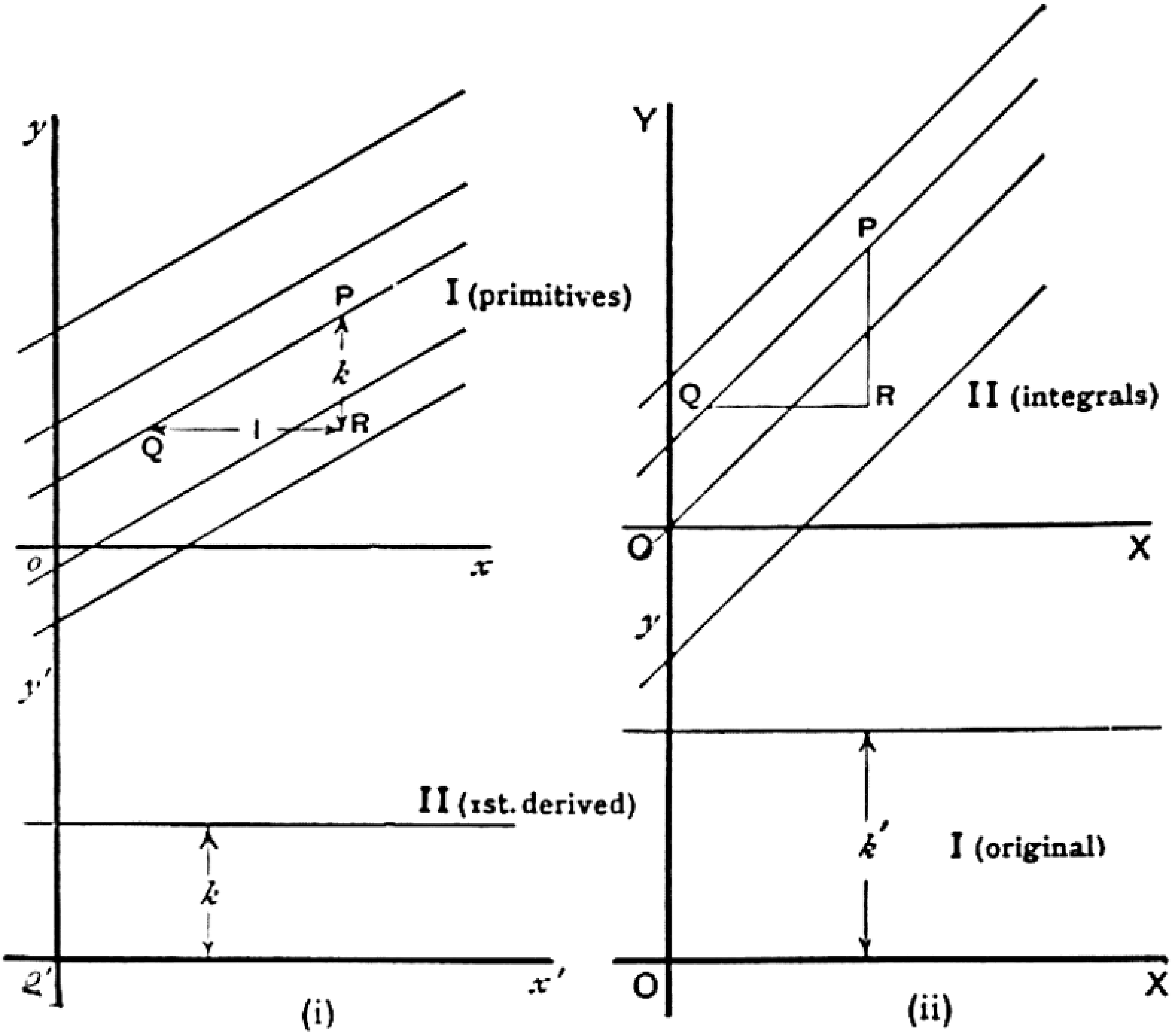
(*a*) $y = 2x^3 - 9x^2 + 12x + 5$, (*b*) $y = x^3 - 12x + 1$,

(*c*) $y = 3x^4 - 8x^3 + 6x^2 + 1$, (*d*) $y = 3x^5 - 5x^3 + 2$,

(*e*) $y = (2 + x)^2(4 - x^2)$.

Integration (First Principles). Having investigated the process by which the first derived of a given primitive (p. 380) is obtained, it is natural to consider the converse problem, viz.: given any curve, to obtain the curve of which it is the first derived. If the given curve be called the original, the curve of which it is the first derived is called the **integral**, and the process by which the integral curve is obtained may be called **integration**. Thus, integration is the *inverse* of differentiation.

Consider the series of primitives I shewn in Fig. 173 (i).



(i) (ii)

FIG. 173.

Since each primitive has the same slope at every point, they will have a common first derived, viz. the graph II, a line parallel to the axis of x at such a distance k that $k = PR/RQ$.

Conversely, in Fig. 173 (ii) any one of the curves II may be regarded as the integral of the line I, since that line is the first derived of any one of the group, provided that

$$k' = PR/RQ.$$

Thus, the problem of integration is of an interrogative character; *i.e.* given any function of x, $f(x)$, we have to find a function $F(x)$, such that $f(x)$ is the differential coefficient of $F(x)$. The notation adopted is

$$F(x) = \int f(x)\, dx.$$

In words, $F(x)$ is the integral of $f(x)$ with respect to x. The symbols $\int$ and 'd' may be translated respectively by the phrases: 'what is the function which gives' and 'when differentiated with respect to'; so that $\int f(x)\, dx$ means: 'what is the function which gives $f(x)$ when differentiated with respect to x?'

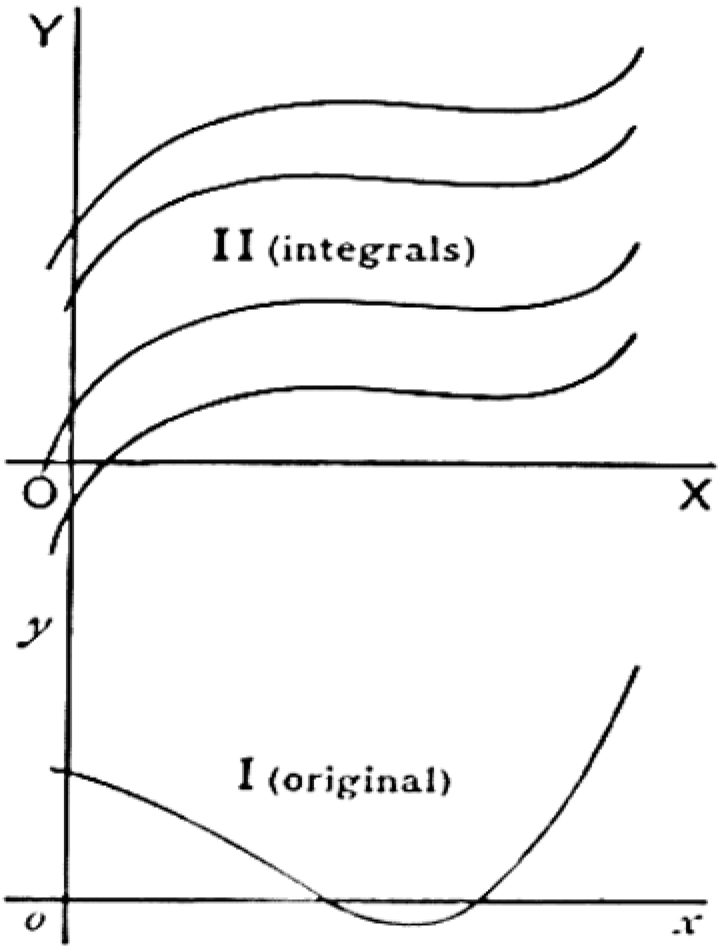


FIG. 174.

In Fig. 174, if I be the graph of $y = f(x)$, the integral curve may be any one of the family of *parallel** curves II,

$$Y = F(x) + C,$$

where C is any arbitrary constant, provided that $F(x)$ is such that $\frac{dF(x)}{dx} = f(x)$.

---

* Parallel because these curves have the same slope at any given value of the abscissa.

Ex. 1. *Integrate, with respect to x, the expression x^3.*

We have to find $$\int x^3 dx.$$

i.e. the expression which, when differentiated, gives x^3.

Now x^4+C when differentiated gives $4x^3$, C being any constant.

So $\frac{1}{4}x^4+C$ when differentiated will give $\frac{1}{4}\times 4x^3$;

$$\therefore \int x^3\,dx = \tfrac{1}{4}x^4 + C.$$

The necessity for introducing the arbitrary constant, C, will be obvious from the last paragraph.

Ex. 2. *Integrate, with respect to x, the expression* $5x^3-2x+1$.

We have to find $$\int(5x^3-2x+1)dx,$$

i.e. the expression which, when differentiated, gives $5x^3-2x+1$.

The required expression is $$\frac{5x^4}{4}-\frac{2x^2}{2}+x+a,$$

where a is any constant.

$$\therefore \int(5x^3-2x+1)dx = \frac{5x^4}{4}-x^2+x+a.$$

N.B.—**The result of an integration should always be tested by differentiating.** Thus, the derivative of $5x^4/4-x^2+x+a$ is $\frac{5}{4}\cdot 4x^3-2x+1$, *i.e.* $5x^3-2x+1$, which is the given expression.

Integration of Powers of x. So long as n is *not equal to* -1,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C.$$

In words, integrate x^n by adding 1 to the index and then dividing by the new index. To this result add a constant.

If this rule be applied to the function x^{-1}, the result will be $\frac{x^{-1+1}}{-1+1}+C$, *i.e.* $\frac{x^0}{0}+C$, which is unintelligible.

Similarly, if n is *not equal to* -1,

$$\int ax^n dx = \frac{ax^{n+1}}{n+1}+C$$

and $$\int(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C.$$

These formulae should be compared with those on p. **377.**

When $n = -1$, the special formula to be used is

$$\int \frac{1}{x}\,dx = \log_e x + C,$$

where e is the base of the Napierian system of logarithms. The proof of this is beyond the scope of this book.

Ex. 1. *If* $\frac{dy}{dx} = a + bx$, *find* y.

We have to find the function, y, whose derivative is $a + bx$.

We have, therefore, to find $\int (a + bx)dx$, which is

$$ax + \tfrac{1}{2}bx^2 + C;$$

$$\therefore\ y = ax + \tfrac{1}{2}bx^2 + C.$$

Ex. 2. *Find the equation of the curve, in which the slope at any point is* $3/x^{\frac{1}{4}}$, *having given that* $y = 2$ *when* $x = 16$.

The slope is $\frac{dy}{dx}$, and we have $\frac{dy}{dx} = 3x^{-\frac{1}{4}}$;

$$\therefore\ y = \int 3x^{-\frac{1}{4}}\,dx = \frac{3x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + C = 4x^{\frac{3}{4}} + C.$$

As $y = 2$ when $x = 16$, $2 = 4 \times 8 + C$; $\therefore\ C = -30$.

The required equation is $y = 4x^{\frac{3}{4}} - 30$.

Ex. 3. *If* $pv = c$, *find* $\int p\,dv$.

Since $pv = c$, we have $p = c/v$;

$$\therefore\ \int p\,dv = \int \frac{c\,dv}{v} = c\log_e v + a, \text{ where } a \text{ is a constant.}$$

Ex. 4. *Find the equation of the curve which passes through the point* (1, 1) *and whose slope at any point is given by* $2/x$.

We have $\frac{dy}{dx} = \frac{2}{x}$; $\therefore\ y = \int \frac{2}{x}dx = 2\log_e x + C$.

Since $x = 1$ when $y = 1$, we have $1 = 2\log_e 1 + C$; $\therefore\ C = 1$.

The required equation is $y = 2\log_e x + 1$, or $e^{y-1} = x^2$.

EXERCISES XLIX.

Integrate the following expressions in Ex. 1·9 with respect to x, y, z, u or t, as the case may be :—

1. $2x$. 2. $z+1$. 3. $\frac{1}{3}x^2+2x$.
4. $3t^2-2t+1$. 5. $\sqrt{u}$. 6. $1/y^{2\cdot 4}$.
7. $1/x^3$. 8. $2t+1/t$. 9. $(3x^2-4x+1)/x^3$.

Find the equations of the curves in Ex. 10-12 if the slope at any point is given by :—

10. $\frac{dy}{dx}=2(x+1)$ and $y=2$ when $x=1$.

11. $\frac{dy}{dx}=\frac{1}{2}\sqrt{x}$ and $y=\frac{8}{3}$ when $x=4$.

12. $\frac{dy}{dx}=1/x^{\frac{1}{2}}+1/x^{\frac{3}{2}}$ and $y=3$ when $x=4$.

13. The acceleration of a point moving in a straight line is $2t$ at time t. Find its velocity v at time t, having given that $v=u$ when $t=0$. Find also its displacement s at time t, having given that $s=0$ when $t=0$.

14. Work Example 13 when the acceleration at time t is $4+2t^2$, the initial conditions being the same as before.

Find the equations of the curves in Ex. 15 and 16 if the slope at any point is given by :—

15. $\frac{dy}{dx}=-\frac{1}{x}$ and $y=-1$ when $x=1$.

16. $\frac{dy}{dx}=x+\frac{2}{x}$ and $y=2$ when $x=2$.

17. Find the equation of the tangent and normal of the curve $y=a\log_e x$ at the point $(1, 0)$.

Area and Integration. Let B and P be two points on a continuous curve (Fig. 175), and let A be the area between the curve, the ordinates BC and PN and the axis of X.

Let (x, y) be the coordinates of P, $(x+\Delta x, y+\Delta y)$ the coordinates of P', another point on the curve.

Complete the rectangles $PNN'Q'$ and $P'N'NQ$.

The area of the rectangle $PNN'Q'$ is $y\,.\,\Delta x$.

The area of the rectangle $P'N'NQ$ is $(y+\Delta y)\,.\,\Delta x$.

In Fig. 175 (i) Δy is positive ; in Fig. 175 (ii) Δy is negative.

The area of the figure $PNN'P'$ is ΔA, being an increment in the area A as P moves to P'.

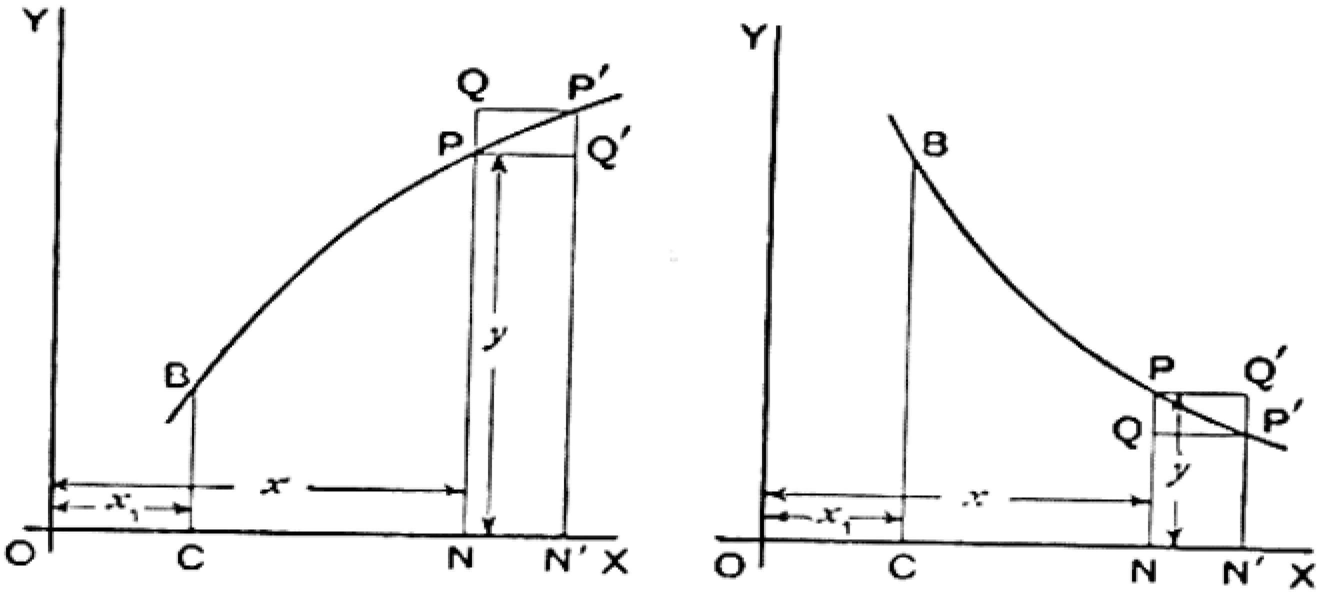


FIG. 175.

Using Fig. 175 (i), $(y+\Delta y)\,.\,\Delta x > \Delta A > y\,.\,\Delta x$;

$$\therefore\ (y+\Delta y) > \frac{\Delta A}{\Delta x} > y, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and, as $\Delta x \to 0$, Δy also $\to 0$,

i.e. the term $(y+\Delta y) \to y$ as $\Delta x \to 0$;

$$\therefore\ \frac{\Delta A}{\Delta x} \to y \text{ also,}$$

$$i.e.\quad \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = y, \quad \text{or} \quad \frac{dA}{dx} = y.$$

Similarly, using Fig. 175 (ii), $(y+\Delta y)\Delta x < \Delta A < y\,\Delta x$;

$$\therefore\ (y+\Delta y) < \frac{\Delta A}{\Delta x} < y.$$

Hence, in the limit, $$\frac{dA}{dx} = y.$$

Illustrations. (i) **Draught and Displacement of Ship.** When the draught of a ship is x ft., let the volume of water it displaces be V cubic ft. Let A sq. ft. be the area of its section by the plane of flotation. When the draught becomes $x+\Delta x$ ft., the volume displaced becomes $V+\Delta V$ cub. ft., and the area of the section by the new plane of flotation becomes $A+\Delta A$ sq. ft.

Just as in the preceding paragraph, it is clear that

$$\Delta V \text{ lies between } A\,.\,\Delta x \quad\text{and}\quad (A+\Delta A)\Delta x;$$

$$\therefore\ \frac{\Delta V}{\Delta x} \text{ lies between } A \quad\text{and}\quad A+\Delta A;$$

$$\therefore\ \text{in the limit,}\qquad \frac{dV}{dx}=A.$$

In practical examples of this kind it is usual to argue more loosely and briefly as follows :—

$$\Delta V \doteqdot A\,\Delta x, \text{ and, in the limit, } \frac{dV}{dx}=A.$$

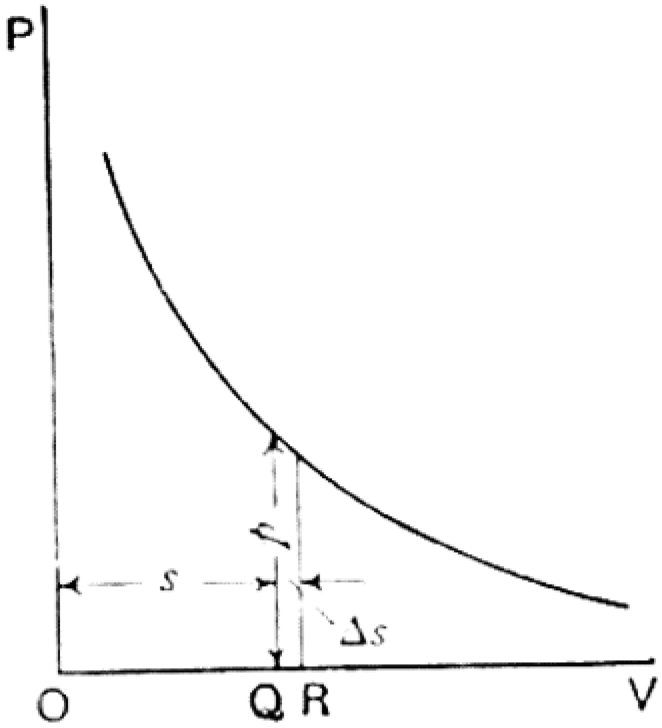


FIG. 176.

(ii) **Displacement and Work Done.** The pressure in a cylinder is a function of the displacement of the piston. When a piston of area A sq. in. moves from Q to R (Fig. 176), the pressure is practically constant and equal to p lb. per sq. in. if QR is small enough. The work done in this displacement is $pA \times QR$, *i.e.* $pA\,.\,\Delta s$ inch-lb.

If W inch-lb. is the work done in the whole displacement s, $\Delta W \doteqdot pA\,.\,\Delta s$, *i.e.* $p\,\Delta V$, where V is the volume.

$$\therefore\ \text{in the limit, } \frac{dW}{dV}=p, \text{ or } W=\int p\,dV.$$

Ex. 1. *Find the area between the parabola* $y=x^2$, *the axis of* X, *and the ordinates at* $x=1$ *and* $x=2$.

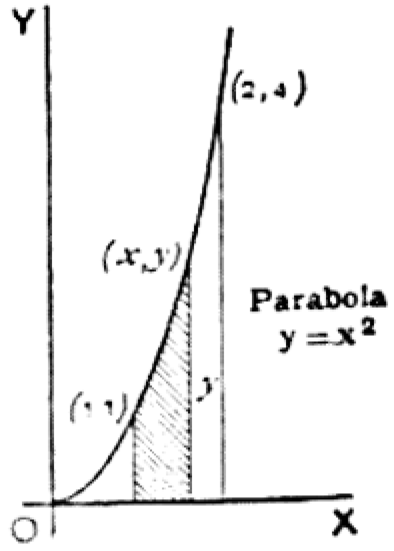


FIG. 177.

Let A be the shaded area in Fig. 177.

$$\text{Then}\qquad \frac{dA}{dx}=y, \quad i.e.\ \frac{dA}{dx}=x^2;$$

$$\therefore\ A=\int x^2\,dx=\frac{x^3}{3}+C.$$

Now, when $x=1$, $A=0$;

$$\therefore\ \tfrac{1}{3}+C=0 \quad\text{or}\quad C=-\tfrac{1}{3};$$

$$\therefore\ A=\frac{x^3-1}{3}; \quad\text{and when } x=2, \quad A=\frac{2^3-1}{3}=2\tfrac{1}{3}.$$

The required area is, therefore, $2\frac{1}{3}$.

Ex. 2. *Find the area between the parabola $y=4-3x+x^2$, the axis of X, and the ordinates at $x=a$ and $x=b$.*

Let A be the shaded area in Fig. 178.

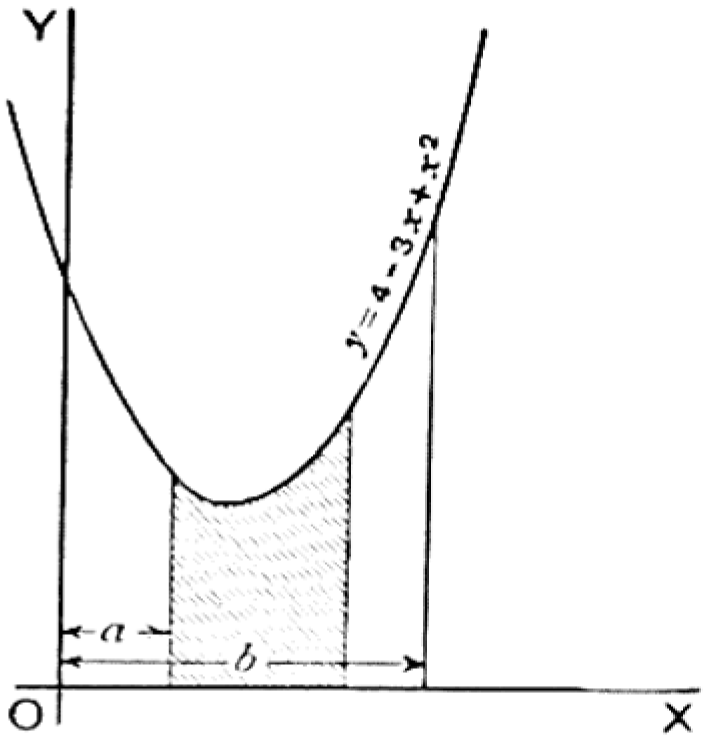


FIG. 178.

Then

$$\frac{dA}{dx}=y=4-3x+x^2;$$

$$\therefore\ A=\int(4-3x+x^2)\,dx$$

$$=4x-\frac{3x^2}{2}+\frac{x^3}{3}+C.$$

When $x=a$, $A=0$;

$$\therefore\ C=-\left(4a-\frac{3a^2}{2}+\frac{a^3}{3}\right);$$

$$\therefore\ A=\left(4x-\frac{3x^2}{2}+\frac{x^3}{3}\right)-\left(4a-\frac{3a^2}{2}+\frac{a^3}{3}\right).$$

$$\text{The area required}=\left(4b-\frac{3b^2}{2}+\frac{b^3}{3}\right)-\left(4a-\frac{3a^2}{2}+\frac{a^3}{3}\right).$$

If $F(x)$ be written for $4x-\frac{3x^2}{2}+\frac{x^3}{3}$, $F(x)$ is called the **indefinite integral** of $4-3x+x^2$, while $F(x)+C$ is called the general integral.

The area is $F(b)-F(a)$, which may be written $\Big[F(x)\Big]_a^b$.

The notation used for the area between a curve $y=f(x)$, the axis of X and the ordinates at $x=a$ and $x=b$, is

$$\int_a^b f(x)\,dx.$$

This is called the **definite integral** of $f(x)$ with respect to x between the limits a and b.

a is the **lower limit** and b the **upper limit** of the integral.

If the indefinite integral of $f(x)$ is $F(x)$, then

$$\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b=F(b)-F(a).$$

Ex. 3. *A gas expands according to the 'law'* $pv^n = k$. *Find the work done while the volume changes from* v_1 *to* v_2.

The work done is represented by the shaded area in Fig. 179.

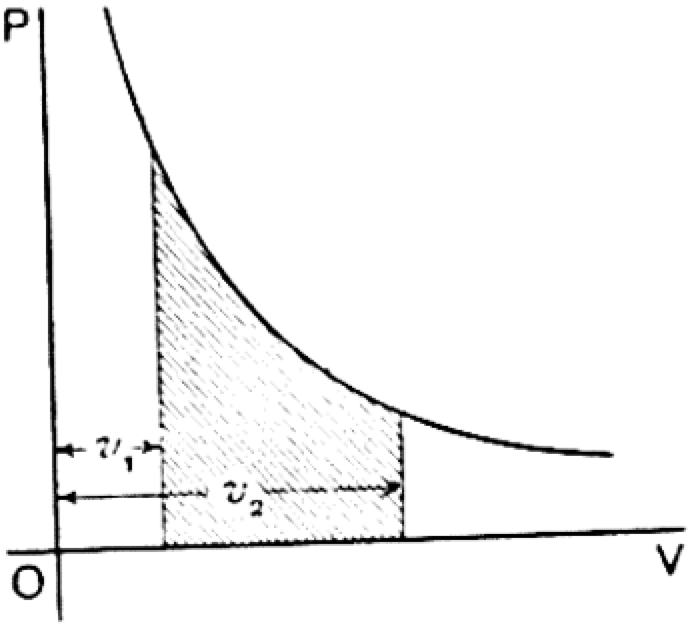


FIG. 179.

This area is
$$\int_{v_1}^{v_2} p\,dv,$$

i.e.
$$\int_{v_1}^{v_2} kv^{-n}\,dv. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The indefinite integral of kv^{-n}, with respect to v, is
$$k\frac{v^{-n+1}}{-n+1};$$

$$\therefore\ (1) \text{ is } \left[k\frac{v^{-n+1}}{-n+1}\right]_{v_1}^{v_2} \quad i.e.\ k\frac{v_2^{-n+1}}{-n+1} - k\frac{v_1^{-n+1}}{-n+1}.$$

The work done is
$$\frac{k}{1-n}(v_2^{1-n} - v_1^{1-n}).$$

Ex. 4. *Let* y_1, y_2, y_3 *be three equidistant ordinates of a parabola whose axis is parallel to the axis of* Y, *and let the distance between consecutive ordinates be* h. *Shew that the area between the parabola, the axis of* X, *and the extreme ordinates* y_1, y_3, *is*
$$\frac{h}{3}(y_1 + y_3 + 4y_2).$$

Reference to pp. 74, 75 and Fig. 43, will shew that the equation of a parabola whose axis is parallel to the axis of Y may be written $y = ax^2 + bx + c$.

In Fig. 180, the mid-ordinate y_2 is chosen as the axis of Y. Writing $x=-h$, $x=0$ and $x=h$ in turn, we have

$$y_1 = ah^2 - bh + c, \quad \text{......................(1)}$$

$$y_2 = c, \quad \text{......................................(2)}$$

$$y_3 = ah^2 + bh + c. \quad \text{......................(3)}$$

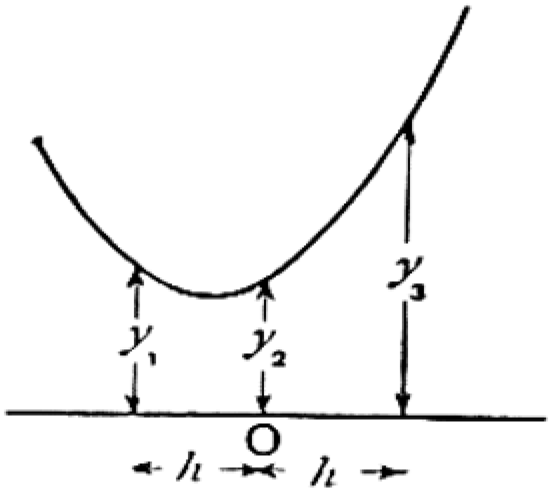


FIG. 180.

Now the area is

$$\int_{-h}^{+h} (ax^2 + bx + c)\,dx, \quad i.e. \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{+h}$$

$$i.e. \qquad \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right),$$

$$i.e. \qquad 2\left(\frac{ah^3}{3} + ch\right),$$

$$\text{or, using (2),} \qquad \frac{h}{3}(2ah^2 + 6y_2).$$

Adding (1) and (3), and using (2), we have

$$y_1 + y_3 = 2ah^2 + 2y_2.$$

Therefore the area is $\frac{h}{3}(y_1 + y_3 + 4y_2)$.

Simpson's Rule. Divide an area, such as that in Fig. 181, into an even number of parts by an odd number of equidistant parallel ordinates. In the figure the number of parts is 10. If successive points of division on the curve are close enough, it can be shewn that a **parabolic** arc, closely approximating to the curve, can be drawn through any three such successive points, and the formula of the last example can be applied.

Thus the area of $CRQB \doteqdot \frac{h}{3}(CR + BQ + 4AP)$

and the area of $C'RQB' \doteqdot \frac{h}{3}(C'R + B'Q + 4A'P)$,

so that the area of the strip between the ordinates y_4 and y_6 is approximately

$$\frac{h}{3}(y_4 + y_6 + 4y_5).$$

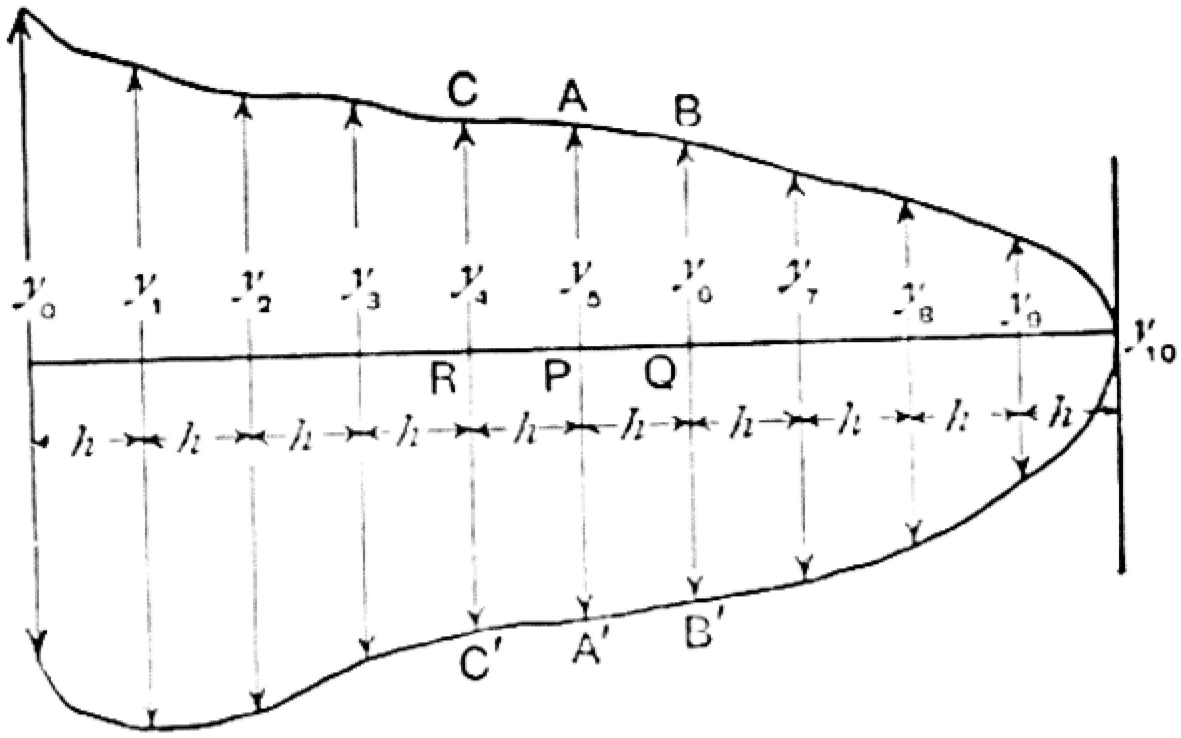


FIG. 181.

The area of each strip of breadth $2h$ can be written down in a similar manner, so that the whole area in Fig. 181

$$\doteqdot \frac{h}{3}(y_0 + y_2 + 4y_1) + \frac{h}{3}(y_2 + y_4 + 4y_3) + \frac{h}{3}(y_4 + y_6 + 4y_5)$$
$$+ \frac{h}{3}(y_6 + y_8 + 4y_7) + \frac{h}{3}(y_8 + y_{10} + 4y_9),$$

i.e.
$$\frac{h}{3}[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)].$$

This is Simpson's rule for finding an approximate value of an area, viz.:

Take the sum of the extreme ordinates, 4 times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last); multiply the total sum by $\frac{1}{3}$ of the distance between each consecutive pair of ordinates.

Note that the lengths of the first and last ordinates may or may not be zero. In Fig. 181 the last ordinate, y_{10}, is zero.

Ex. 5. *The areas A, in thousands of square feet, of the surface of the water in a reservoir at different levels h feet above the lowest point are given. Find the volume of water lost when the level falls from 80 to 50 feet.*

h	0	5	10	21	29	35	41	51	60	71	80
A	0	115	155	260	370	520	620	720	800	890	900

Fig. 182 shews the connection between A and h. Since the ordinates represent *cross sectional areas of the reservoir*, it is clear that the area under the graph between the ordinates at $h = 50$ and $h = 80$ represents the *volume* in the reservoir between the levels 50 and 80.

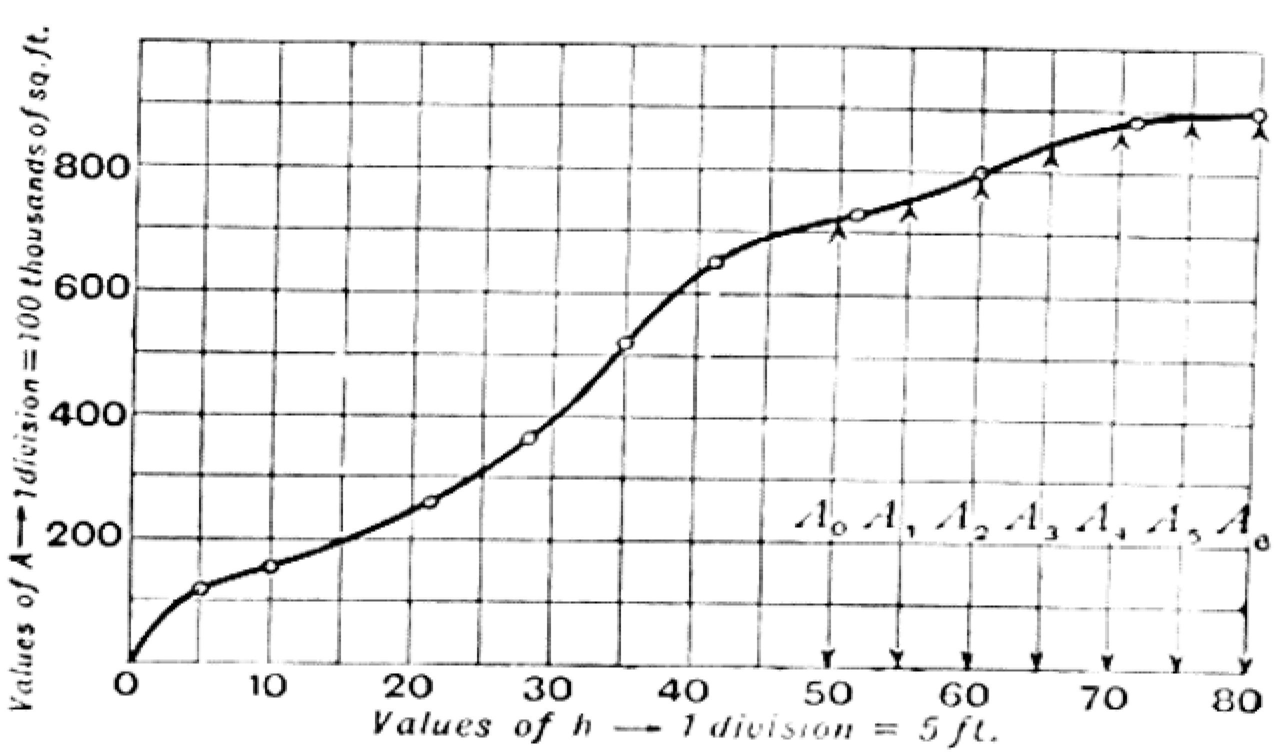


FIG. 182.

The graph between the ordinates at $h = 50$ and $h = 80$ is conveniently divided into 6 equal parts, the corresponding ordinates being

$$A_0 = 718, \quad A_1 = 740, \quad A_2 = 800, \quad A_3 = 850,$$
$$A_4 = 893, \quad A_5 = 900, \quad A_6 = 900.$$

Therefore the volume, in *thousands* of cubic feet, is

$$\tfrac{5}{3} \cdot [718 + 900 + 4(740 + 850 + 900) + 2(800 + 893)],$$

i.e. $\tfrac{5}{3} \cdot [1618 + 4 \times 2490 + 2 \times 1693]$,

i.e. $\tfrac{5}{3} \cdot [1618 + 9960 + 3386]$, *i.e.* $\tfrac{5}{3} \cdot [14964]$ or 24940.

Therefore the volume of water lost is approximately 24,900,000 cubic feet.

EXERCISES L.

1. Find the following definite integrals:

(a) $\int_1^2 dx.$ (b) $\int_{-1}^1 (3x+2)dx.$ (c) $\int_0^1 x^5 dx.$

(d) $\int_1^2 \frac{dx}{x^2}.$ (e) $\int_1^2 x^{1·7}dx.$ (f) $\int_{0·5}^{2·7} x^{-0·4}dx.$

(g) $\int_{-1}^{-2} (2+3x+5x^2)dx.$ (h) $\int_{-0·5}^{0·5} (3x-1)^2 dx.$

2. Find the area bounded by $y=f(x)$, the axis of X, and the ordinates at $x=a$ and $x=b$, where

(a) $f(x)=6+2x+x^2$: $a=0,\ b=3.$
(b) $f(x)=2x-x^3$: $a=1,\ b=2.$
(c) $f(x)=3x^{1·7}-2x^{-1·7}$: $a=2,\ b=4·5.$

3. What does the area under (i) a velocity-time curve and (ii) an acceleration-time curve represent?

4. The speed of a train increases steadily from rest to 15 miles an hour in 10 minutes. Find an expression for the distance it goes in T minutes ($T<10$).

5. Use the formula $\frac{1}{2}\pi r^2$ to find the area of a semicircle of 7·5 cm. radius, correct to four significant figures.

Draw the semicircle and 9 equidistant ordinates. Now calculate its area by Simpson's rule, and find the percentage error.

Use Simpson's rule to find the area of the semicircle when there are 9, 11, 15 and 21 ordinates, and compare the percentage error in each case with the time spent in working.

6. The area of the horizontal section of a reservoir A sq. ft. at the height h ft. from the lowest point is given in the table. What is the volume when h is 30? If the water falls in level from $h=15·5$ to $h=14·5$, what is the loss of volume?

h	0	2·5	4·0	7·0	10·0	12·5	15·0
A	0	2510	3400	4520	5160	5490	5810
h	17·5	20	23	25	28	30	
A	6210	6890	7810	8270	8670	8780	

7. Draw the graph of $y=2x^{\frac{1}{2}}$ for values of x from 0 to 10. Draw 11 ordinates and apply Simpson's rule to find the area between the curve and the axis of X and the ordinate $x=10$.

Now apply the formula, area $=\frac{4}{3}x^{\frac{3}{2}}$, where $x=10$, and compare your results.

8. When a body has moved through a distance s feet, the force acting on it is equal to the weight of P lb. The following are corresponding values of P and s :

P	520	504	486	468	448	408	368	360
s	400	480	600	696	760	880	1040	1120

The expression $P \, . \, \Delta s$ gives approximately the work done in foot-pounds as the body increases its displacement by the amount Δs, —the approximation being closer the smaller the interval Δs is.
Find in foot-lb. the work done while the displacement increases from 450 to 1000 ft.

9. When a body starting from rest has been moving for t seconds, the force acting on it is P lb. weight.

P	520	504	486	468	448	408	368	360	360
t	0	8·0	17·4	23·5	27·2	34·0	42·6	47·3	51·8

The expression $\Sigma P \, . \, \Delta t$, calculated for the time t, is approximately proportional to the velocity V of the body at the end of that time.
Compare the velocity after 25 with that after 50 seconds.

10. When a body has moved through x feet the force acting on it is P pounds.

P	358	342	301	259	222	195	156	124	57	0
x	0	11	40	65	78	89	98	106	117	127

Find approximately the work done from $x = 0$ to $x = 50$ and the work done in the whole displacement to $x = 100$.

11. When a body starting from rest has been moving for t seconds, the force acting on it is P pounds.

P	130	122	119	110	105	100	95	88	79	70
t	0	1·5	2·1	4·5	5·7	7·2	8·5	10·0	11·2	12

Compare the velocity after 5 seconds with that after 12 seconds.

12. The length of an indicator diagram is 3·7 inches, the end ordinates are 0·82 and 0·20, and the other ordinates are 0·83, 0·76, 0·65, 0·45, 0·35, 0·32, 0·29, 0·26, 0·22 inch respectively. The scale of pressure is 80 lb. per sq. in. to 1 inch. Find the mean pressure (i) by the mid-ordinate rule, (ii) by Simpson's rule.

Graphic Integration. A graphic method of obtaining the integral of a given function is sometimes useful. To a close degree of approximation, the integral may be assumed to be a continuous curve made up of circular arcs. The method of graphic integration adopted depends on the following property of a circle: the bisector of the external angle between two tangents to a circle is parallel to their chord of contact. Hence, if a circular arc passes through a point P (Fig. 183) and cuts a line AA' in Q, and if the tangents at P and Q are parallel to the lines Y and Z respectively, the point Q may be found by drawing through P a line parallel to the bisector of the angle between Y and Z.

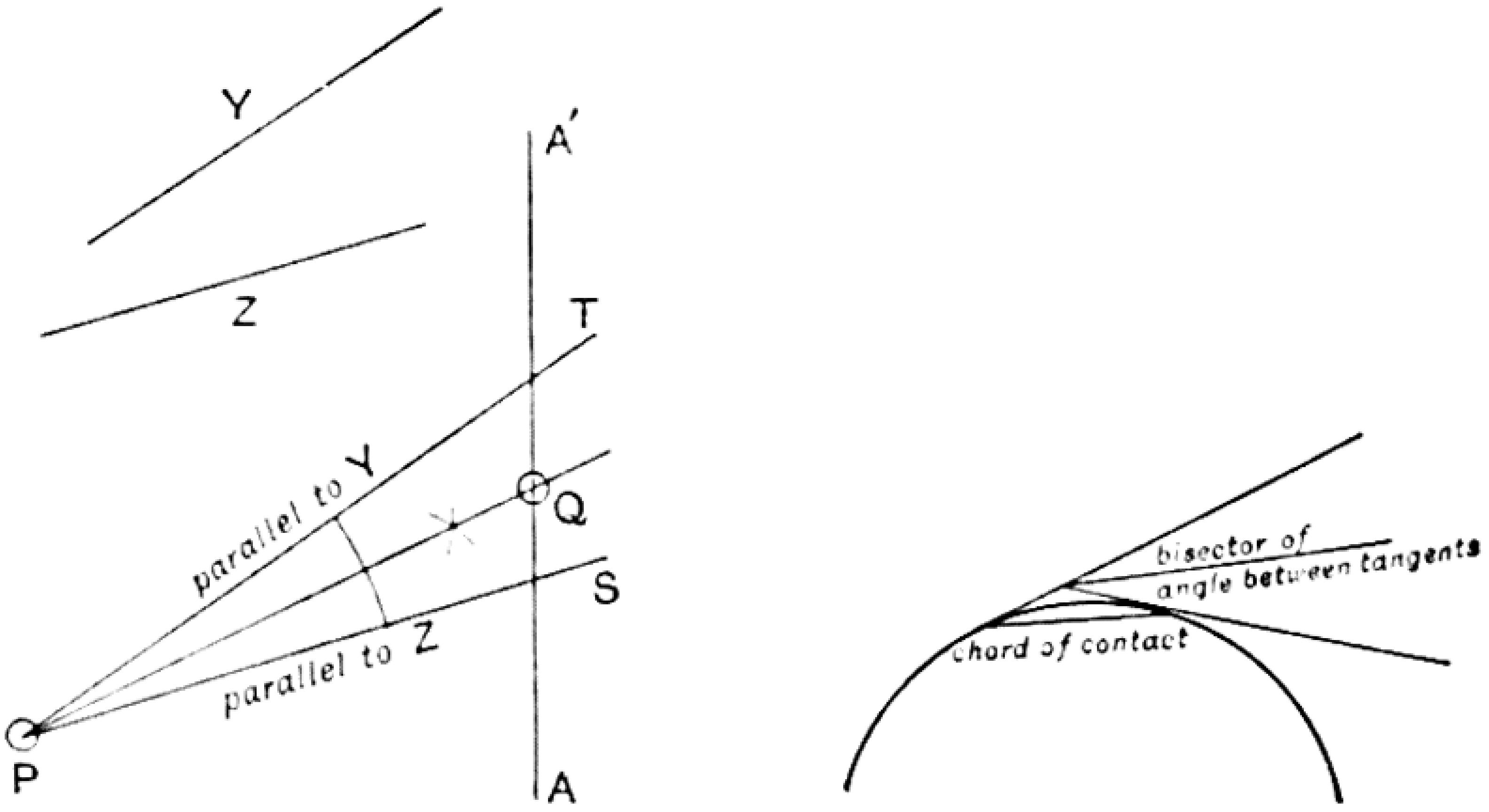


FIG. 183.

Ex. 1. *Construct the integral of the curve PQRS..., it being given that $y=0$ when $x=0$.*

Draw any convenient axis $O'X'$ (Fig. 184) and any convenient ordinate $Apqr...$, on which to measure slopes. O' is the first point on the integral curve, and the slope there is zero; *i.e.* the tangent at O' is $O'X'$. The point P shews that the slope at the corresponding point on the integral is 1·6. Since $OA=5$, take $Ap=5\times1{\cdot}6=8{\cdot}0$. Then $O'p$ gives the direction of the tangent to the integral curve at the point where it cuts the ordinate $x=1$.

Bisect the angle $AO'p$ by $O'P'$, cutting $x=1$ in P'. By the preceding paragraph, P' is a point on the integral.

The point Q shews that the slope at the corresponding point on the integral is 2·55. Take $Aq = 5 \times 2{\cdot}55 = 12{\cdot}75$. Oq gives the direction of the tangent to the integral curve at the point where it cuts the ordinate $x = 2$.

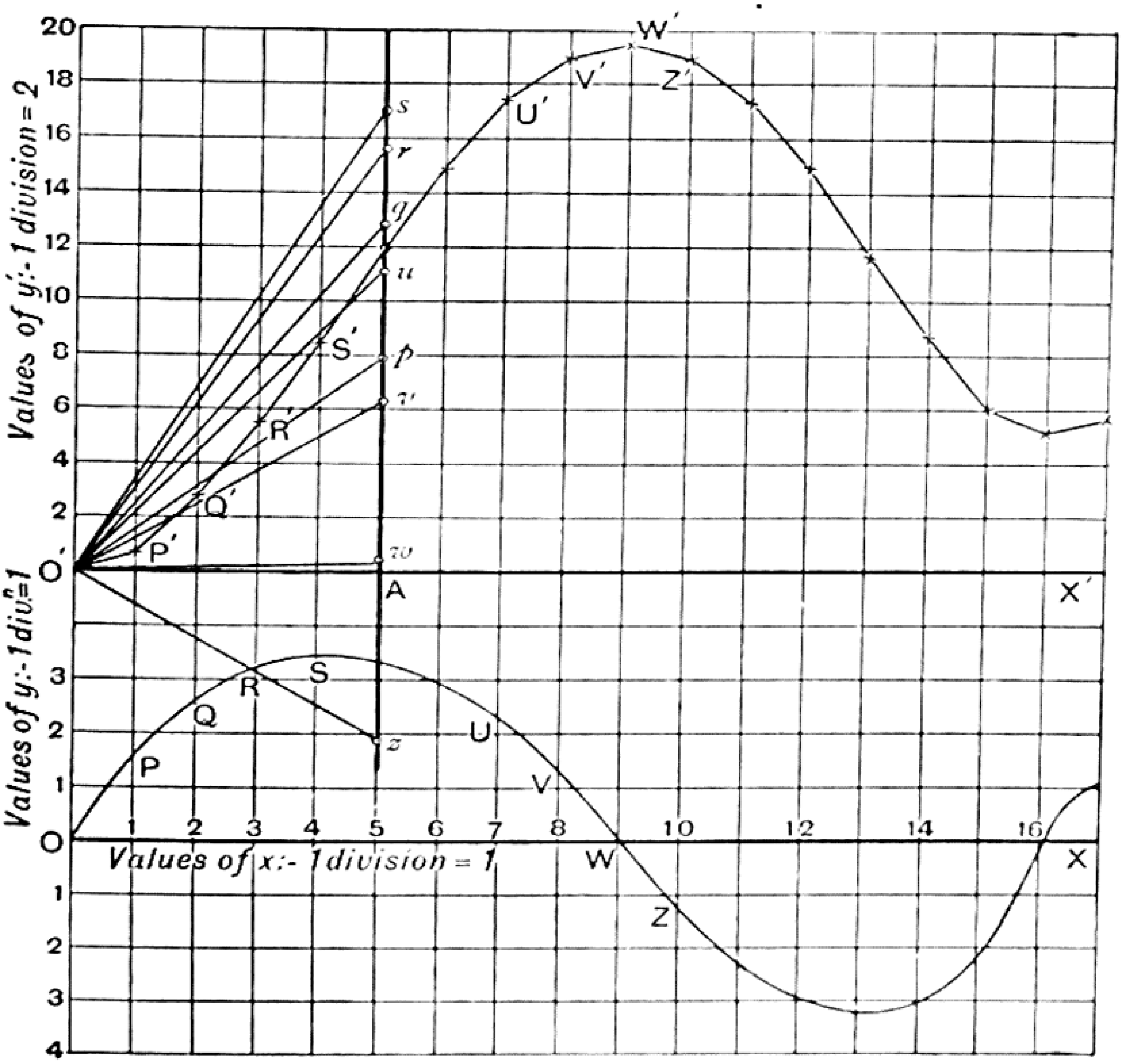


FIG. 184.

Through P' draw $P'Q'$ *parallel to the bisector of the angle* $pO'q$, cutting $x = 2$ in Q'. Q' is another point on the integral. Points R', S', etc., are obtained similarly, and a smooth curve (not shewn in Fig. 184) is drawn.

A check on the accuracy of the work is obtained by seeing that the integral has maximum or minimum ordinates at points corresponding to the intersections of the given curve with the axis of X. Compare the points W' and W.

Integration as a Summation. In Fig. 185, let the equation of the curve be

$$y = f(x).$$

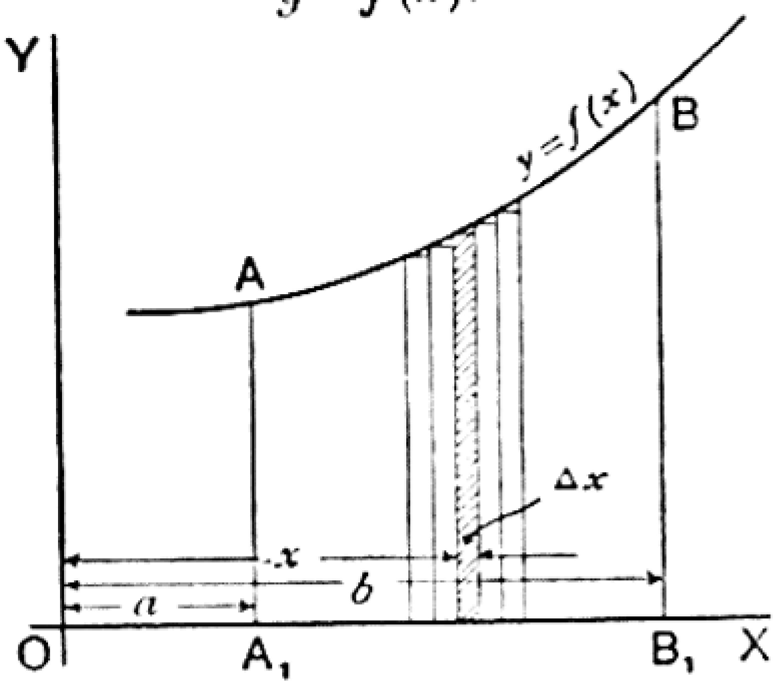


FIG. 185.

The area AA_1B_1B is $\int_a^b y\,dx$.

The area of the shaded rectangle is $y\,\Delta x$.

The sum of a number of rectangles like this extending from the ordinate AA_1 to the ordinate BB_1 is written $\sum_{x=a}^{x=b} y\,\Delta x$. A few of these rectangles are shewn.

$\sum_{x=a}^{x=b} y\,\Delta x$ is read 'Sigma $y\,\Delta x$ from $x=a$ to $x=b$.'

Now, clearly, the smaller Δx is, the more closely does $\sum_{x=a}^{x=b} y\,\Delta x$ approximate to the area given by $\int_a^b y\,dx$.

In the limit we have

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} y\,\Delta x = \int_a^b y\,dx.$$

Hence $\int_a^b y\,dx$ may be taken to mean the limit of the sum, between the ordinates $x=a$ and $x=b$, of areas of which $y\,\Delta x$ is the type.

The two aspects of integration considered above, viz. integration as the inverse of differentiation and integration as the limit of a sum, must be borne in mind.

In the first aspect of integration, given a function $f(x)$, then $\int f(x)dx$ is another function such that its derivative is $f(x)$.

In the second aspect of integration, $\int f(x)\,dx$ is the limit of a sum, which measures an area bounded by the graph of $f(x)$, the axis of x, and two ordinates.

In Fig. 186 the curve $y=f(x)$ is shewn and four integral curves, whose equations are of the form $Y=F(x)+c$, where $F(x)=\int f(x)\,dx$.

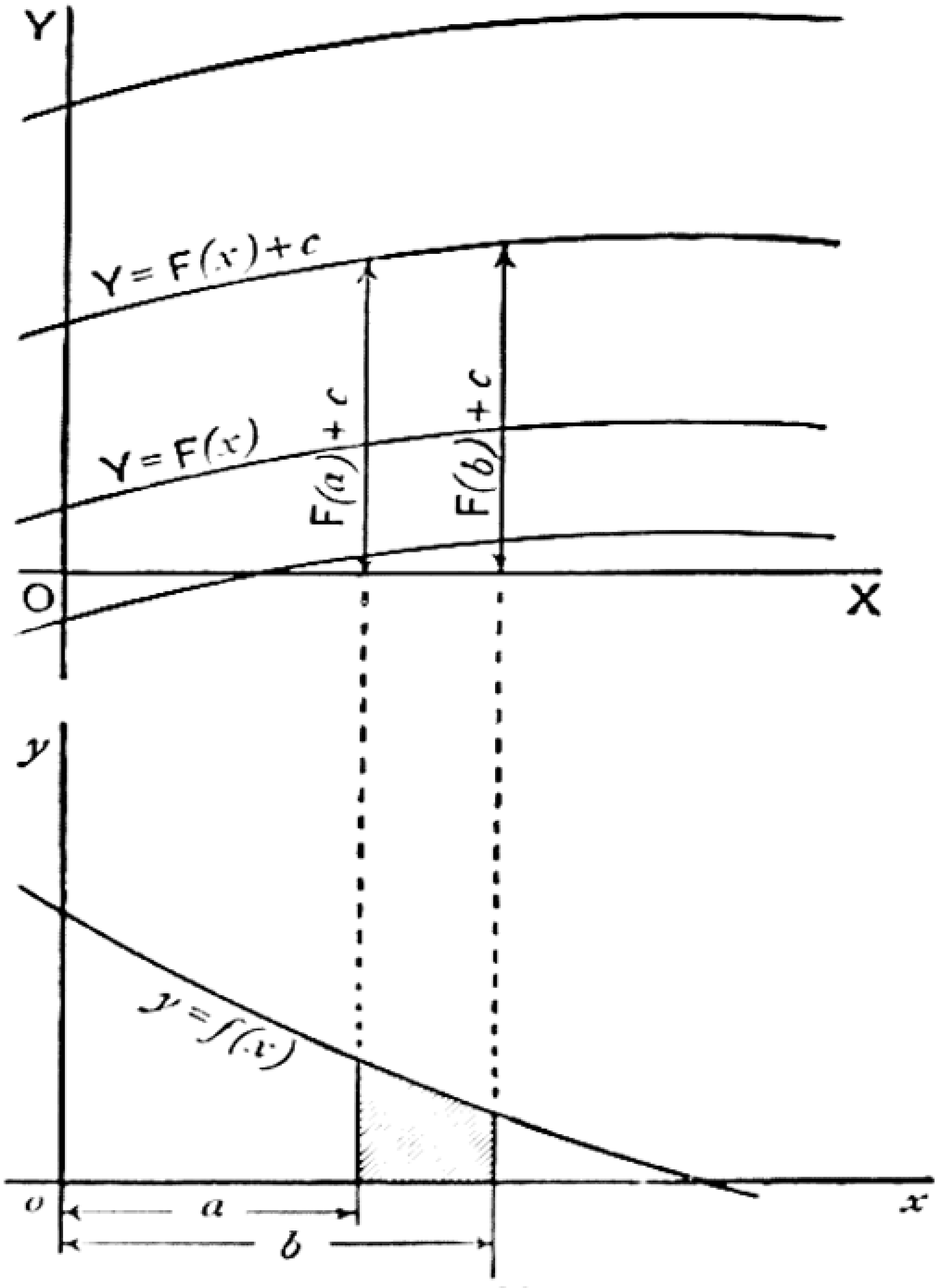


FIG. 186.

From the first aspect of integration the conclusion is that the slope at any point of an integral curve is equal to the corresponding ordinate of the original.

From the second aspect of integration, the conclusion is that the difference $F(b)-F(a)$ between any two ordinates of an integral curve is equal in magnitude to the corresponding area under the original curve

Tabular Integration. The approximate value of a definite integral can be effected arithmetically by calculating $\Sigma y\,\Delta x$.

Ex. 1. *Find approximately* $\int_1^3 x^{1\cdot5}dx$. Let y equal $x^{1\cdot5}$ and y_m be the mean between two consecutive calculated values of y.

x	y	Δx	y_m	$y_m\Delta x$
1·0	1·000	0·2	1·158	0·232
1·2	1·315	0·2	1·486	0·297
1·4	1·657	0·2	1·841	0·368
1·6	2·024	0·2	2·220	0·444
1·8	2·415	0·2	2·622	0·524
2·0	2·828	0·2	3·046	0·609
2·2	3·263	0·2	3·496	0·699
2·4	3·718	0·2	3·956	0·791
2·6	4·193	0·2	4·440	0·888
2·8	4·686	0·2	4·941	0·988
3·0	5·196			5·84 $\doteqdot \int_1^3 x^{1\cdot5}dx$.

Ex. 2. *Find approximately* $\int_{0\cdot5}^{1\cdot5} 0\cdot37e^{1\cdot57x}\,dx$. Proceeding as above

x	$1\cdot57\,x\log e$	y	y_m	Δx	$y_m\Delta x$
0·5	0·3410	0·8114	0·8803	0·1	0·088
0·6	0·4091	0·9491	1·030	0·1	0·103
0·7	0·4773	1·110	1·205	0·1	0·121
0·8	0·5455	1·299	1·410	0·1	0·141
0·9	0·6137	1·520	1·649	0·1	0·165
1·0	0·6819	1·778	1·930	0·1	0·193
1·1	0·7501	2·081	2·258	0·1	0·226
1·2	0·8182	2·434	2·641	0·1	0·264
1·3	0·8864	2·848	3·091	0·1	0·309
1·4	0·9546	3·333	3·616	0·1	0·362
1·5	1·0228	3·899			1·97 $\doteqdot \int_{0\cdot5}^{1\cdot5} 0\cdot37\,e^{1\cdot57x}dx$.

Ex. 3. *The following numbers give v the speed of a train in miles per hour at the time t minutes after leaving a railway station. In each interval of time, what is the distance passed over by the train?*

v	0	0·192	0·736	1·584	2·688	4·000
t	0	0·2	0·4	0·6	0·8	1·0
v	5·472	7·056	8·704	10·37	12·00	
t	1·2	1·4	1·6	1·8	2·0	

At each of the times tabulated, what is x, the distance from the station? Tabulate your answers.

If the speed at the middle of any interval of 0·2 minute is v_m miles per hour, the distance traversed in this interval is approximately $v_m \times 0{\cdot}2/60$ miles. Tabulating, we have

v	t	v_m	Δt	$\frac{v_m \times 0\cdot2}{60}$	x
0	0				
		0·096	0·2	0·0003	
0·192	0·2				0·0003
		0·464	0·2	0·0015	
0·736	0·4				0·0018
		1·160	0·2	0·0039	
1·584	0·6				0·0057
		2·136	0·2	0·0071	
2·688	0·8				0·0128
		3·344	0·2	0·0111	
4·0	1·0				0·0239
		4·736	0·2	0·0158	
5·472	1·2				0·0397
		6·264	0·2	0·0209	
7·056	1·4				0·0606
		7·880	0·2	0·0263	
8·704	1·6				0·0869
		9·537	0·2	0·0318	
10·37	1·8				0·1187
		11·185	0·2	0·0373	
12·0	2·0				0·1560

The last column but one gives displacements in successive intervals. The last column gives the total displacement in miles from the starting point at the end of each interval.

Ex. 4. *The following table connects p, the pressure in lb. per sq. in.*, and v, *the volume of steam in cubic ft., in a cylinder. Find the work done for the whole change of volume.*

p	3·97	3·79	3·65	3·52	3·40	3·28	3·17	3·08	2·99	2·90	2·82
v	3·9	4·1	4·3	4·5	4·7	4·9	5·1	5·3	5·5	5·7	5·9

If p_m is the mean pressure during the small change of volume Δv, the work done during that change of volume is approximately $144 p_m \Delta v$ ft.-lb.

Tabulating, we have

p	p_m	Δv	$p_m \Delta v$
3·97	3·88	0·2	0·776
3·79	3·72		0·744
3·65	3·59		0·718
3·52	3·46		0·692
3·40	3·34		0·668
3·28	3·23		0·646
3·17	3·13		0·626
3·08	3·04		0·608
2·99	2·95		0·590
2·90	2·86		0·572
2·82			6·64

$\therefore$ work done $\doteqdot 144 \times 6{\cdot}64$ ft.-lb. $\doteqdot 956$ ft.-lb.

Further Illustrations. (i) Area of Circle.—*Use the integral notation, and find expressions for the area of a circle of radius a.*

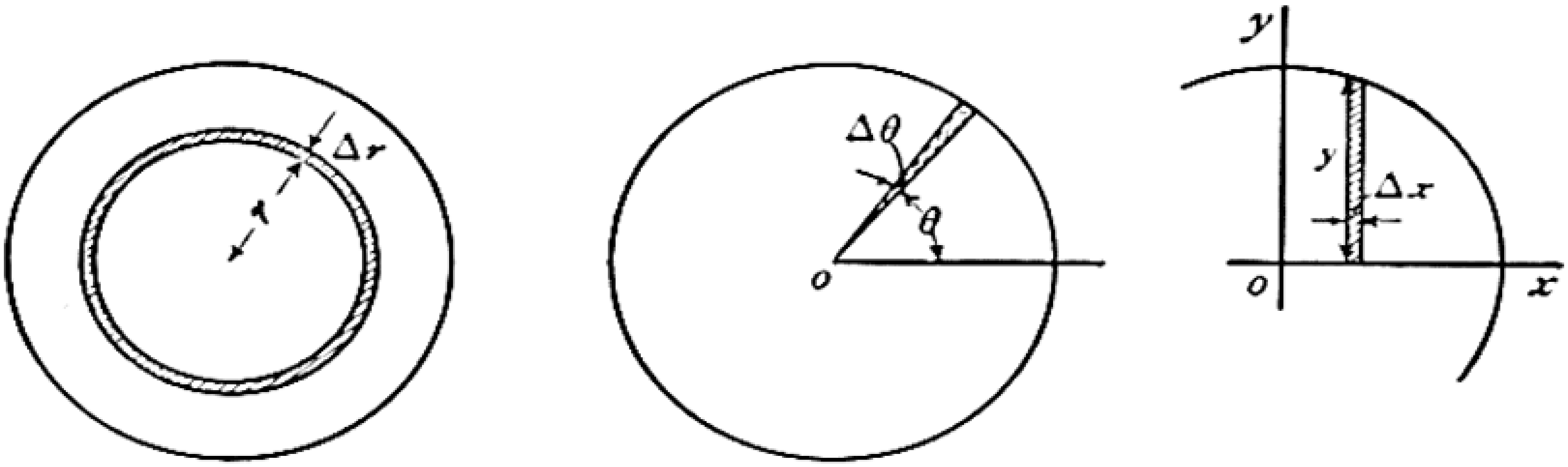
The area may be split up into elementary areas in several ways (Fig. 187).

(a) Imagine the circle to be composed of a series of concentric narrow rings.

The area of the ring shewn $\doteqdot 2\pi r\, \Delta r$;

$\therefore$ area of circle $= \int_0^a 2\pi r\,dr$, since the radius increases from 0 to a.

The area is $\Big[\pi r^2\Big]_0^a$, i.e. πa^2.



(a) (b) (c)

FIG. 187.

(b) Imagine the circle to be composed of a number of small sectors.

The area of the sector shewn $= \frac{1}{2}a^2\Delta\theta$;

$$\therefore \text{ area of circle} = \int_0^{2\pi} \tfrac{1}{2}a^2\,d\theta = \Big[\tfrac{1}{2}a^2\theta\Big]_0^{2\pi} = \pi a^2.$$

(c) Consider a quadrant as composed of a series of very narrow rectangles.

The area of the rectangle shewn $\doteqdot y\Delta x$;

$$\therefore \text{ area of circle} = 4\int_0^a y\,dx.$$

(ii) **Work done in Stretching a Wire.** *Assume that the tension in a wire is proportional to its extension, that the work done in a small extension is measured approximately by the product, tension × extension, and find an expression for the work done in stretching a wire an amount a from its natural length.*

Let x denote the extension when the tension is T, so that $T = Ex$, where E is a constant.

The work done in the small extension Δx is approximately $T\Delta x$, i.e. $Ex\,\Delta x$.

$\therefore$ the work done in stretching a wire from its natural length ($x = 0$) until $x = a$, is

$$\int_0^a Ex\,dx, \quad \text{i.e. } \tfrac{1}{2}Ea^2.$$

(iii) *Interpret the expression* $\int_p^q x\,dy$.

Suppose I. (Fig. 188) is the curve $x = f(y)$.

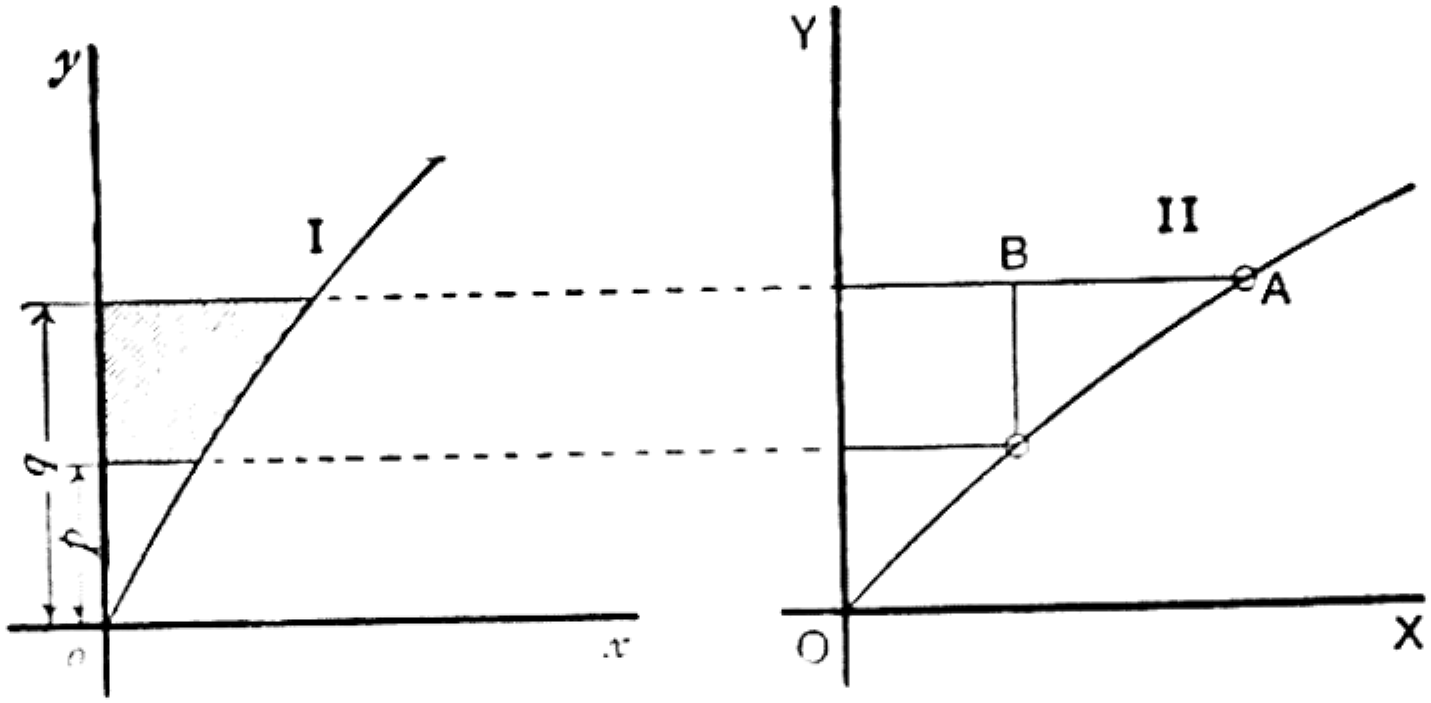


FIG. 188.

Then $\int_p^q x\,dy$ means the shaded area in Fig. 188.

If II. is an integral curve, drawn as explained on p. 404,

$$\int_p^q x\,dy = AB.$$

The position of II. relatively to I. must be noted.

(iv) **Bending Moment and Shearing Force in Beams.** *Find alternative expressions to those found on* pp. 375, 376 *for the connection between* W, *the load,* M, *the bending moment, and the shearing force,* S, *at any point of a beam.*

From illustration (viii), p. 375, $\Delta S = W\Delta x$,

$$\therefore \quad \int dS = \int W\,dx.$$

Therefore, if S_0 is the shear at the origin and S the shear at a point distant x from the origin,

$$S - S_0 = \int_0^x W\,dx, \quad \text{........................(3)}$$

The interpretation of (3) is that the difference of the shears at any two points of a beam loaded in any manner is the total load on the beam between the two points.

Again, $\quad \Delta M = S\,\Delta x. \quad \therefore \int dM = \int S\,dx.$

If M_0 is the bending moment at the origin and M the bending moment at a point distant x from the origin,

$$M - M_0 = \int_0^x S\,dx. \quad \text{.......................}(4)$$

Equations (1) and (2), pp. 375, 376, and (3) and (4) above, may be illustrated graphically. Thus, consider the case of a beam loaded as in Fig. 189.

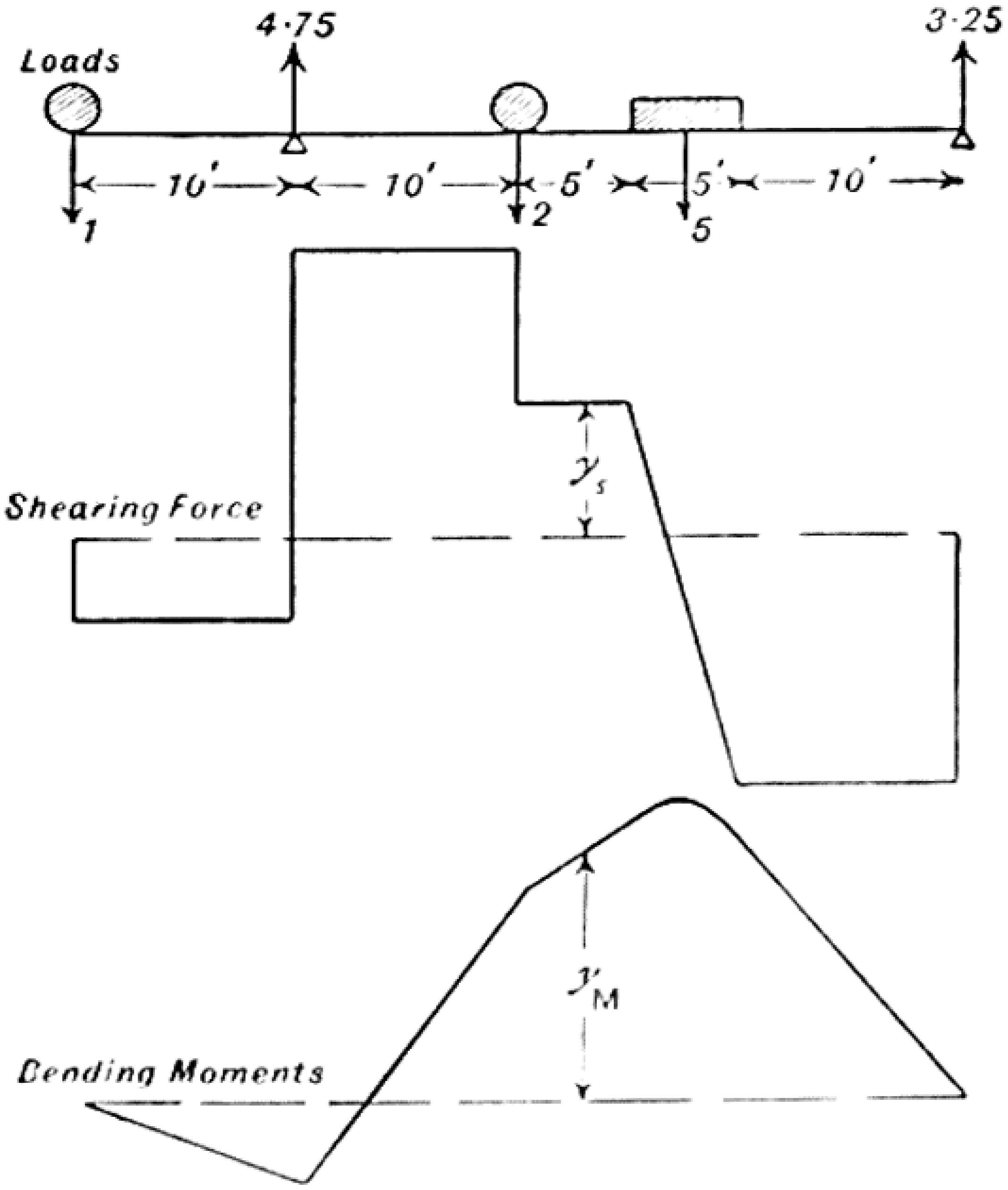


FIG. 189.

The reactions at the supports are found to be 4·75 and 3·25 tons.

From the interpretation of equation (3), the method of drawing the S.F. diagram should be clear, any ordinate y_s being the algebraic sum of the loads to the left of that ordinate.

From (4), in the same way, y_M is $\int_0^x y_s\,dx$, *i.e.* the algebraic sum of the areas of parts of the S.F. diagram to the left of y_s.

Note that the part of the B.M. diagram corresponding to uniform loading is parabolic.

(v) **Surface of Cone and Frustum.** *Find an expression for the surface of a right circular cone, slant side L, base radius r. Deduce a formula for the surface of a frustum of a cone, radius of top r_1, radius of bottom r, slant side L_1.*

The surface of the cone (Fig. 190) is the sum of the areas of all the elementary triangles bounded by two near generators, like VPQ.

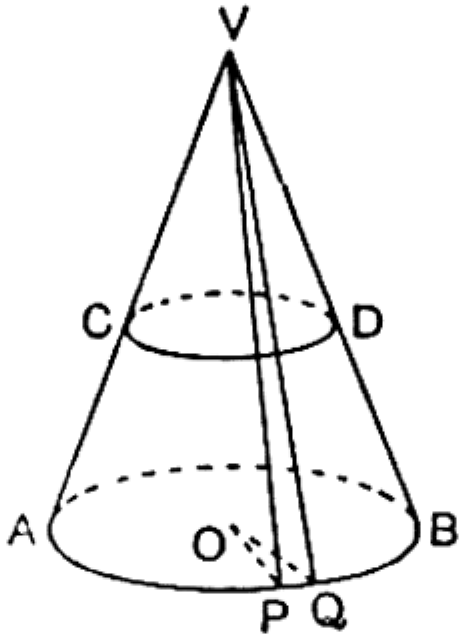


FIG. 190.

If $\qquad PQ = \Delta s$, area $VPQ \doteqdot \frac{1}{2} L \, \Delta s$.

$\therefore$ surface of cone $\doteqdot \Sigma \frac{1}{2} L \, \Delta s$, the summation extending round the whole circumference of the base.

Now the value of $\Sigma \Delta s = 2\pi r$;

$$\therefore \text{ surface} = \tfrac{1}{2} L (2\pi r) = \pi r L.$$

Also the surface of the cone

$$VCD = \pi r_1 \,.\, VD,$$

and the surface of the cone

$$VAB = \pi r \,.\, VB;$$

$\therefore$ the surface of the frustum $= \pi (r \,.\, VB - r_1 \,.\, VD)$.

By similar triangles,

$$\frac{VD}{r_1} = \frac{VB}{r} = \frac{L_1}{r - r_1};$$

$$\therefore \text{ the surface of the frustum} = \frac{\pi L_1}{r - r_1} (r^2 - r_1{}^2) = \pi L_1 (r + r_1).$$

Hence, surface of frustum = slant side × mean of circumferences of top and bottom.

(vi) **Surface of Revolution.** *Find an expression for the area of the surface generated by the revolution of the curve PQ about an axis OY in its plane* (Fig. 191).

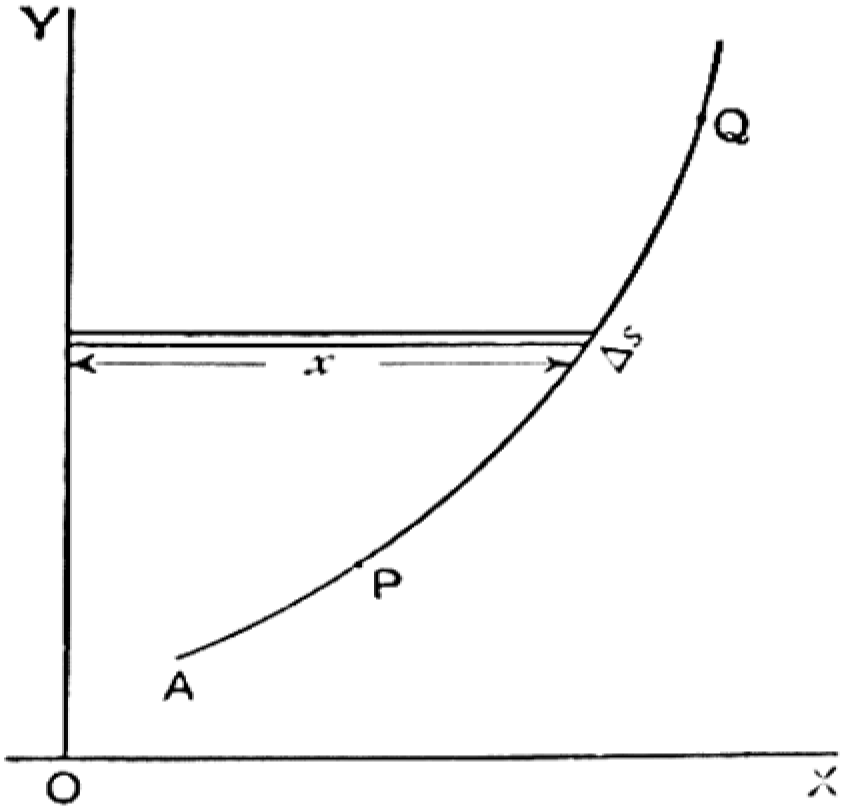


FIG. 191.

Let A be any arbitrary point on the curve, and let the arc $AP = s_1$ and the arc $AQ = s_2$.

The ring generated by the revolution of the element Δs is approximately the frustum of a cone of slant side Δs, the radii of top and bottom being $x + \Delta x$ and x respectively.

By illustration (v),

$$\text{the area of this ring} \doteqdot 2\pi \, \Delta s \, (x + \tfrac{1}{2}\Delta x) \doteqdot 2\pi x \, \Delta s$$

(neglecting $\Delta x \,.\, \Delta s$ in comparison with Δs).

$$\text{Hence the required area} = \int_{s_1}^{s_2} 2\pi x \, ds.$$

This can usually be evaluated if x and s can be expressed in terms of the same variable.

(vii) **Volume of Solid of Revolution.** *Find an expression for the volume of the solid generated by the revolution of the curve PQ about the axis OY* (Fig. 191).

The thin disc generated by the revolution about OY of the two lines of length x and $x + \Delta x$, shewn in the figure, is practically a cylinder of radius x and thickness Δy.

The volume of this disc $\doteqdot \pi x^2 \Delta y$;

$\therefore$ the volume of the solid generated by PQ is

$$\int_{y_1}^{y_2} \pi x^2 dy,$$

where y_1, y_2 are the ordinates of P, Q respectively.

Similarly, the volume generated by the revolution of PQ about OX is

$$\int_{x_1}^{x_2} \pi y^2 dx,$$

where x_1, x_2 are the abscissae of P, Q respectively.

These expressions for the volume can usually be evaluated when one variable has been expressed in terms of the other.

Ex. 1. *Find the volume of the solid of revolution generated by the rotation about the axis of X of that portion of the parabola* $y^2 = 4ax$ *cut off by the ordinate* $x = h$.

In the rotation, the ordinate PQ generates a circle of area πy^2 (Fig. 192).

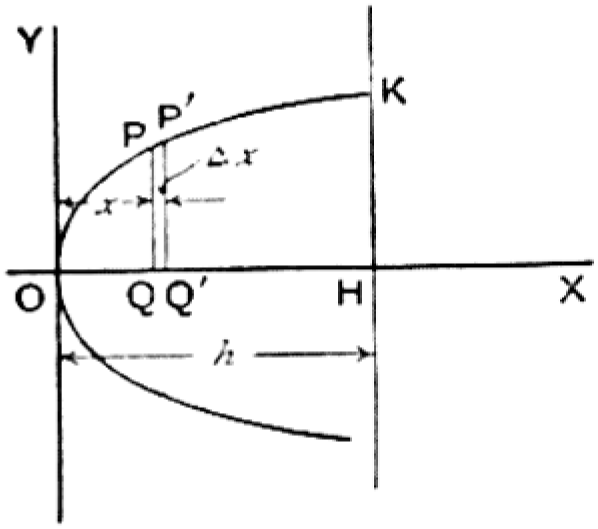


FIG. 192.

The volume of the elementary slice generated by $PQQ'P'$ is

$$\pi y^2 \Delta x, \text{ approximately.}$$

The volume required is

$$\int_0^h \pi y^2 dx, \qquad i.e. \int_0^h 4\pi ax\, dx, \qquad i.e. \left[2\pi ax^2\right]_0^h.$$

Therefore, the volume is $2\pi ah^2$.

Ex. 2. *Find the moment of inertia of a uniform rod of length* $2a$ *about an axis through one end perpendicular to its length.*

The moment of inertia of a body about an axis is Σmr^2, where m is the mass of an element of the body distant r

from the axis, and the summation extends over the whole body.

Let OA be the rod and let the distances from O of two near points P, Q, on it be x and $x+\Delta x$ (Fig. 193).

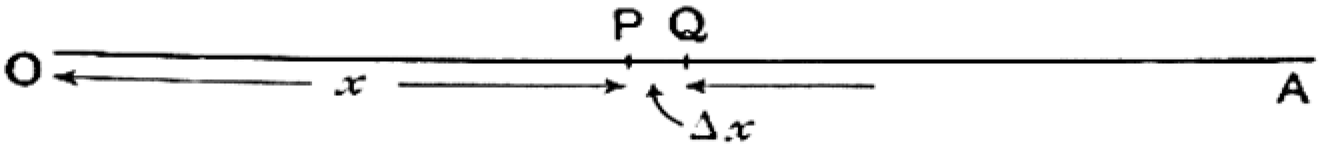


FIG. 193.

If m is the mass per unit length of the rod, the mass of the element PQ is $m\,\Delta x$.

The moment of inertia of the rod about O is approximately $\Sigma mx^2\Delta x$, the summation extending over the whole rod.

The required moment of inertia is

$$\int_0^{2a} mx^2dx, \quad i.e. \left[\frac{mx^3}{3}\right]_0^{2a}, \quad i.e. \frac{8ma^3}{3},$$

Therefore the moment of inertia is $4Ma^2/3$, where $M=2ma$, the mass of the whole rod.

Centre of Gravity. If m is the mass of an element of a body and x its distance from a fixed line or plane, then $\bar{x}$, the distance of the centre of gravity of the body from that fixed line or plane, is given by the formula

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},$$

where the summation extends over the whole body.

Ex. 3. *Find the coordinates of the centre of gravity of the portion OKH of the parabola shewn in Fig.* 192.

If m is the mass per unit of area, the mass of the element $PP'Q'Q$ is $my\,\Delta x$, and may be regarded as concentrated at the point whose coordinates are $(x, \frac{1}{2}y)$.

The centre of gravity of the area OKH coincides with the centre of gravity of the series of masses of which $my\,\Delta x$, situated at the point $(x, \frac{1}{2}y)$, is typical.

If $(\bar{x}, \bar{y})$ is the centre of gravity of the area OKH, then, by the above formula,

$$\bar{x}=\Sigma my\,\Delta x\,.\,x/\Sigma my\,\Delta x \quad \text{and} \quad \bar{y}=\Sigma my\,\Delta x\,.\,\tfrac{1}{2}y/\Sigma my\,\Delta x,$$

where the summation extends over the whole area.

Using the integral notation, we have

$$\bar{x} = \int_0^h mxy\,dx \Big/ \int_0^h my\,dx \quad \text{and} \quad \bar{y} = \int_0^h \tfrac{1}{2}my^2\,dx \Big/ \int_0^h my\,dx.$$

Since $y^2 = 4ax$,

$$\int_0^h my\,dx = \int_0^h 2ma^{\frac{1}{2}}x^{\frac{1}{2}}\,dx = \tfrac{4}{3}ma^{\frac{1}{2}}h^{\frac{3}{2}},$$

$$\int_0^h mxy\,dx = \int_0^h 2ma^{\frac{1}{2}}x^{\frac{3}{2}}\,dx = \tfrac{4}{5}ma^{\frac{1}{2}}h^{\frac{5}{2}},$$

and

$$\int_0^h \tfrac{1}{2}my^2\,dx = \int_0^h 2max\,dx = mah^2;$$

$$\therefore \quad \bar{x} = \tfrac{4}{5}ma^{\frac{1}{2}}h^{\frac{5}{2}} \big/ \tfrac{4}{3}ma^{\frac{1}{2}}h^{\frac{3}{2}} = \frac{3h}{5},$$

$$\bar{y} = mah^2 \big/ \tfrac{4}{3}ma^{\frac{1}{2}}h^{\frac{3}{2}} = \tfrac{3}{4}a^{\frac{1}{2}}h^{\frac{1}{2}}.$$

Ex. 4. *Shew that the area of the zone of a sphere of radius a, which is bounded by two parallel planes distant h apart, is $2\pi ah$.*

In Fig. 194, HH', KK' are the traces of the given planes. The problem is to find the area of the surface generated by the revolution of the arc AB about OX.

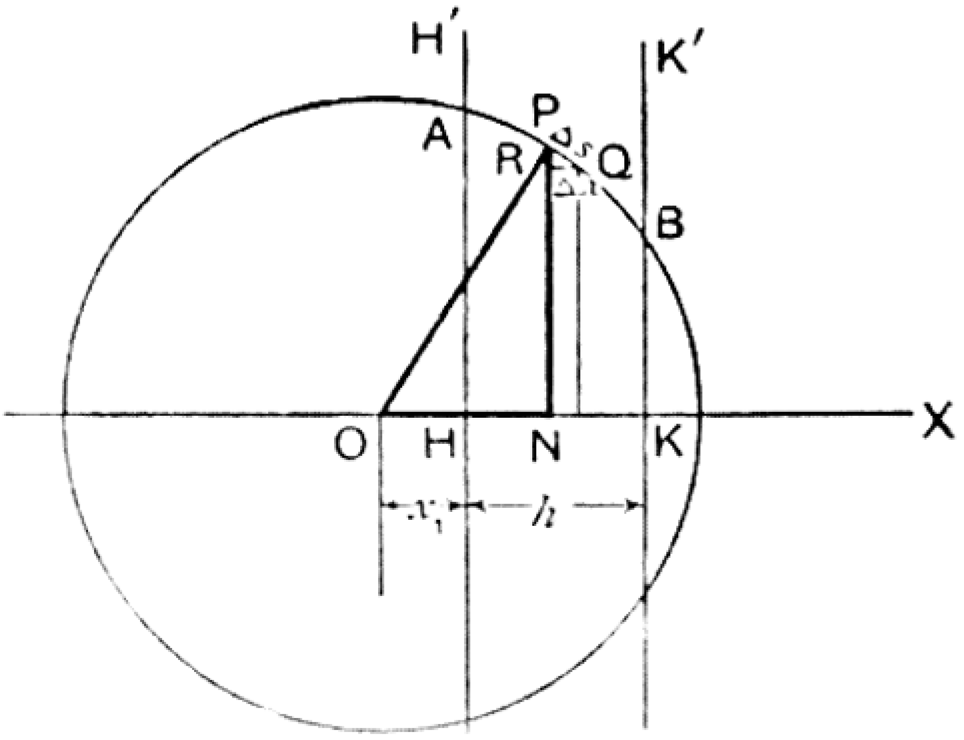


FIG. 194.

Let P, Q be neighbouring points in the arc AB. Draw PN perpendicular to OX and QR perpendicular to PN.

Using the notation indicated in the figure, we may regard the arc PQ as approximately a straight line of length Δs, so that OP may be taken as perpendicular to PQ.

Further, PN is perpendicular to QR and ON to PR.

Therefore the triangles OPN and PQR are practically similar;

$$\therefore \frac{PN}{OP} \doteqdot \frac{QR}{PQ},$$

$$i.e. \quad \frac{y}{a} \doteqdot \frac{\Delta x}{\Delta s}, \qquad \text{or} \quad y\,\Delta s \doteqdot a\,\Delta x.$$

Now, by illustration (vi), p. 413, the surface of the zone is

$$\Sigma 2\pi y\,\Delta s, \quad i.e.\ \Sigma 2\pi a\,\Delta x.$$

With the usual notation, the surface is

$$\int_{x_1}^{x_1+h} 2\pi a\,dx, \qquad i.e. \left[2\pi a x\right]_{x_1}^{x_1+h}, \qquad i.e.\ 2\pi a h.$$

EXERCISES LI.

1. Draw an integral curve of each of the following :

(i) $y=1{\cdot}5$; (ii) $y=0{\cdot}73$; (iii) $y=a$;

(iv) $y=x+1{\cdot}3$; (v) $y=0{\cdot}5x$.

What can be inferred from the results obtained in cases (i), (ii) and (iii) ?

2. Write down an expression for the area between the curve $y=ax^{0\cdot 5}$ and the axis of x from the origin to the ordinate $x=b$.

3. Write down an expression for the area between the curve $y=ax^{0\cdot 5}$ and the axis of y from the origin to the line $y=c$.

4. Assume that the area of the surface of a sphere of radius r is $4\pi r^2$, and write down an expression for the volume of a sphere of radius a.

5. Draw an integral curve of the cosine curve. What may be inferred ? Test your inference by evaluating $\Sigma \cos\theta\,\Delta\theta$ from $\theta=0$ to $\theta=\frac{1}{2}\pi$.

6. What does any integral of an acceleration-time curve represent ?

7. Find $\int_1^2 \frac{1}{x}\,dx$ by the tabular method, taking successive differences of 0·1 in x. Compare your answer with the value of $\log_e 2$. What inference appears probable from this comparison ? Test this inference by evaluating $\int_1^{3\cdot 5} x^{-1}dx$ and comparing with $\log_e 3{\cdot}5$.

Use the tabular method to work Examples 8-18.

8. Find $\int_{0\cdot5}^{2\cdot8} 2\cdot4x^2dx$.

9. Find $\int_{0}^{3} (0\cdot075x^2+2)dx$.

10. Find $\int_{-1}^{1} 0\cdot5e^{x}dx$.

11. Find $\int_{-0\cdot5}^{1} e^{1\cdot4x}dx$.

12. The following values of y and x being given, tabulate $\Delta y/\Delta x$ and $y\,\Delta x$ in each interval, and A or the sum of such terms as $y\,\Delta x$. Of course A is the approximate area of the curve whose ordinate is y.

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
y	0	0·1736	0·3420	0·5000	0·6428	0·7660	0·8660	0·9397	0·9848	1·0000

[B.E.]

13. By tabulation, give approximately a table of values of $A=\int y\,dx$, if the following values of x and y are given. Let A be 0 when x is 0. Plot y and x on squared paper.

x	0	0·1	0·2	0·3	0·4	0·5	0·6
y	1·5663	1·6774	1·8002	1·9391	2·1000	2·2918	2·5281

Draw a curve shewing A and x. [B.E.]

14. The following values of y and x being given, tabulate $\Delta y/\Delta x$ and ΔA in each interval, ΔA being the area in the interval between two ordinates. Tabulate the values of A if $A=0$ when $x=3$.

x	3	4	5	6	7	8	9	10	11	12	13
y	1·75	10·45	19·08	27·56	35·84	43·84	51·50	58·87	65·61	71·93	77·71

[B.E.]

15. Draw the curve $y=10\sqrt{x}$ from $x=1$ to $x=9$. The curve rotates about the axis of x, generating a surface of revolution. Imagine the solid divided up into a sufficient number of thin slices by planes at right angles to x: add up the volumes of these slices from $x=1$ to $x=9$. [B.E.]

16. The following numbers give v the speed of a train in miles per hour at the time t hours since leaving a railway station. In each interval of time, what is the distance passed over by the train?

v	0	2·4	4·7	7·2	9·6	12·0	14·3	16·9	18·9	20·7	22·2	23·4	24·3	24·9
t	0·00	0·04	0·08	0·12	0·16	0·20	0·24	0·28	0·32	0·36	0·40	0·44	0·48	0·52

At each of the times tabulated, what is x, the distance from the station? Tabulate your answers. [B.E.]

17. The following values of x and y being given, tabulate $\Delta y/\Delta x$ in each interval, also $\Delta A = y\Delta x$ and $A=\int y\,dx$. Shew in curves how the value of y, $\frac{dy}{dx}$ and A depend on x.

x	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
y	6·428	7·071	7·660	8·192	8·660	9·063	9·397	9·659	9·848

18. A motor car weighing 8·5 cwt. starts from rest, and the effective pull p of its engine in lb. weight after moving for t seconds is as follows:

p	14	27	44	63	68	61	52·5	49
t	0	2	4	6	8	10	12	14

Assume that $\int_0^t p\,dt = mv$, where m is the mass of the car in engineers' units, *i.e.* $\frac{112\times 8\cdot 5}{32\cdot 2}$, and v ft. per sec. is the speed at the end of t seconds. Find the velocity after 14 seconds. [B.E.]

19. A cone is generated by the rotation about the axis of X of the portion of the line $y=mx$ between the origin and the ordinate $x=h$. Find its volume, and from your result deduce the formula

$$V=\tfrac{1}{3}\pi r^2 h.$$

20. A sphere is generated by the rotation about the axis of X of the circle $x^2+y^2=a^2$. Find its volume.

21. The curve $y=a+bx+cx^2$ rotates about the axis of X. Find the volume of the frustum cut off by the planes defined by $x=h_1$ and $x=h_2$.

22. The portion of the parabola $y^2=4ax$ between the origin and the ordinate at $x=h$ rotates about the axis of Y. Find the volume of the spindle generated.

23. The hyperbola $xy=c^2$ rotates about (i) the axis of X, (ii) the axis of Y. Find the volume of the solid generated by the arc between the points $(ch, c/h)$ and $(ck, c/k)$.

24. A weight of 10 lb. when attached to an elastic wire is found to stretch it 0·1 inch. Find the work done in stretching this wire 0·5 inch beyond its natural length. (See p. 409.)

25. Find the moment of inertia of a uniform rod of mass M and length $2l$ about an axis through its middle point perpendicular to its length.

26. Find the moment of inertia of a circular disc of mass M and radius a about an axis through its centre perpendicular to its plane. (See Fig. 187 (a), p. 409.)

27. Find the moment of inertia of a sphere of mass M and radius a about a diameter.

28. The density of a rod of length l at a point varies as the distance of that point from one end. Find the distance of the centre of gravity of the rod from the end where the density is greatest.

29. Shew that the centre of gravity of a uniform solid cone is at a point on its axis which divides it in the ratio of 3 : 1.

30. Find the position of the centre of gravity of the paraboloid of revolution defined in Ex. 1, p. 414.

31. Shew that the centre of gravity of a uniform solid hemisphere of radius a is at a distance of $\tfrac{3}{8}a$ from the plane base.

CHAPTER XIV.

DIFFERENTIATION AND INTEGRATION. GENERAL FORMULAE AND FURTHER APPLICATIONS.

Differentiation of a Sum or Difference. If y, u, v are functions of x, and $y = u + v$, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

The values of the functions corresponding to the value $x + \Delta x$ of the variable x are

$$y + \Delta y, \quad u + \Delta u, \quad v + \Delta v;$$

$$\therefore \; y + \Delta y = (u + \Delta u) + (v + \Delta v) = y + \Delta u + \Delta v;$$

$$\therefore \; \Delta y = \Delta u + \Delta v;$$

$$\therefore \; \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now let $\Delta x \to 0$; then proceeding to the limit, (1) becomes

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Similarly, if $y = u - v$, then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

Differentiation of a Product. If y, u, v are functions of x, and $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The values of the functions corresponding to the value $x + \Delta x$ of the variable x are

$$y + \Delta y, \quad u + \Delta u, \quad v + \Delta v;$$

$$\therefore\ y+\Delta y=(u+\Delta u)(v+\Delta v)=uv+u\,\Delta v+v\,\Delta u+\Delta u\,.\,\Delta v;$$

$$\therefore\ \Delta y=u\,\Delta v+v\,\Delta u+\Delta u\,.\,\Delta v;$$

$$\therefore\ \frac{\Delta y}{\Delta x}=u\frac{\Delta v}{\Delta x}+v\frac{\Delta u}{\Delta x}+\Delta u\,.\,\frac{\Delta v}{\Delta x}. \quad \text{..............(1)}$$

Now, as $\Delta x \to 0$, Δy, Δu and Δv all $\to 0$.

Further, $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, $\frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$ and $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$;

$\therefore$ proceeding to the limit, (1) becomes

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}. \quad \text{.........................I.}$$

N.B.—If $y=av$, where a is a constant, then

$$\frac{dy}{dx}=a\frac{dv}{dx}, \quad \text{since } \frac{da}{dx}=0.$$

Differentiation of a Quotient. If y, u, v are functions of x and $y=u/v$, then

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

We have $$y+\Delta y=\frac{u+\Delta u}{v+\Delta v};$$

$$\therefore\ \Delta y=\frac{u+\Delta u}{v+\Delta v}-\frac{u}{v}=\frac{v\,\Delta u-u\,\Delta v}{v(v+\Delta v)};$$

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{v\frac{\Delta u}{\Delta x}-u\frac{\Delta v}{\Delta x}}{v(v+\Delta v)};$$

$\therefore$ proceeding to the limit, $$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}. \quad \text{.................II.}$$

Important Formula. If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Corresponding to the value $x+\Delta x$ of the variable x, u has the value $u+\Delta u$ and y has the value $y+\Delta y$.

Clearly $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}$, since Δu may be cancelled ;

$\therefore$ proceeding to the limit, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. ....III.

In the same way, $\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = 1$;

$\therefore$ proceeding to the limit, $\frac{dy}{dx} \times \frac{dx}{dy} = 1$;

$$\therefore \frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}. \text{ ....IV.}$$

These results can be illustrated graphically. The interpretation of the last formula is that if PT is a tangent to the curve $y = f(x)$, then the tangent of the angle between PT and OY is the reciprocal of the tangent of the angle between PT and OX.

Differentiation of x^n.

(*a*) n a positive integer.

Let $y = x^n$, then $y + \Delta y = (x + \Delta x)^n$;

$$\therefore \Delta y = (x + \Delta x)^n - x^n.$$

Using the binomial theorem,

$$\Delta y = nx^{n-1} . \Delta x + {}_nC_2 x^{n-2} (\Delta x)^2 + \ldots + (\Delta x)^n ;$$

$$\therefore \frac{\Delta y}{\Delta x} = nx^{n-1} + \Delta x \{ {}_nC_2 x^{n-2} + \ldots + (\Delta x)^{n-2} \} ;$$

$\therefore$ proceeding to the limit, $\frac{dy}{dx} = nx^{n-1}$.

(*b*) n a positive fraction equal to p/q.

Let $y = x^{p/q}$, then $y^q = x^p$.

Let $z = y^q$. ....(1)

Then $z = x^p$. ....(2)

From (1) and (*a*), $\frac{dz}{dy} = qy^{q-1}$.

From (2), and (*a*), $\frac{dz}{dx} = px^{p-1}$.

But $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}, \quad \text{using III.};$$

∴ substituting, $$\frac{dy}{dx} = \frac{1}{qy^{q-1}} \cdot (px^{p-1}), \quad \text{using IV.},$$

i.e. $$\frac{dy}{dx} = \frac{p}{q} \frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{x^{p-\frac{p}{q}}} = \frac{p}{q} x^{\frac{p}{q}-1} :$$

$$\therefore \frac{dy}{dx} = nx^{n-1}.$$

(*c*) *n* a negative number, say $-m$, where m is positive.

Let $y = x^{-m}$; ∴ $y = \dfrac{1}{x^m}$;

$$\therefore \text{ using II., } \frac{dy}{dx} = \frac{0 - mx^{m-1}}{x^{2m}} = -mx^{-m-1};$$

$$\therefore \frac{dy}{dx} = nx^{n-1}.$$

Ex. 1. *Differentiate* $(ax+b)^n$.

Let $y = (ax+b)^n$, and let $u = ax + b$.

Then $$y = u^n.$$

Now $\dfrac{du}{dx} = a$, since $u = ax + b$; and $\dfrac{dy}{du} = nu^{n-1}$, since $y = u^n$.

But $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx};$$

$$\therefore \frac{dy}{dx} = nu^{n-1} \cdot a = an(ax+b)^{n-1}.$$

Ex. 2. *Differentiate* $\sqrt{x^2+1}$.

Let $y = \sqrt{x^2+1}$, and let $z = x^2 + 1$.

Then $$y = \sqrt{z}.$$

Now $\dfrac{dz}{dx} = 2x$, since $z = x^2 + 1$; and $\dfrac{dy}{dz} = \frac{1}{2}z^{-\frac{1}{2}}$ since $y = \sqrt{z}$.

But $$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx};$$

$$\therefore\ \frac{dy}{dx}=\tfrac{1}{2}z^{-\frac{1}{2}}\,.\,2x=x/\sqrt{x^2+1}.$$

It is often useful to write $\dfrac{dy}{dx}$ as Dy.

Thus, Formula I. p. 422, can be written $D(uv)=u\,Dv+v\,Du$.

,, ,, II. p. 422, ,, ,, $D\dfrac{u}{v}=\dfrac{v\,Du-u\,Dv}{v^2}$.

Again, $$Dx^n=nx^{n-1}.$$

Ex. 3. *Differentiate* $\dfrac{x}{\sqrt{ax^2+bx+c}}$.

Let $y=\dfrac{x}{\sqrt{ax^2+bx+c}}$; then, by Formula II., we have

$$Dy=\frac{\sqrt{ax^2+bx+c}\,.\,Dx-x\,D\sqrt{ax^2+bx+c}}{ax^2+bx+c}. \quad \text{.........(1)}$$

To find $D\sqrt{ax^2+bx+c}$, write $z=ax^2+bx+c$.

Then $$\frac{dz}{dx}=2ax+b.$$

Now $D\sqrt{ax^2+bx+c}=D(z^{\frac{1}{2}})=\dfrac{d(z^{\frac{1}{2}})}{dx}=\dfrac{d(z^{\frac{1}{2}})}{dz}\cdot\dfrac{dz}{dx}$.

Therefore, $D\sqrt{ax^2+bx+c}=\tfrac{1}{2}z^{-\frac{1}{2}}(2ax+b)=\dfrac{2ax+b}{2\sqrt{ax^2+bx+c}}$;

Using (1),

$$Dy=\frac{(\sqrt{ax^2+bx+c})\times 1-x\left(\dfrac{2ax+b}{2\sqrt{ax^2+bx+c}}\right)}{ax^2+bx+c};$$

$$\therefore\ Dy=\frac{2(ax^2+bx+c)-(2ax^2+bx)}{2(ax^2+bx+c)^{\frac{3}{2}}}$$

$$=\frac{bx+2c}{2(ax^2+bx+c)^{\frac{3}{2}}}.$$

Ex. 4. *Water is being poured at the rate of u cubic feet per second into an inverted conical vessel, the height of which is h ft. and the radius of whose base is r ft. Find the rate at which the water is rising in the vessel when its depth is* $\frac{1}{2}h$ *ft.*

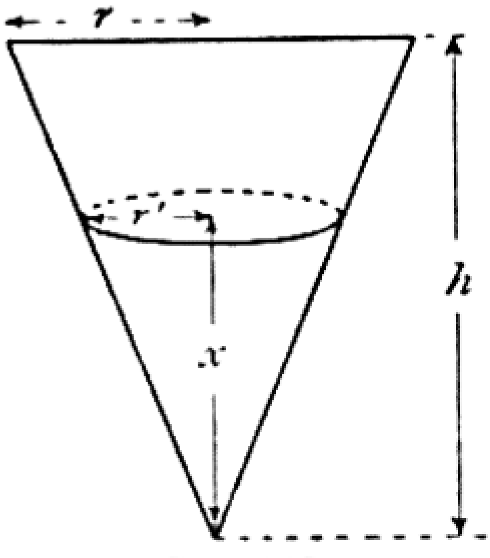


FIG. 195.

The volume of the cone is $\frac{1}{3}\pi r^2 h$ (Fig. 195).

When the depth of water is x ft., the water occupies a cone the radius of whose base is r', where $r'/r = x/h$.

The volume of water in the cone is then V cubic feet, where

$$V = \tfrac{1}{3}\pi r'^2 x = \tfrac{1}{3}\pi \frac{r^2}{h^2} x^3;$$

$$\therefore \frac{dV}{dt} = \tfrac{1}{3}\pi \frac{r^2}{h^2}(3x^2) \cdot \frac{dx}{dt}.$$

Now $\frac{dV}{dt}$ is the rate of increase of volume with time, *i.e.* the rate at which the water is being poured in, *i.e.* $\frac{dV}{dt} = u$;

$$\therefore \quad u = \frac{\pi r^2 x^2}{h^2}\frac{dx}{dt}, \quad \text{or} \quad \frac{dx}{dt} = \frac{uh^2}{\pi r^2 x^2}.$$

The rate at which the water is rising when the depth is x ft. is therefore $uh^2/\pi r^2 x^2$ ft. per sec.

When $x = \frac{1}{2}h$, the water is rising at a rate of $4u/\pi r^2$ ft. per sec.

Ex. 5. *Shew that the circle,* $x^2 + y^2 - x - y = 0$, *and the parabola,* $4y = x^2 - 4x$, *touch at the origin. Find the common tangent.*

The equation $x^2 + y^2 - x - y = 0$ means that x and y vary in such a way that the expression on the left is always zero. Therefore the rate of variation of this expression is also zero;

$$i.e. \qquad 2x + \frac{d(y^2)}{dx} - 1 - \frac{dy}{dx} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now $\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$. Therefore (1) becomes

$$2x + 2y\frac{dy}{dx} - 1 - \frac{dy}{dx} = 0; \qquad i.e. \ \frac{dy}{dx} = \frac{1-2x}{2y-1};$$

$\therefore$ the slope of the circle at the origin (0, 0) is -1.

Similarly, for the parabola,

$$4\frac{dy}{dx}=2x-4.$$

$\therefore$ the slope of the parabola at the origin is -1.

Hence the circle and the parabola have the same tangent at the origin.

The equation of the tangent is $x+y=0$, since the slope is -1.

Ex. 6. *Shew that the portion of the tangent to the curve* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ *intercepted between the axes is of constant length.*

Differentiating, as in Ex. 5, we have

$$\tfrac{2}{3}x^{-\frac{1}{3}}+\tfrac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0\,; \quad \therefore\ \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

The slope of the tangent at a point $(h,\ k)$ on the curve is therefore

$$-\left(\frac{k}{h}\right)^{\frac{1}{3}}.$$

The equation of the tangent at $(h,\ k)$ is

$$y-k+\left(\frac{k}{h}\right)^{\frac{1}{3}}(x-h)=0,$$

$$i.e. \qquad h^{\frac{1}{3}}y+k^{\frac{1}{3}}x-h^{\frac{1}{3}}k^{\frac{1}{3}}(h^{\frac{2}{3}}+k^{\frac{2}{3}})=0,$$

$$\text{or} \qquad h^{\frac{1}{3}}y+k^{\frac{1}{3}}x=h^{\frac{1}{3}}k^{\frac{1}{3}}a^{\frac{2}{3}},$$

since $(h,\ k)$ lies on the curve.

This meets the axes of X and Y in the points A, B where

$$OA=h^{\frac{1}{3}}a^{\frac{2}{3}} \quad \text{and} \quad OB=k^{\frac{1}{3}}a^{\frac{2}{3}}\,;$$

$$\therefore\ AB^2=(h^{\frac{2}{3}}+k^{\frac{2}{3}})a^{\frac{4}{3}}=a^2\,; \quad \therefore\ AB=a.$$

N.B.—If $y=\log_e x$, then $\dfrac{dy}{dx}=\dfrac{1}{x}$. This is an important formula, the proof of which is beyond the scope of this book. The converse, *i.e.* $\int dx/x=\log_e x+c$, was mentioned on p. 391.

Ex. 7. *Differentiate* $\log_e(x - \sqrt{x^2 - 1})$.

Let $$y = \log_e(x - \sqrt{x^2 - 1});$$

$$\therefore \frac{dy}{dx} = \frac{d \log_e(x - \sqrt{x^2 - 1})}{d(x - \sqrt{x^2 - 1})} \cdot \frac{d(x - \sqrt{x^2 - 1})}{dx}$$

$$= \frac{1}{x - \sqrt{x^2 - 1}} \times \left\{1 - \frac{d(\sqrt{x^2 - 1})}{dx}\right\}$$

$$= \frac{1}{x - \sqrt{x^2 - 1}} \left\{1 - \frac{\frac{1}{2} 2x}{\sqrt{x^2 - 1}}\right\};$$

$$\therefore \frac{dy}{dx} = \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} - x}{\sqrt{x^2 - 1}} = -\frac{1}{\sqrt{x^2 - 1}}.$$

EXERCISES LII.

Differentiate the functions in Ex. 1-21 :

1. $\sqrt{2 - x^2}$.
2. $\sqrt{2x^2 - 1}$.
3. $\sqrt{x^2 + a^2}$.
4. $\sqrt{ax^2 + 2bx + c}$.
5. $\frac{1}{\sqrt{1 + x^2}}$.
6. $\sqrt[3]{x^3 - 1}$.
7. $(ax^3 + 3bx^2 + 3cx + d)^{-\frac{1}{3}}$.
8. $(x + 1)(x - 3)$.
9. $x^2(x^2 + 1)$.
10. $(ax + b)^3(cx + d)^4$.
11. $x\sqrt{x^2 + 1}$.
12. $(2x + 1)\sqrt[3]{x^3 - 1}$.
13. $(ax + b)\sqrt{cx + d}$.
14. $\frac{x - 1}{x - 2}$.
15. $\frac{x}{x^2 - 1}$.
16. $\frac{px + q}{rx + s}$.
17. $\frac{(5x + 2)^2}{3 - x}$.
18. $\frac{\sqrt{x + 1}}{x - 1}$.
19. $\sqrt{\frac{x + 1}{x - 1}}$.
20. $\sqrt{\frac{x^2 + 1}{x^2 - 1}}$.
21. $\log_e(ax + b)$.
22. $\log_e(x + \sqrt{x^2 + 1})$.

23. A spherical balloon is being blown up so that its volume increases at the rate of U cubic feet per sec. How fast is the radius increasing when the radius is r ?

24. The cross-section of a trough is an equilateral triangle of side 18 in., and its length is 6 feet. If water is poured in at the rate of 2 cubic feet a minute, at what rate is the level of the water rising in the trough when its depth is one foot ?

25. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point P $(at^2, 2at)$.

Let the tangent cut the axis of X at T and the axis of Y at Z, let the normal cut the axis of X at G, let S be the point $(a, 0)$, and let N be the foot of the ordinate at P. Prove that (i) the origin bisects TN; (ii) $ST = SG$; (iii) SZ is perpendicular to PT.

26. Shew that the ellipse $2x^2 + 3y^2 = 5$ and the circle

$$3x^2 + 3y^2 - 2x - 4 = 0$$

touch one another at the points $(1, \pm 1)$. Find the equations of the common tangents.

27. Shew that the equation of the tangent to the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad \text{at} \quad P(x', y') \quad \text{is} \quad xx'/a^2 + yy'/b^2 = 1.$$

If C is the origin, T the point where the tangent at P meets the axis of X and N the foot of the ordinate at P, prove that

$$CN \,.\, CT = a^2.$$

28. If, in Ex. 27, the tangent at P meets the axis of Y at t and n is the foot of the perpendicular from P on the axis of Y, prove that $Cn \,.\, Ct = b^2$.

29. Prove that the equation of the tangent and normal, at the point $(a \cos\theta, b \sin\theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$, are respectively $x \cos\theta/a + y \sin\theta/b = 1$ and $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$.

30. If the tangent at any point P on the curve $x^m y^n = c^{m+n}$ meet the axes of OX, OY in the points T, T', prove that PT/PT' is constant.

Second Differential Coefficient or Second Derivative. If y is a function of x, say $f(x)$, $\dfrac{dy}{dx}$ is its first derivative; the derivative of $\dfrac{dy}{dx}$ with respect to x is written $\dfrac{d^2y}{dx^2}$, and is the **second derivative** of y with respect to x.

Just as $\dfrac{dy}{dx}$ measures the rate of increase of y with respect to x, so $\dfrac{d^2y}{dx^2}$ measures the rate of increase of $\dfrac{dy}{dx}$ with respect to x.

Therefore, if $\dfrac{d^2y}{dx^2}$ is positive, $\dfrac{dy}{dx}$ increases with x; *i.e.* the slope of the curve $y = f(x)$ increases with x.

This is illustrated by Figs. 16 and 17, p. 38. In the latter $\dfrac{dy}{dx}$ is negative, but increases algebraically as x increases. In both figures, the curve is concave upwards.

Similarly, if $\dfrac{d^2y}{dx^2}$ is negative, $\dfrac{dy}{dx}$ decreases as x increases, *i.e.* the slope of $y=f(x)$ decreases as x increases. This is illustrated by Figs. 18 and 19, p 39, both of which are convex upwards.

Near a point of inflexion, the curve changes from convex upwards to concave upwards, or *vice versa*. In other words, $\dfrac{d^2y}{dx^2}$ changes sign as we pass through a point of inflexion.

If $y=f(x)$, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$ may be written $f'(x)$ and $f''(x)$.

Applications to Maxima and Minima. If $y=f(x)$ and y is a **minimum** when $x=a$, we know that $\mathbf{f'(a)=0}$, and that $f'(a-h)$ is negative and $f'(a+h)$ is positive, where $a-h$, a and $a+h$ are neighbouring and increasing values of x,

i.e. $f'(x)$ is increasing in the neighbourhood of $x=a$;

$\therefore$ **f′(a) is positive.**

Similarly, if $f(b)$ is a **maximum**, $\mathbf{f'(b)=0}$ and $f'(b-h)$ is positive while $f'(b+h)$ is negative, where $b-h$, b and $b+h$ are neighbouring and increasing values of x,

i.e. $f'(x)$ is decreasing in the neighbourhood of $x=b$;

$\therefore$ **f″(b) is negative.**

The following table gives the conditions for a maximum or minimum:

y	$\dfrac{dy}{dx}$	$\dfrac{d^2y}{dx^2}$
minimum	zero	positive
maximum	zero	negative

The method of discriminating between maxima and minima by examining the sign of $\dfrac{d^2y}{dx^2}$ fails when $\dfrac{d^2y}{dx^2}=0$. When this happens, it is best to proceed as shewn on p. 382.

Ex. 1. *Find the maximum and minimum values of* $\dfrac{x}{1+x^2}$.

Let $$f(x)=\frac{x}{1+x^2};$$

$$\therefore\ f'(x)=\frac{1\times(1+x^2)-2x\times x}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=\frac{(1-x)(1+x)}{(1+x^2)^2}.$$

Now, when $x=1$, $f'(x)=0$.

Writing $f'(x)$ in the form $(1-x)\phi(x)$, where $\phi(x)=\dfrac{1+x}{(1+x^2)^2}$, we have

$$f''(x)=-1\times\phi(x)+(1-x)\phi'(x)\ ;$$

$$\therefore\ f''(1)=-\phi(1)=-\frac{1+1}{(1+1^2)^2},\ \text{which is negative ;}$$

$$\therefore\ \text{if } x=1,\ \frac{x}{1+x^2}\ \text{is a maximum, viz. } \tfrac{1}{2}.$$

Again, when $x=-1$, $f'(x)=0$.

Writing $f'(x)$ in the form $(1+x)\psi(x)$, where $\psi(x)=\dfrac{1-x}{(1+x^2)^2}$, we have

$$f''(x)=1\times\psi(x)+(1+x)\psi'(x)\ ;$$

$$\therefore\ f''(-1)=\psi(-1)=\frac{1-(-1)}{\{1+(-1)^2\}^2},\ \text{which is positive ;}$$

$$\therefore\ \text{if } x=-1,\ \frac{x}{1+x^2}\ \text{is a minimum, viz. } -\tfrac{1}{2}.$$

The method employed here will be found useful.

Ex. 2. *Find the maximum and minimum values of*

$$(1+x)^2(1-x^2).$$

Let $f(x)=(1+x)^2(1-x^2)$;

$$\therefore\ f'(x)=2(1+x)(1-x^2)+(1+x)^2(-2x)$$

$$\therefore\ f'(x)=2(1+x)\{1-x-2x^2\}=2(1+x)^2(1-2x).$$

Take $x=\frac{1}{2}$, then $f'(x)=0$.

Writing $f'(x)$ in the form $(1-2x)\,\phi(x)$, where $\phi(x)=2(1+x)^2$, we have

$$f''(x)=-2\phi(x)+(1-2x)\,\phi'x\ ;$$

$$\therefore\ f''(\tfrac{1}{2})=-2\,\phi(\tfrac{1}{2})=-4(\tfrac{3}{2})^2,\ \text{which is negative ;}$$

$$\therefore\ \text{if } x=\tfrac{1}{2},\ (1+x)^2(1-x^2)\ \text{is a maximum, viz. } \tfrac{27}{16}.$$

Take $x=-1$, then $f'(x)=0$.

Writing $f'(x)$ in the form $(1+x)^2\psi(x)$, where $\psi(x)=2(1-2x)$, we have

$$f''(x)=2(1+x)\,\psi(x)+(1+x)^2\psi'(x)\ ;$$

$$\therefore\ f''(-1)=0.$$

Consequently it is impossible to discover whether $x = -1$ gives a maximum or a minimum by this method.

Using the method of p. 382, we have

x	$f'(x)$	
$-1-h$	$2(-h)^2(3+2h)$,	which is positive.
-1	0	
$-1+h$	$2(h)^2(3-2h)$,	which is positive.

$\therefore\ x = -1$ gives neither a maximum nor a minimum, but a point of inflexion.

Simple Differential Equations. If $y = f(x)$, any relation connecting y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, etc., is called a differential equation. When a differential equation is given, it is in some cases possible to find the relation between the variables.

Ex. 1. *Find the equation of the curve in which the subnormal is constant.*

If P is any point on the curve $y = f(x)$ (Fig. 196), PN the ordinate at P and G the point where the normal cuts the axis of X, then NG is called the **subnormal.**

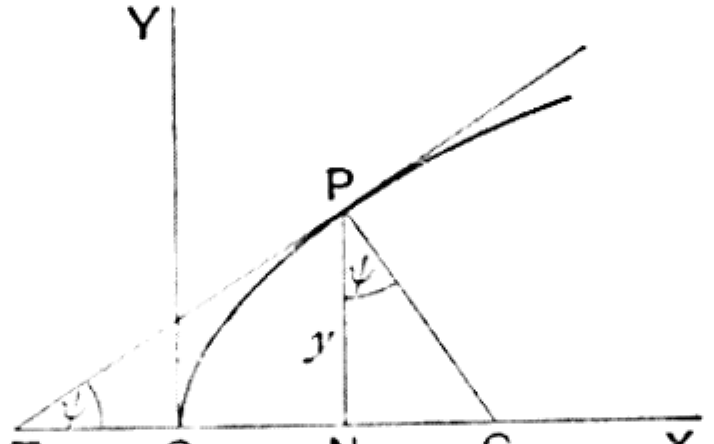


FIG. 196.

Using the figure, $\tan\psi =$ the slope at $P = \dfrac{dy}{dx}$.

The angle $NPG = \psi$; $\therefore\ NG = y \tan\psi$.

If the subnormal is constant, we have $y\dfrac{dy}{dx} = c$;

$$\therefore\ 2y\frac{dy}{dx} = 2c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now the expression which, on differentiating with respect to x, gives $2y\dfrac{dy}{dx}$ is, clearly, y^2+a; and the expression which gives $2c$ is $2cx+b$, where a and b are constants:

$$\therefore\ y^2+a=2cx+b \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is equivalent to (1).

Now (2) can be written $y^2=2cx+d$, where d is a constant.

If we choose the origin on the axis of x so that $x=0$ when $y=0$, then $d=0$, and the simplest form of the equation of the curve for which the subnormal is constant is

$$y^2=2cx.$$

Ex. 2. *The acceleration of a point moving in a straight line is* $(2t+1)$ *ft. per sec. per sec. at time* t. *Find the distance travelled in* t *seconds if the initial velocity is 2 ft. per sec.*

Let s ft. be the distance travelled in t seconds.

Then $\dfrac{ds}{dt}$, being the rate of increase of s with t, measures the velocity, and $\dfrac{d^2s}{dt^2}$ measures the rate of increase of the velocity.

We have $$\frac{d^2s}{dt^2}=2t+1;$$

$$\therefore\ \frac{ds}{dt}=t^2+t+c, \text{ where } c \text{ is a constant.}$$

Now when $t=0$, $\dfrac{ds}{dt}=2$; $\therefore\ 2=c$; and $\dfrac{ds}{dt}=t^2+t+2$.

$\therefore\ s=\frac{1}{3}t^3+\frac{1}{2}t^2+2t+a$, where a is a constant.

Now when $t=0$, $s=0$, and therefore $a=0$,

$$i.e.\qquad s=\tfrac{1}{3}t^3+\tfrac{1}{2}t^2+2t.$$

Ex. 3. *Solve the differential equation* $\dfrac{d^2y}{dx^2}=\dfrac{1}{x^2}$, *given that when* $x=1$, $y=0$ *and* $\dfrac{dy}{dx}=-1$.

$\dfrac{d^2y}{dx^2}=\dfrac{1}{x^2}$; $\therefore\ \dfrac{dy}{dx}=-\dfrac{1}{x}+a$. But $\dfrac{dy}{dx}=-1$ when $x=1$; $\therefore\ a=0$.

$\dfrac{dy}{dx}=-\dfrac{1}{x}$; $\therefore\ y=-\log_e x+b$. But $y=0$ when $x=1$; $\therefore\ y=-\log_e x$.

Ex. 3. **Bending of Beams.** *Find an expression for the deflection at any point of a beam of length 2L ft. supported at its ends and carrying a uniformly distributed load of w lb. per ft. run.*

Let OX be the horizontal tangent at the lowest point O of the beam, and draw OY perpendicular to OX (Fig. 197).

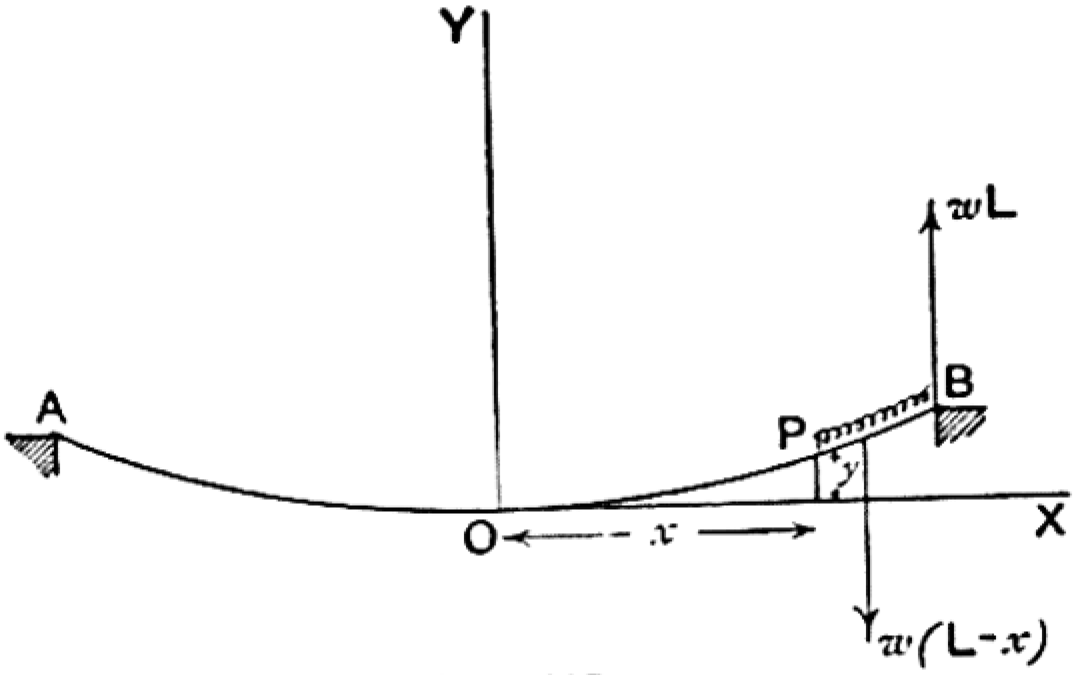


FIG. 197.

The curve AB is assumed to be so flat that any arc OP can be taken as equal to x, the abscissa of P.

If P is the point x, y, it can be shewn that

$$\frac{d^2y}{dx^2}=\frac{w}{2EI}(L^2-x^2).$$

Integrating with respect to x,

$$\frac{dy}{dx}=\frac{w}{2EI}\left(L^2x-\frac{x^3}{3}\right)+C.$$

Now, since $\frac{dy}{dx}=0$ when $x=0$, $\therefore\ C=0$.

Integrating again, $y=\frac{w}{2EI}\left(\frac{L^2x^2}{2}-\frac{x^4}{12}\right)$,

there being no constant, since $y=0$ when $x=0$.

Again, when $x=L$, $y_B{}^*=\frac{w}{2EI}\cdot\frac{5L^4}{12}=\frac{5}{24}\frac{wL^4}{EI}$.

Hence, the deflection at $P=y_B-y=\frac{w}{24EI}\{5L^4-6L^2x^2+x^4\}$.

The maximum deflection is at O, and is equal to y_B.

Any case of bending may be investigated in a similar manner, provided that an algebraic expression for the bending moment at any point can be found.

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* *i.e.* the value of y at B.

EXERCISES LIII.

1. Find the first, second and third derivatives of the following functions :—

$$x^3,\quad 3x^2-2x+4,\quad \frac{1}{x},\quad \frac{1}{(x+1)^2},\quad \frac{ax+b}{cx+d},\quad \sqrt{x+1},$$

$$x^2(1-x),\quad (x+2)^3(x-1)^2.$$

2. Examine whether the following curves are convex or concave upwards:

(i) $y=4x^2$, at the points $(1, 4)$ and $(-1, 4)$.
(ii) $y^2=4x$, at the points $(1, 2)$ and $(1, -2)$.
(iii) $y=2x^3+x$, at the points $(1, 3)$ and $(-1, -3)$.
(iv) $y=(x+1)/x$, at the points $(1, 2)$ and $(-3, \frac{2}{3})$.

3. What value of x makes $x(1-x^2)$ a maximum ?

4. When is the sum of a number and its reciprocal a maximum ? When is it a minimum ?

5. What is the rectangle of greatest area which can be inscribed in a circle ?

6. Find the dimensions of the greatest open box of square base which can be made from a square sheet of tin of side a inches.

7. X, Y are two points on a straight shore line distant 8 miles apart. A man in a boat at Z, distant 5 miles from X, wishes to reach Y in the shortest possible time (ZXY is 90°). He cannot walk more than 4 miles per hour or row more than 3 miles per hour. At what point P between X and Y should he land ?

8. Given that the combined length and girth of the greatest parcel which may be sent by parcel post must not exceed 6 feet, find (i) the dimensions of the largest box with square base and vertical sides ; (ii) the length of side and volume of the largest cube ; (iii) the diameter and volume of the largest sphere ; (iv) the diameter, length and volume of the largest cylinder which can be sent by parcel post.

9. Shew that the surface of a cone of given volume V cub. ft. is least when the height h ft. of the cone is $r\sqrt{2}$, where r ft. is the radius of the base.

10. A small magnet is placed on the axis of a circular coil of radius a, through which an electric current is flowing. If x is the distance of the magnet from the plane of the coil, the force exerted on the magnet is given by $F=x/(x^2+a^2)^{\frac{5}{2}}$. Find when F is a maximum.

11. The intensity of illumination at a point distant d ft. from a light of candle-power c is c/d^2. There are two lights P and Q 12 ft. apart: P is 16 C.P., Q is 25 C.P. Find the point in PQ at which the intensity of illumination is a minimum.

12. A beam AB of length L is supported at the ends and carries a moving load of amount W uniformly distributed over a length l. Shew that the bending moment at a point C on the beam is a maximum when C divides the lengths l and L in the same ratio.

13. Solve the following differential equations:

(i) $\frac{dy}{dx}=\frac{1}{x^2}$, where $x=1$ when $y=1$.

(ii) $\frac{d^2y}{dx^2}=a+bx$, where $y=0$ and $\frac{dy}{dx}=c$ when $x=0$.

(iii) $\frac{d^2y}{dx^2}=\frac{w}{2}(l-x)^2$, if $\frac{dy}{dx}=0$ when $x=0$ and $y=0$ when $x=l$.

14. Solve the equation $\frac{d^2y}{dx^2}=a+bx+cx^2$, given that $\frac{dy}{dx}=0$ when $x=0$ and $y=d$ when $x=0$.

15. Solve the equation $\frac{d^2y}{dx^2}=6x$, given that $y=2$ when $x=0$ and when $x=1$.

Problems on Beams.

16. Cantilever of length l ft. with load of W lb. concentrated at the free end. The deflection y at a distance x from the fixed end is given by

$$\frac{d^2y}{dx^2}=\frac{W}{EI}(l-x).$$

Solve the equation. Shew that the maximum deflection is $\frac{1}{3}Wl^3/EI$, and occurs when $x=l$.

17. Cantilever of length l ft. with load w lb. per ft. Given that

$$\frac{d^2y}{dx^2}=\frac{w}{2EI}(l-x)^2,$$

solve the equation, and find the maximum deflection.

18. Beam of length $2l$ ft., with ends fixed horizontally, carrying a load of w lb. per ft. The deflection y at a distance x from one end is given by

$$\frac{d^2y}{dx^2}=\frac{w}{6EI}\{2l^2-6lx+3x^2\}.$$

Find the maximum deflection.

Integration by Substitution. To effect an integration, it is sometimes convenient to change the variable of integration.

Ex. 1. *Find* $\int (ax+b)^n dx$.

Put $$z=(ax+b) \quad \text{and} \quad y=\int (ax+b)^n dx.$$

Then $$\frac{dz}{dx}=a \quad \text{and} \quad \frac{dy}{dx}=(ax+b)^n=z^n.$$

Now $$\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=\frac{z^n}{a};$$

$$\therefore\ y=\frac{z^{n+1}}{a(n+1)}=\frac{(ax+b)^{n+1}}{a(n+1)}.$$

Ex. 2. *Integrate* $\dfrac{x}{(x^2+1)^{\frac{3}{2}}}$.

Put $$z=\sqrt{x^2+1} \quad \text{and} \quad y=\int \frac{x\,dx}{(x^2+1)^{\frac{3}{2}}}.$$

Then $$\frac{dz}{dx}=\frac{x}{\sqrt{x^2+1}} \quad \text{and} \quad \frac{dy}{dx}=\frac{x}{(x^2+1)^{\frac{3}{2}}}.$$

Now $$\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=\frac{1}{x^2+1}, \quad \text{or} \quad \frac{dy}{dz}=\frac{1}{z^2};$$

$$\therefore\ y=-\frac{1}{z}=-\frac{1}{\sqrt{x^2+1}}.$$

Ex. 3. *Integrate* $\dfrac{x^3}{\sqrt{x-1}}$.

Put $$z=\sqrt{x-1} \quad \text{and} \quad y=\int \frac{x^3 dx}{\sqrt{x-1}}.$$

Then $$\frac{dz}{dx}=\frac{1}{2\sqrt{x-1}} \quad \text{and} \quad \frac{dy}{dx}=\frac{x^3}{\sqrt{x-1}}.$$

Now $$\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=2x^3=2(1+z^2)^3;$$

$$\therefore\ \frac{dy}{dz}=2(1+3z^2+3z^4+z^6);$$

$$\therefore\ y=2\left(z+z^3+\frac{3z^5}{5}+\frac{z^7}{7}\right)$$
$$=2\sqrt{x-1}\,\{1+(x-1)+\tfrac{3}{5}(x-1)^2+\tfrac{1}{7}(x-1)^3\}.$$

On reduction,

$$y = \tfrac{2}{35}\sqrt{x-1}\,\{16 + 8x + 6x^2 + 5x^3\}.$$

No general rule for choosing the new variable z can be given. The only method is one of trial, but, after practice, inspection generally reveals the correct substitution.

Ex. 4. *Find the area between the curve $y^3(7x+1) = 1$, the axis of x and the ordinates $x = 0$ and $x = 9$.*

We have
$$y = \frac{1}{(7x+1)^{\frac{1}{3}}}.$$

The required area is $\displaystyle\int_0^9 \frac{dx}{(7x+1)^{\frac{1}{3}}}$.

Put
$$z = 7x + 1 \quad\text{and}\quad Y = \int \frac{dx}{(7x+1)^{\frac{1}{3}}};$$

$$\therefore\ \frac{dz}{dx} = 7 \qquad\text{and}\qquad \frac{dY}{dx} = \frac{1}{(7x+1)^{\frac{1}{3}}} = \frac{1}{z^{\frac{1}{3}}};$$

$$\therefore\ \frac{dY}{dz} = \frac{dY}{dx}\Big/\frac{dz}{dx} = \frac{1}{7z^{\frac{1}{3}}};$$

$$\therefore\ Y = \frac{1}{7}\,\frac{z^{-\frac{1}{3}+1}}{(-\frac{1}{3}+1)} = \frac{3}{14}(7x+1)^{\frac{2}{3}}.$$

The required area is $\left[\dfrac{3}{14}(7x+1)^{\frac{2}{3}}\right]_0^9$, *i.e.* $\dfrac{3}{14}\left[(64)^{\frac{2}{3}} - 1^{\frac{2}{3}}\right]$.

The area is 45/14.

If an algebraic fraction is to be integrated, it is advisable to use partial fractions, as follows:—

Ex. 5. *Integrate* $\dfrac{5}{2x^2 - 3x - 2}$.

$$2x^2 - 3x - 2 = (2x+1)(x-2).$$

Using partial fractions, we find (see p. 228)

$$\frac{5}{2x^2 - 3x - 2} = \frac{1}{x-2} - \frac{2}{2x+1};$$

$$\therefore\ \int \frac{5\,dx}{2x^2 - 3x - 2} = \int \frac{dx}{x-2} - \int \frac{2\,dx}{2x+1}.$$

If $y = \int \frac{dx}{x-2}$ and $z = x - 2$, we have

$$\frac{dy}{dx} = \frac{1}{x-2} \quad \text{and} \quad \frac{dz}{dx} = 1 ; \quad \therefore \frac{dy}{dz} = \frac{dy}{dx} \Big/ \frac{dz}{dx} = \frac{1}{x-2} = \frac{1}{z} ;$$

$$\therefore\ y = \log z = \log_e(x-2), \quad i.e. \int \frac{dx}{x-2} = \log_e(x-2).$$

Similarly, $$\int \frac{2\,dx}{2x+1} = \log_e(2x+1) ;$$

$$\therefore \int \frac{5\,dx}{2x^2-3x-2} = \log_e(x-2) - \log_e(2x+1) = \log_e \frac{x-2}{2x+1}.$$

EXERCISES LIV.

Integrate :—

1. $\frac{x}{x^2+1}$. 2. $\frac{x}{(x^2+a^2)^2}$. 3. $\frac{ax+b}{\sqrt{ax^2+2bx+c}}$.

4. $x\sqrt{1-x^2}$. 5. $\frac{4x-5}{(2x^2-5x+8)^3}$. 6. $\frac{2x-3}{\sqrt{x^2-3x+1}}$.

7. $x^2\sqrt{x^3+1}$. 8. $x^5 . \sqrt[3]{x^6+2}$. 9. $\frac{x^2}{\sqrt{x+1}}$.

Find the value of the following definite integrals :—

10. $\int_1^2 (2x+1)^2 dx$. 11. $\int_1^2 \frac{dx}{(2x+1)^2}$. 12. $\int_{-1}^4 \sqrt{x+5}\,dx$.

13. $\int_4^8 \frac{dx}{x^2-9}$. 14. $\int_3^5 \frac{x+1}{x^2-4}\,dx$. 15. $\int_3^5 \frac{x}{x^2-3x+2}\,dx$.

16. $\int_0^1 x\sqrt{1-x^2}\,dx$. 17. $\int_0^b x\sqrt{b^2-x^2}\,dx$. 18. $\int_0^a \frac{x\,dx}{(x^2+a^2)^{\frac{3}{2}}}$.

Differentiation and Integration of the Circular Functions.

Let $y = \sin x$, then $y + \Delta y = \sin(x + \Delta x)$;

$$\therefore\ \Delta y = \sin(x+\Delta x) - \sin x$$
$$= 2\cos(x + \tfrac{1}{2}\Delta x)\sin \tfrac{1}{2}\Delta x \quad \text{(see p. 316)} ;$$

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{2\cos(x+\frac{1}{2}\Delta x)\sin\frac{1}{2}\Delta x}{\Delta x} = \cos(x+\tfrac{1}{2}\Delta x) \times \frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}.$$

Now, as $\Delta x \to 0$,

$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, $\cos(x + \frac{1}{2}\Delta x) \to \cos x$ and $\frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \to 1$ (see p. 161);

$$\therefore \frac{dy}{dx} = \cos x;$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{I.}$$

Again, let $y = \cos x$, then $y + \Delta y = \cos(x + \Delta x)$;

$$\therefore \Delta y = \cos(x + \Delta x) - \cos x = -2\sin(x + \tfrac{1}{2}\Delta x)\sin\tfrac{1}{2}\Delta x;$$

$$\therefore \frac{\Delta y}{\Delta x} = -\sin(x + \tfrac{1}{2}\Delta x) \times \frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}.$$

Whence, as before, $\frac{dy}{dx} = -\sin x$;

$$\therefore \frac{d}{dx}(\cos x) = -\sin x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{II.}$$

From II., we have $\int \sin x \, dx = -\cos x$; $\ldots\ldots\ldots\ldots\ldots\ldots$III.

and from I., $\int \cos x \, dx = \sin x$. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots$IV

Let $y = \tan x$, then $y + \Delta y = \tan(x + \Delta x)$;

$$\therefore \Delta y = \tan(x + \Delta x) - \tan x$$

$$= \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x}$$

$$= \frac{\sin(x + \Delta x - x)}{\cos(x + \Delta x)\cos x};$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\cos(x + \Delta x)\cos x} \times \frac{\sin\Delta x}{\Delta x};$$

and, in the limit, $\frac{dy}{dx} = \frac{1}{\cos^2 x}$,

$$i.e. \quad \frac{d}{dx}(\tan x) = \sec^2 x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{V.}$$

From V., we have $\int \sec^2 x \, dx = \tan x$. $\ldots\ldots\ldots\ldots\ldots\ldots$VI.

Ex. 1. *Differentiate* $\sec x$, $\operatorname{cosec} x$ *and* $\cot x$.

If $y = \sec x$, $\qquad y = (\cos x)^{-1}$;

$$\therefore \frac{dy}{dx} = \frac{d(\cos x)^{-1}}{d\cos x} \times \frac{d\cos x}{dx}.$$

This is an example of the formula $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$,

where $\qquad z = \cos x$;

$$\therefore \frac{dy}{dx} = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

If $y = \operatorname{cosec} x$, $\qquad y = (\sin x)^{-1}$;

$$\therefore \frac{dy}{dx} = -(\sin x)^{-2} \times \cos x = -\operatorname{cosec} x \cot x.$$

If $y = \cot x$, $\qquad y = (\tan x)^{-1}$;

$$\therefore \frac{dy}{dx} = -(\tan x)^{-2} \sec^2 x$$

$$= -\operatorname{cosec}^2 x.$$

Ex. 2. *Integrate* $\dfrac{1}{x^2 + a^2}$.

To evaluate $\displaystyle\int \frac{dx}{x^2 + a^2}$, let $x = a\tan\theta$ and $y = \displaystyle\int \frac{dx}{x^2 + a^2}$;

$$\therefore \frac{dx}{d\theta} = a\sec^2\theta \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{x^2 + a^2} = \frac{1}{a^2\tan^2\theta + a^2} = \frac{1}{a^2\sec^2\theta};$$

$$\therefore \frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta} = \frac{1}{a}; \qquad \therefore y = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Ex. 3. *Integrate* $\dfrac{1}{\sqrt{a^2 - x^2}}$.

Here let $x = a\sin\theta$ and $y = \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$;

$$\therefore \frac{dx}{d\theta} = a\cos\theta \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{a^2(1 - \sin^2\theta)}} = \frac{1}{a\cos\theta};$$

$$\therefore \frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta} = 1; \qquad \therefore y = \theta = \sin^{-1}\frac{x}{a}.$$

Ex. 4. *Differentiate* $\sin^{-1}\dfrac{x}{a}$ *with respect to* x.

Let $y=\sin^{-1}\dfrac{x}{a}$; $\quad\therefore\ a\sin y=x.$

Differentiating with respect to y, we have $a\cos y=\dfrac{dx}{dy}$;

$$\therefore\ \frac{dy}{dx}=\frac{1}{a\cos y}=\frac{1}{\sqrt{a^2-a^2\sin^2 y}}=\frac{1}{\sqrt{a^2-x^2}}.\quad \text{(Cf. with Ex. 3.)}$$

Ex. 5. *Integrate* $\cos^2 x$.

$$\int\cos^2 x\,dx=\int\frac{1+\cos 2x}{2}dx=\int\frac{dx}{2}+\frac{1}{2}\int\cos 2x\,dx;$$

$$\therefore\ \int\cos^2 x\,dx=\frac{1}{2}x+\frac{1}{2}\,\frac{\sin 2x}{2}=\frac{2x+\sin 2x}{4}.$$

EXERCISES LV.

Differentiate the functions in Ex. 1-18 :—

1. $\sin(ax+b)$.
2. $\cos(ax-b)$.
3. $\tan(px+q)$.
4. $\cot 2x$.
5. $\sec(3x+4)$.
6. $\operatorname{cosec}\frac{1}{2}x$.
7. $\sin^2 x$.
8. $\cos^3(ax+h)$.
9. $\tan^2(lx-m)$.
10. $\sin 2x\cos x$.
11. $\cos ax\cos bx$.
12. $x^2\sin(ax+b)$.
13. $\dfrac{\cos x}{x}$.
14. $\dfrac{ax+b}{\sin x}$.
15. $\dfrac{1-\sin x}{1+\sin x}$.
16. $\cos^{-1}(ax+b)$.
17. $\tan^{-1}x$.
18. $\sec^{-1}x$.

Find the value of $\dfrac{d^2y}{dx^2}$ where y has the values given in Ex. 19-27:

19. $\sin x$.
20. $\cos x$.
21. $\tan x$.
22. $\sin^2 x$.
23. $a\sin x+b\cos x$.
24. $\log\sin x$.
25. $\log(\sec x+\tan x)$.
26. $\log\tan\dfrac{x}{2}$.
27. $\log\dfrac{1-\cos x}{1+\cos x}$.
28. If $y=a\sin(px+q)$, shew that y satisfies the equation

$$\frac{d^2y}{dx^2}+p^2y=0.$$

Integrate with respect to x the functions in Ex. 29-40 :

29. $\sin 2x$. **30.** $\cos(ax+b)$. **31.** $\sec^2(px+q)$.

32. $\sin^2 x$. **33.** $\sin^2(ax+b)$. **34.** $\sin 2x \cos x$.

35. $\sin x \cos 2x$. **36.** $\sin ax \cos bx$. **37.** $\cos ax \cos bx$.

38. $\sin^2 x \cos x$. **39.** $\cos^3 x \sin x$. **40.** $\tan x$.

Integrate the functions in Ex. 41-44 with respect to x, by making the substitutions suggested :—

41. $\dfrac{1}{\sqrt{b^2-a^2x^2}}$. (Put $ax = b\sin\theta$.)

42. $\dfrac{1}{b^2+a^2x^2}$. (Put $ax = b\tan\theta$.)

43. $\dfrac{1}{\sqrt{4x-x^2}}$. (Put $x-2 = 2\sin\theta$.)

44. $\dfrac{1}{4x^2+4x+17}$. (Put $2x+1 = 4\tan\theta$.)

Find the value of the following :—

45. $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{3-4x^2}}$. **46.** $\displaystyle\int_0^3 \frac{dx}{9+x^2}$. **47.** $\displaystyle\int_0^1 \frac{dx}{2+2x+x^2}$.

48. $\displaystyle\int_1^{2\cdot5} \frac{dx}{\sqrt{2+2x-x^2}}$.

Miscellaneous Examples on the Calculus.

Ex. 1. *Find, from first principles, the derivative of* $\tan^{-1}x$.

Let $y = \tan^{-1}x$, then $y + \Delta y = \tan^{-1}(x+\Delta x)$;

$$\therefore \quad \Delta y = \tan^{-1}(x+\Delta x) - \tan^{-1}x. \quad \ldots\ldots\ldots\ldots(1)$$

Let $\alpha = \tan^{-1}(x+\Delta x)$ and $\beta = \tan^{-1}x$.

Then $x+\Delta x = \tan\alpha$ and $x = \tan\beta$;

$$\therefore \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\Delta x}{1+x(x+\Delta x)}.$$

Using (1), $$\Delta y = \alpha - \beta = \tan^{-1}\frac{\Delta x}{1+x(x+\Delta x)};$$

$$\therefore\ \tan\Delta y = \frac{\Delta x}{1+x(x+\Delta x)};\qquad \therefore\ \frac{\tan\Delta y}{\Delta y}\cdot\frac{\Delta y}{\Delta x} = \frac{1}{1+x(x+\Delta x)}.$$

Now, as $\Delta y \to 0$, $\Delta x \to 0$,

$$\frac{\tan\Delta y}{\Delta y} \to 1 \quad\text{and}\quad \frac{1}{1+x(x+\Delta x)} \to \frac{1}{1+x^2}.$$

In the limit, therefore, $$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Ex. 2. *The coordinates of any point on a cycloid are given by* $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$. *Find the slope at any point on the curve in terms of* θ. *What are the equations of the tangent and normal at the point determined by* θ ?

Since $\quad x = a(\theta - \sin\theta)\quad$ and $\quad y = a(1-\cos\theta)$,

we have $$\frac{dx}{d\theta} = a(1-\cos\theta) \quad\text{and}\quad \frac{dy}{d\theta} = a\sin\theta;$$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \cot\tfrac{1}{2}\theta.$$

The slope of the curve at the point determined by θ is, therefore, $\cot\frac{1}{2}\theta$. The equation of the tangent, being the line through $\{a(\theta-\sin\theta),\ a(1-\cos\theta)\}$ at slope $\cot\frac{1}{2}\theta$, is

$$Y - a(1-\cos\theta) = \cot\tfrac{1}{2}\theta\,\{X - a(\theta - \sin\theta)\},$$

i.e. $$Y\sin\tfrac{1}{2}\theta - X\cos\tfrac{1}{2}\theta$$
$$= a(1-\cos\theta)\sin\tfrac{1}{2}\theta - a(\theta-\sin\theta)\cos\tfrac{1}{2}\theta$$
$$= a(\sin\tfrac{1}{2}\theta - \theta\cos\tfrac{1}{2}\theta) + a(\sin\theta\cos\tfrac{1}{2}\theta - \cos\theta\sin\tfrac{1}{2}\theta),$$

i.e. $$Y\sin\tfrac{1}{2}\theta - X\cos\tfrac{1}{2}\theta = 2a\sin\tfrac{1}{2}\theta - a\theta\cos\tfrac{1}{2}\theta.$$

The normal is the line at right angles to this through the point determined by θ; its equation is

$$\{Y - a(1-\cos\theta)\}\cos\tfrac{1}{2}\theta + \{X - a(\theta-\sin\theta)\}\sin\tfrac{1}{2}\theta = 0,$$

i.e. $$Y\cos\tfrac{1}{2}\theta + X\sin\tfrac{1}{2}\theta$$
$$= a(\cos\tfrac{1}{2}\theta + \theta\sin\tfrac{1}{2}\theta) - a(\cos\theta\cos\tfrac{1}{2}\theta + \sin\theta\sin\tfrac{1}{2}\theta),$$

i.e. $$Y\cos\tfrac{1}{2}\theta + X\sin\tfrac{1}{2}\theta = a\theta\sin\tfrac{1}{2}\theta.$$

Ex. 3. *Find the maximum and minimum values of*

$$a \sin^2\theta + b \cos^2\theta.$$

Let $$y = a\sin^2\theta + b\cos^2\theta;$$

$$\therefore \frac{dy}{d\theta} = 2a\sin\theta\cos\theta - 2b\cos\theta\sin\theta = (a-b)\sin 2\theta;$$

$$\therefore \frac{dy}{d\theta} = 0, \text{ when } \sin 2\theta = 0, \text{ i.e. when } 2\theta = 0,\ \pi,\ 2\pi, \text{ etc.,}$$

$$\text{i.e. when } \theta = 0,\ \pi/2,\ \pi,\ 3\pi/2, \text{ etc.}$$

Now $$\frac{d^2y}{d\theta^2} = 2(a-b)\cos 2\theta.$$

When $\theta = 0,\ \pi,\ 2\pi$ etc., $\dfrac{d^2y}{d\theta^2} = 2(a-b)$,

and when $\theta = \pi/2,\ 3\pi/2,\ 5\pi/2$, etc., $\dfrac{d^2y}{d\theta^2} = -2(a-b)$.

If $a > b$, $\dfrac{d^2y}{d\theta^2}$ is positive when θ is $n\pi$,

and $\dfrac{d^2y}{d\theta^2}$ is negative when θ is $(2n+1)\pi/2$;

$\therefore$ $a\sin^2\theta + b\cos^2\theta$ is a maximum when θ is $(2n+1)\pi/2$, and its maximum value is a; and it is a minimum when $\theta = n\pi$ and its minimum value is b.

Ex. 4. *A circular arc is of constant length l; shew that the area of the segment bounded by l is a maximum when the arc is a semicircle.*

Let r be the radius of the circle and θ the angle subtended by the arc l at the centre. Then $r\theta = l$.

If A is the area of the segment,

$$A = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \text{ (Fig. 198).}$$

Substituting for r,

$$2A = \frac{l^2}{\theta} - \frac{l^2\sin\theta}{\theta^2};$$

$$\therefore \frac{2}{l^2}A = \frac{\theta - \sin\theta}{\theta^2}.$$

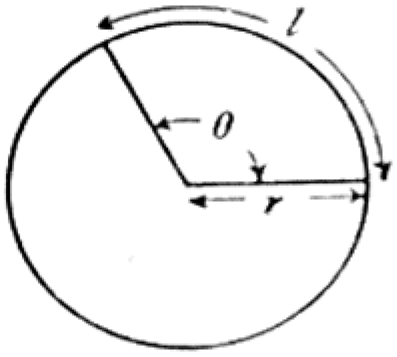

FIG. 198.

Differentiating with respect to θ,

$$\frac{2}{l^2}\frac{dA}{d\theta}=\frac{(1-\cos\theta)\theta^2-2\theta(\theta-\sin\theta)}{\theta^4}=\frac{2\sin\theta-\theta(1+\cos\theta)}{\theta^3};$$

$$\therefore\ \frac{2}{l^2}\frac{dA}{d\theta}=\frac{\cos\frac{1}{2}\theta\{4\sin\frac{1}{2}\theta-2\theta\cos\frac{1}{2}\theta\}}{\theta^3};$$

$$\therefore\ \frac{2}{l^2}\frac{dA}{d\theta}=\cos\tfrac{1}{2}\theta\,.\,U,\quad\text{where } U=\frac{4\sin\frac{1}{2}\theta-2\theta\cos\frac{1}{2}\theta}{\theta^3}.$$

When $\theta=\pi$, $\cos\frac{1}{2}\theta=0$, and $\therefore\ \dfrac{dA}{d\theta}=0.$

Again, $$\frac{2}{l^2}\frac{d^2A}{d\theta^2}=-\frac{1}{2}\sin\tfrac{1}{2}\theta\,.\,U+\cos\tfrac{1}{2}\theta\,.\,\frac{dU}{d\theta};$$

$$\therefore\ \text{when } \theta=\pi,\ \frac{2}{l^2}\frac{d^2A}{d\theta^2}=-\frac{1}{2}\times(\text{value of } U \text{ when } \theta=\pi)$$

$$=-\frac{1}{2}\left(\frac{4}{\pi^3}\right),\ \text{which is negative};$$

$\therefore\ \theta=\pi$ gives a maximum value for A, *i.e.* the arc l is a semicircle.

Ex. 5. *Find the turning points between $x=0$ and $x=\pi$ of*

$$y=\cos x+\tfrac{1}{2}\cos 2x.$$

We have $$\frac{dy}{dx}=-(\sin x+\sin 2x);$$

$$\therefore\ \frac{dy}{dx}=-\sin x\{1+2\cos x\};\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$\therefore\ \dfrac{dy}{dx}=0$, when $\sin x=0$ and when $\cos x=-\frac{1}{2}$,

i.e. when $x=0$ or π and when $x=\frac{2}{3}\pi$.

Writing $\dfrac{dy}{dx}=-\sin x\,.\,U$, where $U=1+2\cos x$,

$$\frac{d^2y}{dx^2}=-\cos x\,.\,U-\sin x\frac{dU}{dx};$$

$\therefore$ when $x=0$,

$$\frac{dy}{dx}=0 \quad \text{and} \quad \frac{d^2y}{dx^2}=-1\times(\text{value of } U \text{ when } x=0)$$

$$=-1\times 3, \text{ which is negative;}$$

$$\therefore\ x=0 \text{ makes } y \text{ a maximum.}$$

When $x=\pi$,

$$\frac{dy}{dx}=0 \quad \text{and} \quad \frac{d^2y}{dx^2}=-(-1)\times(\text{value of } U \text{ when } x=\pi)$$

$$=1\times(1-2), \text{ which is negative;}$$

$$\therefore\ x=\pi \text{ makes } y \text{ a maximum.}$$

Again, writing $\frac{dy}{dx}=-(1+2\cos x)\sin x$,

$$\frac{d^2y}{dx^2}=2\sin x(\sin x)-(1+2\cos x)\cos x;$$

$\therefore$ when $x=\frac{2}{3}\pi$,

$$\frac{dy}{dx}=0 \quad \text{and} \quad \frac{d^2y}{dx^2}=2\sin^2\tfrac{2}{3}\pi, \text{ which is positive;}$$

$$\therefore\ x=\tfrac{2}{3}\pi \text{ makes } y \text{ a minimum}$$

The graph is shewn in Fig. 199.

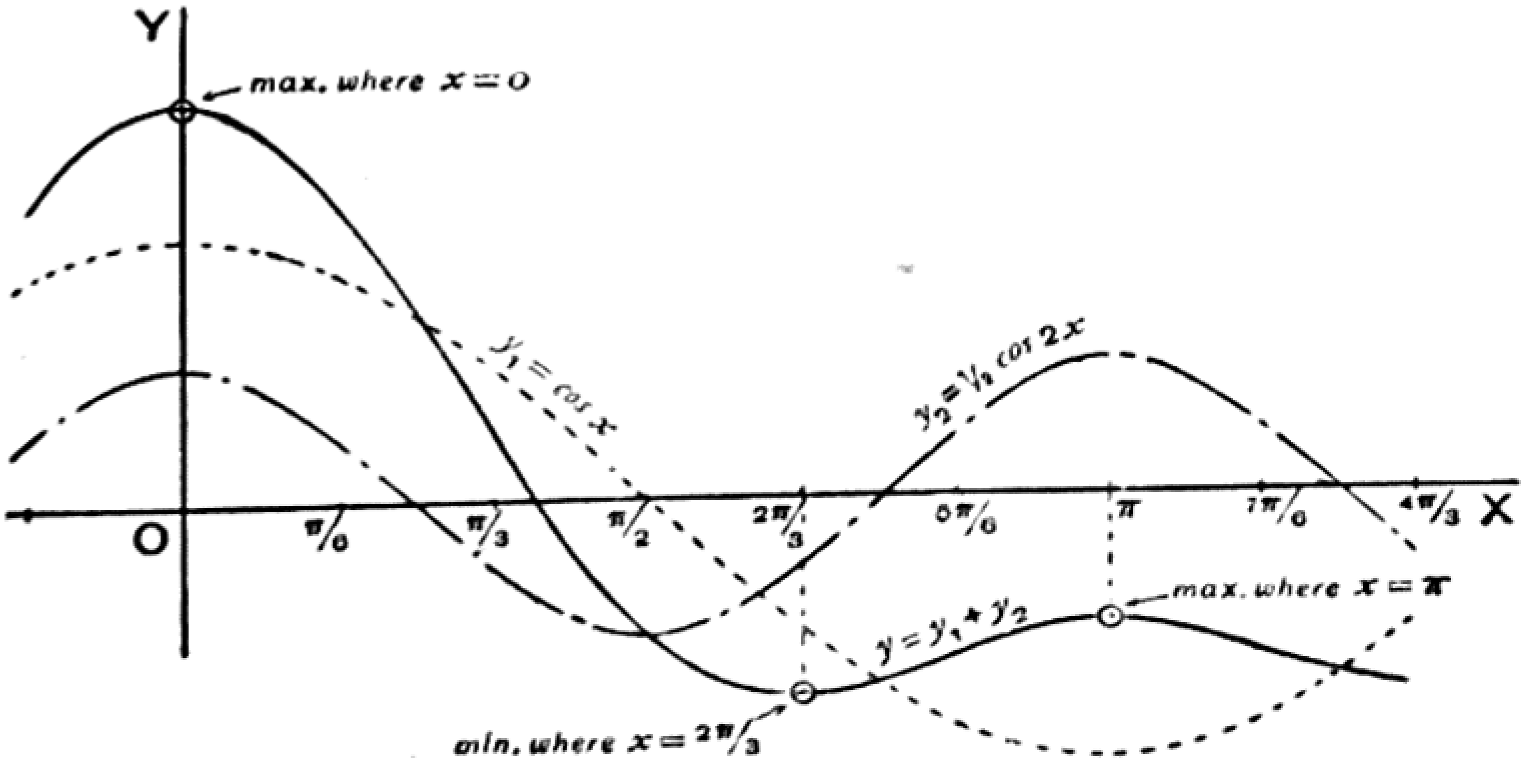

FIG. 199.

Ex. 6. *Sketch the curve* $y^2 = x^2(4 - x^2)$, *and find the area of one loop.*

For real values of y, $4 - x^2$ must be positive; $\therefore\ 0 < x^2 < 4$, *i.e.* x must lie between -2 and $+2$.

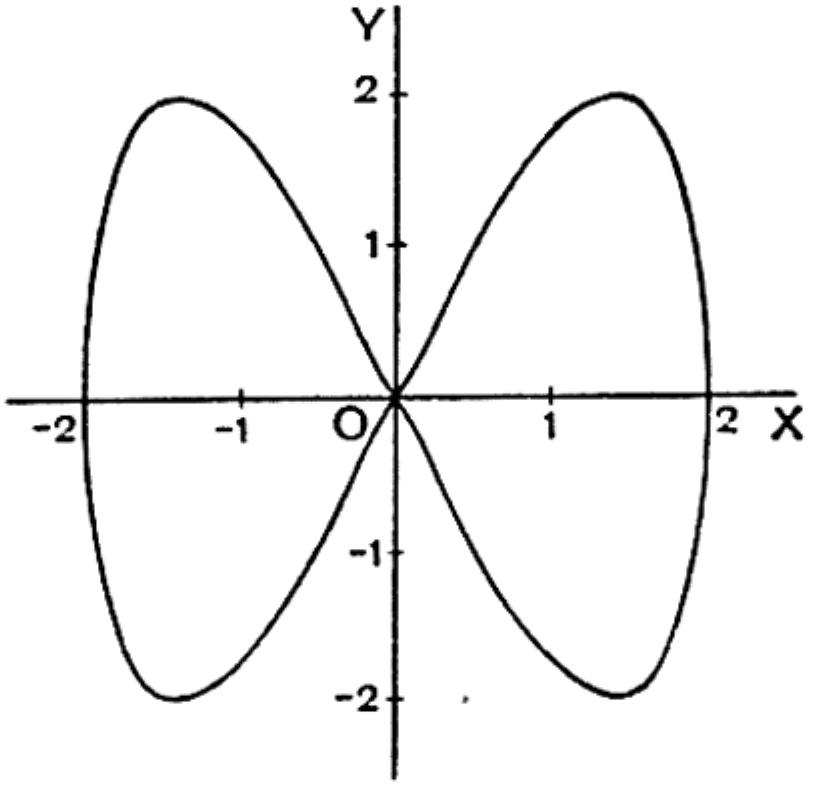

FIG. 200.

The curve clearly passes through the origin, and is symmetrical about the axis of X and the axis of Y. The sketch is shewn in Fig. 200. The area of the right-hand loop is

$$2\int_0^2 y\,dx, \quad i.e. \quad 2\int_0^2 x\sqrt{4 - x^2}\,dx.$$

Let $x = 2\sin\theta$ and $A = \int x\sqrt{4 - x^2}\,dx$.

Then

$$\frac{dx}{d\theta} = 2\cos\theta \quad \text{and} \quad \frac{dA}{dx} = x\sqrt{4 - x^2} = 2\sin\theta \,.\, 2\cos\theta\,;$$

$$\therefore\ \frac{dA}{d\theta} = \frac{dA}{dx}\cdot\frac{dx}{d\theta} = 8\sin\theta\cos^2\theta :$$

$$\therefore\ A = \int 8\sin\theta\cos^2\theta\,d\theta = -\frac{8}{3}\cos^3\theta = -\tfrac{1}{3}(4 - x^2)^{\frac{3}{2}};$$

$\therefore$ the area of one loop is $2\left[-\tfrac{1}{3}(4 - x^2)^{\frac{3}{2}}\right]_0^2$, *i.e.* 16/3.

The substitution $4 - x^2 = u$ would also give a solution.

MISCELLANEOUS EXERCISES.

1. Differentiate $x^4 + 5x + 2$.

Write down the differential coefficient of $\dfrac{x+1}{x+2}$ in the form of a limit, and evaluate this limit. [B.U.]

2. Obtain from first principles the derivative of $1/x$.

Differentiate $\frac{1}{3}x^3 - 2x^{\frac{1}{2}}$, $(2x+3)^{-4}$. [Leeds.]

3. Differentiate with respect to x :

$$\text{(i)}\ \frac{1}{\sqrt[3]{x^2}}, \qquad \text{(ii)}\ x\sqrt{(4x+7)}, \qquad \text{(iii)}\ \frac{\sin x}{x^3}.$$

Shew that the point $(1, -5)$ lies on the curve $y = 2x^2 + 5x - 12$. Find the equation of the tangent to the curve at this point, and deduce the intercepts which the tangent makes upon the axes of coordinates. Check your results by a diagram drawn to scale. [S.W.]

4. Find from first principles the differential coefficients of the functions $\sin x$, $\tan(x+1)$.

Find also the differential coefficients in the case of the functions

$$\sqrt{\frac{x+1}{x-1}}, \quad \tan^{-1}\left(\frac{1}{x}\right), \quad \frac{x+2}{x^2 - x - 1}. \qquad \text{[Durham.]}$$

5. Define the differential coefficient of a function, and from the definition differentiate $\dfrac{u}{v}$, where u and v are functions of x.

Shew that the curves $y = 6 + x - x^2$ and $y(x-1) = x + 2$ touch at the point $(2, 4)$, and find the equation of the common tangent at this point. [Durham.]

6. Explain the geometrical significance of the differential coefficient, and prove the rule for differentiating a product of two functions of x.

Given (i) $\dfrac{dy}{dx} = (x^2 - 1)^2$, find $\dfrac{d^2y}{dx^2}$ and y.

(ii) $y = a\sin^2 x + b\cos^2 x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [Durham.]

7. Prove that the curve

$$y = x^3 - 9x^2 + 24x - a$$

has one maximum and one minimum point.

For what values of a will it touch the x-axis? [Leeds.]

8. Explain the use of the second differential coefficient in connection with maxima and minima.

Find the volume of the maximum cone which can be inscribed in a hemisphere of radius 20 in. with its vertex at the centre of the base of the hemisphere. [Durham.]

9. If y is a continuous function of x, shew that when x increases y increases or diminishes according as $\frac{dy}{dx}$ is positive or negative. Deduce the conditions that y should have a maximum or minimum value.

Find for what values of x between the limits 0 and π the function $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ increases with x, and where are its stationary values? [Durham.]

10. Shew that when the function y has a maximum value $\frac{d^2y}{dx^2}$ is negative.

A circular sector has a given perimeter. Shew that its area is a maximum when the angle of the sector is two radians. [Durham.]

11. Using scales of $\frac{1}{2}$ inch = unity along each axis, sketch the curve

$$5y = (9 - x)(x + 3)$$

from $x = -4$ to $x = 10$. Calculate the area of that portion of the curve which lies above the x-axis. [Leeds.]

12. Integrate $(2x + 1)^{\frac{1}{2}}dx$, and calculate

$$\int_1^2 (x^3 + x^{-3})\,dx.$$

Find the total weight of a rod 28 inches long if the weight w lb. per unit length at distance x from one end is given by

$$w = \tfrac{1}{12}(x + 10)^{\frac{1}{2}}.$$ [Leeds.]

13. The area bounded by the curve $y^2 = 4ax$, the y-axis and the lines $y = 3a$, $y = -3a$ revolves about the y-axis. Find the volume generated. [Leeds.]

14. In the curve $y = 4x - x^2$, obtain the area cut off by the axis of x, and the volume formed when this area rotates about the axis of x. [Durham.]

15. Integrate $\frac{2}{(6x-5)^3}$, $\sqrt{2x-3}$, $\sin^2 x$.

Find the volume of the solid obtained by rotating the ellipse $4x^2 + 9y^2 = 36$ about the x-axis. [Durham.]

16. Prove that the area of a zone of a sphere is proportional to the distance between the planes which bound the zone.

Calculate the number of ordinary square miles in the tropics (that is, between latitude $23\frac{1}{2}°$ N. and $23\frac{1}{2}°$ S.) given that the Admiralty nautical mile is 6080 ft. long and that this arc subtends an angle of $1'$ at the earth's centre. [S.W.]

17. Prove that the curved area of any spherical segment is equal to that of the circle whose radius is the distance from the boundary of its base to its highest point.

How far from the centre of a sphere, radius 10 inches, is the base of a segment whose total area is equal to that of the hemisphere ? [S.W.]

18. If A denotes the area between a curve in the first quadrant, the axes OX, OY and the ordinate drawn to the point $P(x, y)$ on the curve, prove that $\frac{dA}{dx} = y$; and shew how to calculate the area when the equation of the curve is given.

Find the area between the curve $y = \frac{1}{2}(x^3 + 3x^2 + 4)$, the x-axis and the ordinates $x = 2$, $x = 5$. [S.W.]

19. A curve goes through the origin, and its slope at any point (x, y) is $4x - \frac{3}{2}x^2$. Prove that its equation is $2y = x^2(4 - x)$.

Find the area intercepted between the curve and the axis of x, and the volume of the solid formed by the rotation of the same area about the axis of x. [Durham.]

20. Find y in terms of x, given :

(i) $\frac{d^3y}{dx^3} = 60x^2 - 48x + 30$,

(ii) $y = 0$ when $x = 0$,

(iii) $y = 8$ when $x = 1$,

and (iv) $\frac{d^2y}{dx^2} = 102$ when $x = 2$.

[Leeds.]

21. What does the area under (i) a force-time, (ii) a force-space curve represent ?

22. (*a*) A body of weight 200 lb. moves in a straight line under a force which varies with the time according to the following table. What is the velocity of the body after 30 secs.?

Time in seconds -	0	5	10	15	20	25	30
Force in pounds -	10·0	9·0	8·2	7·4	6·7	6·0	5·4

(*b*) The accelerating force of a train weighing 290 tons varies with the distance according to the following table. Find the velocity of the train after travelling 1000 ft.

Distance in feet -	0	100	200	300	400	500	600	700	800	900	1000
Force in tons -	5·5	5·5	4·7	3·3	2·4	1·8	1·3	1·0	0·6	0·3	0

23. If the tangent and normal at P to the curve $ay^2 = x^3$ meet the axis of X in the points T and G respectively, and if N is the foot of the ordinate at P, prove that $8aNG = 27NT^2$.

24. Find the equation of the tangent and normal at the point $(a\cos^3\theta, b\sin^3\theta)$ on the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

25. A chord of length $2c$ divides a circle of radius a into two segments. If A is the area of either segment and θ the angle which its arc subtends at the centre, prove that $\frac{dA}{d\theta} = c^2$.

26. CP, PQ are the crank and connecting rod of a steam engine, C being the centre of the crank-shaft and Q the crosshead. If $CP = a$, $PQ = c$, $CQ = r$ and $QCP = \theta$, express $\cos\theta$ in terms of a, c and r, and find $\frac{dr}{d\theta}$.

27. Find the average height of the portion of the curve

$$y = a\sin^2 x$$

lying between $x = 0$ and $x = \pi$.

28. If the tangent at any point P of the curve $xy = c^2$ meets the axes of coordinates in T, T', shew that P is the mid-point of TT' and that the area of the triangle which TT' cuts off from the axes is constant.

29. Shew, by drawing figures, that

$$\text{(i)}\quad \int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx\,:$$

$$\text{(ii)}\quad \int_a^b f(x)dx = -\int_b^a f(x)dx\,;$$

$$\text{(iii)}\quad \int_0^a f(x)dx = \int_0^a f(a-x)\,dx.$$

30. If $f(x)$ contains $(x-a)^n$ as a factor, prove that $f'(x)$ contains $(x-a)^{n-1}$ as a factor.

Factorise $x^4 - 4x^3 + 16x - 16$, given that it has three equal factors.

31. If s, v, be the displacement in feet and velocity in feet per second of a point moving in a straight line t seconds after the start, prove that the acceleration is $v\,dv/ds$ ft. per sec. per sec.

If $v^2 = a^2 - s^2$, where a is a constant, shew that the retardation varies as s.

32. If Δs is an element of a plane area, x its distance from a fixed line and p the intensity of pressure at that element, then the centre of pressure is at a distance $\bar{x}$ from the line, where

$$\bar{x} = \int px\,dS \Big/ \int p\,dS.$$

Find the depth of the centre of pressure in the following cases:—

(i) Rectangle, sides a, b, whose plane is vertical when the side a is on the surface of a liquid.

(ii) Triangle, base a, height h, whose plane is vertical when (1) its base, (2) its vertex is on the surface of the liquid.

(iii) Circle of radius r, plane vertical, just totally immersed.

33. Work Exercise 32, (i) when the upper edge a is immersed to a depth c; (ii) (1) when the base is immersed to a depth c; and (ii) (2) when the vertex is immersed to a depth c.

34. Find the moments of inertia of:

(i) A rectangle, sides a and b, about a line through its centre parallel to side a.

(ii) A rectangle, sides a and b, about one of the sides b.

(iii) A hoop of radius r about a tangent.

(iv) A disc of radius r about a diameter.

35. O is the centre of a disc of radius r, OA is perpendicular to its plane and of length a. Find the moment of inertia of the disc about a line through A parallel to the plane of the disc.

36. Use the result of Exercise 35 to find the moment of inertia of a cylinder of length $2l$ and radius r about a line through its centre perpendicular to its length.

37. Shew that the rhombus of least perimeter that can be described about a circle is a square.

38. Shew that the vertical angle of the cone of maximum volume which can be inscribed in a sphere is $\cos^{-1} 1/3$.

39. A closed curve $r=f(\theta)$ contains the origin. Shew that its area is given by $\frac{1}{2}\int_0^{2\pi} r^2 d\theta$. Deduce the area of a circle of radius r.

40. Trace, roughly, the curve $r=4+3\cos\theta$, and find its area.

41. Draw the curve $y^2=x^2(x+1)$, and find the area of the loop.

42. Trace the curve $y^2=x(x-1)(2-x)$, and find the volume generated by the rotation of the loop about the axis of X.

43. Sketch the curve $y^2=\frac{x^2(1+x)}{1-x}$. Find the area of its loop. (Put $x=\cos 2\theta$.) Find also the volume generated by the rotation of this loop about the axis of X.

44. Through the point (a, b) a line PQ is drawn cutting the axes OY, OX in P and Q respectively. Shew that the area of the triangle POQ is a minimum when (a, b) is the mid-point of PQ.

45. In Exercise 44, find the angle OQP and the length of PQ when that length is a minimum.

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ANSWERS.

EXERCISES I. Page 7.

1. $x=2,\ y=3$.
2. $x=3,\ y=1$.
3. $x=7,\ y=3$.
4. $x=6,\ y=4$.
5. $x=3,\ y=1/3$.
6. $x=1,\ y=2$.
7. $x=3/2,\ y=5/2$.
8. 11, 8.
9. 30·1, 15·6.
10. 20°, 20°, 140°.
11. A, 4 m.p.h. ; B, 3 m.p.h.
12. 2*s*., 2*s*. 6*d*. per lb.
13. $x=3\cdot23,\ y=1\cdot75$.
14. 44·52, 27·84 years.
15. $x=8/3,\ y=2$; $7\frac{1}{3}$ in.
16. $a\doteqdot-179,\ b\doteqdot3\cdot369$; 837.
17. $a=0\cdot5,\ b=0\cdot6$; 6·5 lb.
18. $x=1,\ y=-1,\ z=2$.
19. $x=-1,\ y=1,\ z=-2$.
20. $x=11/7,\ y=0,\ z=13/7$.
21. $x=13/16,\ y=-11/8,\ z=31/16$.
22. $x=247/57,\ y=52/57,\ z=-125/57$.
23. $x=1,\ y=-1,\ z=2$.
24. $x=2,\ y=3,\ z=4$.
25. $x=1,\ y=\frac{1}{2},\ z=1$.
26. $a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2=0$.
28. $(WV'-W'V)/VV'(V-V')$.
29. $a\doteqdot3\cdot40,\ b\doteqdot-1\cdot10,\ c\doteqdot0\cdot25$.
30. (i) $a\doteqdot-2\cdot17\times10^{-4},\ b\doteqdot2\cdot42\times10^{-6}$;
(ii) $a\doteqdot-5\cdot92\times10^{-4},\ b\doteqdot4\cdot85\times10^{-6}$.
31. $a\doteqdot0\cdot014,\ b\doteqdot0\cdot0042,\ c\doteqdot0\cdot00027$.

EXERCISES II. Page 18.

1. (i) $7\frac{1}{2}$; (ii) $-3,\ 9/5$.
2. $0\cdot31,\ -0\cdot81$.
3. (i) $6,\ 2\frac{1}{2}$; (ii) $(\pm4/3,\ \pm5/3)$.
4. $a=-2,\ b=-5,\ x=-3\frac{1}{2}$; $a=-13/5,\ b=-13/2,\ x=-26/7$.
6. $x^2-2(a+b)x+4ab=0$.
7. $289(x^2-x)+42=0$.
8. $a^2x^2-(b^2-2ac)x+c^2=0$.
9. $q+r=-2(a+b)p/a,\ qr=p^2+c/a$.
10. $(8\pm3\sqrt{6},\ -5\mp2\sqrt{6})$.

11. (i) $(3, 1), (-1\frac{1}{2}, -2)$; (ii) $(\pm 1, \pm 2), (\pm 7\sqrt{13}/13, \pm 2\sqrt{13}/13)$.

12. (i) $(\frac{1}{4}, \frac{1}{2}), (\frac{1}{9}, -\frac{1}{3})$; (ii) $(\pm 2, \pm 1), (\pm 3, \pm\frac{1}{2})$.

13. $\{(77 \pm 3\sqrt{2233})/46, (-28 \pm \sqrt{2233})/23\}$.

14. $(2q^2 - 4qp^2 + p^4)/q^4$; (i) $2/q^2$, (ii) $17/4q^2$.

15. $(1, -2), (-2\frac{1}{2}, 8\frac{1}{2})$.

17. $(2\alpha - 5\beta)(4\alpha - \beta) = 0$.

18. $(p, q), (-p, 2p + q)$.

19. (i) $(6, 3), (3, 6)$; (ii) $(6, 5), (-5, -6)$;
(iii) $(1\frac{1}{2}, -2), (-11, 11/7)$; (iv) $(\pm 2, \mp 1), (\pm\frac{1}{2}, \mp\frac{1}{2})$.

20. (i) $x^2 - 2lx + 4m = 0$; (ii) $mx^2 - lx + 1 = 0$;
(iii) $x^2 - plx + p^2m = 0$; (iv) $x^2 - (2p + l)x + p^2 + pl + m = 0$;
(v) $x^2 - (l^2 - 2m)x + m^2 = 0$; (vi) $mx^2 - (l^2 - 2m)x + m = 0$.

21. (i) $\pm 1, \pm 2$; (ii) $-1, -1, 5, -7$; (iii) $\pm 1, \pm\frac{2}{3}$.

22. (i) $ax^2 + (2a + b)x + a + b + c = 0$;
(ii) $(a + b + c)x^2 - 2(a - c)x + a - b + c = 0$.

23. $(0, 1), (23/11, -58/11)$.

24. $a^2cx^2 + 2x(ac^2 - 2b^2c + a^2b) + c^3 - 2abc + a^3 = 0$.

25. $16/7$, $-49/32$.

28. $x^2 + xy - y^2 + 2$; 0.

29. $(ab' - a'b)^2 + 4(ha' - h'a)(hb' - h'b) = 0$; $k = 4$, $(3, -5/3), (3, -7/4)$;
$k = 8$, $(-1, 15/11), (-1, 21/16)$.

30. $2n/(x + y)$ hours.

31. 12 days.

EXERCISES IV. Page 48.

2. (i) $y = 6{\cdot}0 - 2{\cdot}08x$; (ii) $y = 4{\cdot}3 + 1{\cdot}40x$; (iii) $y = 0{\cdot}60 + 0{\cdot}53x$;
(iv) $y = -0{\cdot}80 + 0{\cdot}40x$; (v) $y = 2{\cdot}6 - 0{\cdot}45x$; (vi) $y = -2 - 0{\cdot}80x$.

3. 4.

4. 1·85.

5. (i) 1·2 ; (ii) 17 ; (iii) 33 ; (iv) $-2{\cdot}75$.

6. (i) $x = 7$, $y = -10$; (ii) $x = 3$, $y = 2$;
(iii) $x = 4{\cdot}36$, $y = -4{\cdot}86$; (iv) $x = 1{\cdot}31$, $y = 3{\cdot}09$.

7. $x = -4{\cdot}15$, $y = -2{\cdot}46$.

8. $x - y = 0$, $x + y = a$.

9. $y = 8 + 1{\cdot}5x$.

10. $y = +14x - 26$; $y = -0{\cdot}071x - 1{\cdot}60$.

11. 21, 24, 21 ; 24.

12. (i) 27 ; (ii) 29·5 ; (iii) 33.

13. $\frac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$.

14. 46·5.

EXERCISES V. Page 58.

1. $N \doteqdot 0{\cdot}63O + 15$; £34, £50, £78.

2. £135, £30.

3. $W = 26P + 2500$; 4320.

4. 119, 161, 213.

5. $a \doteqdot 0{\cdot}016$, $b \doteqdot 75$; 6·8*d.* a yard.

6. $a \doteqdot -43$, $b \doteqdot 11{\cdot}5$.

7. $a \doteqdot -11{\cdot}5$, $b \doteqdot 16{\cdot}1$.

8. $a \doteqdot 11{\cdot}0$, $b \doteqdot 15{\cdot}3$; 16, -14, 22, -20, 31, -10, 49, -35.

9. $W \doteqdot 20P + 250$. **10.** $C \doteqdot 0{\cdot}0167Y + 65{\cdot}5$; 7·2*d.* a yard.

11. $a \doteqdot 8{\cdot}3$, $b \doteqdot 0{\cdot}086$. **12.** $y \doteqdot 46{\cdot}8x + 110$; 10.

13. $K \doteqdot 1{\cdot}55L + 2{\cdot}76$; 7·4, 10·5, 15·2.

14. (0·844, 1·65) ; 58°, 145°, 87°.

15. $W \doteqdot 0{\cdot}285 + 0{\cdot}620U$; $T \doteqdot 0{\cdot}481 + 0{\cdot}970U$; 3·39, 5·33.

16. From $x \doteqdot 4{\cdot}8$ to $x \doteqdot 30$. **17.** (1) 308 min., (2) 43°.

18. $a \doteqdot 2{\cdot}05$, $b \doteqdot -0{\cdot}16$. **19.** $y = 2{\cdot}04 + 0{\cdot}039x$; 2·87, 2·04.

20. $a \doteqdot 0{\cdot}53$, $b \doteqdot 0{\cdot}24$. **21.** $a = 1{\cdot}50$, $b = 0{\cdot}20$.

EXERCISES VI. Page 78.

1. (i) (2, 0) ; 2. (ii) $(-1, 2)$; 3. (iii) $(\frac{1}{2}, -\frac{1}{2})$; $\frac{1}{2}\sqrt{2}$.

2. $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = 2$; 15/8. **3.** 18·2. **4.** 1·7.

5. 0·67, 1·8, $-0{\cdot}58$, $-2{\cdot}9$. **6.** -5 ; 2·29, $-0{\cdot}29$; $-1{\cdot}36$; yes.

7. 3·39. **8.** $-0{\cdot}96$, 2·59.

9. £6800, £10,420, £9100, £6875, £4400 ; about 3*s.* 1*d.*

12. $-4{\cdot}68$, 0·45, 4·23. **13.** $-1{\cdot}52$. **14.** 1, 0·39. **15.** -2, 0·70.

16. (*a*) $-2{\cdot}18$; (*b*) -1 ; (*c*) 1·49, 0·20, $-1{\cdot}69$. **18.** 18, 2.

20. 3, $\frac{1}{3}$. **22.** $-3{\cdot}71$, $-0{\cdot}86$, 1·57. **23.** $\frac{1}{2}$, $-4\frac{1}{2}$.

24. £540 ; 18 days.

26. Cut at $(-1, -1)$, (1, 1) ; touch at $(-\frac{1}{2}, -2)$.

27. 4, 6, 10 in. **29.** $a = -1\frac{1}{2}$, $b = 5\frac{1}{2}$, $c = -2$

30. Max. $-4\sqrt{7}$; Min. $+4\sqrt{7}$; no.

31. $a = 3$, $b = 11$, $c = -4$; $A = 1/13$, $B = -1/13$; $\alpha = \frac{1}{3}$, $\beta = -4$;
2, 2, $\frac{1}{3}$, -4.

32. $6b^2 = 25ac$; $4ax^2 + 2bx + c = 0$.

34. 0·32. **35.** Max. $1 - 2\sqrt{2}$; Min. $1 + 2\sqrt{2}$.

37. $\pm b(b^2 - 2ac)\sqrt{b^2 - 4ac}/a^4$; 1·37, $-4{\cdot}37$.

38. (i) 0 ; (ii) (2, 0), (4/5, 18/5).

39. 2·61 feet. **40.** 1/13 and 5.

EXERCISES VII. Page 87.

1. $6\sqrt{2}$, $6\sqrt{3}$, $2\sqrt[3]{2}$, $5\sqrt[3]{2}$. **2.** $37 - \sqrt{35}$, $3\sqrt{6} - 2\sqrt{3}$.

3. $2\sqrt{6}$, $7\sqrt{3} + 2\sqrt{2}$. **4.** $4\sqrt{7}$, $5\sqrt[3]{2}$, $3\sqrt[4]{3}$.

5. $22\sqrt{3}-27\sqrt{2}$, $\frac{1}{2}(6-\sqrt{10})$, $(12+7\sqrt{2}-5\sqrt{3}-\sqrt{6})/23$, $(a-\sqrt{b})/(a^2-b)$, $(c+\sqrt{d})(\sqrt{a}+\sqrt{b})/(a-b)$.

6. $\frac{1}{2}(5\sqrt{3}+6\sqrt{2}-1)$.

7. $\sqrt{5}+\sqrt{2}$, $5+2\sqrt{6}$, $\frac{1}{2}(\sqrt{14}+\sqrt{22})$, $\frac{1}{2}(3\sqrt{2}+\sqrt{38})$, $\frac{1}{2}(5\sqrt{2}-\sqrt{38})$.

8. 7, -1, 3, $-q$.

9. 2, 7, $\frac{1}{2}(9\pm\sqrt{5})$; -2, 3, $\frac{1}{2}(1\pm\sqrt{61})$; $\frac{1}{2}(2\pm\sqrt{21})$. 10. ± 2.

12. $p \doteqdot -18{\cdot}47$ or $-0{\cdot}53$. If $b>0$, $k^2>1/2b$; if $b<0$, k can have any value.

13. (a) $a\sqrt{b}-2b\sqrt{a}$; (b) $\{(a+b)\sqrt{ab}+2ab\}/ab$; (c) $(x+y)\sqrt{x^2+y^2}$; (d) $(x+\sqrt{x^2-y^2})/y$.

14. 3. 15. 1. 16. $\frac{1}{2}(\sqrt{14}+\sqrt{6})$. 17. 0·168. 18. $\pm(\sqrt{3}\pm\sqrt{2})$.

EXERCISES VIII. Page 93.

1. $a^{\frac{3}{8}}$, $1/a^{\frac{1}{6}}$, $2a^{\frac{7}{3}}$, $\frac{1}{2}xy$. 2. $a^2/b^{\frac{5}{2}}$, $xy^{\frac{1}{2}}$, $(p-q)^2$, 1. 3. b, $y/x^{\frac{5}{2}}$.

4. $9x-4x^{-\frac{1}{3}}$, $x^{3a}-3x^{a}+3x^{-a}-x^{-3a}$, $x^2+2x+3+2x^{-1}+x^{-2}$, $x-y$, x^3-2.

5. $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$. 6. $2ab+2bc+2ca-a^2-b^2-c^2$.

7. $x^{\frac{1}{2}}-y^{\frac{1}{2}}$, $a^{\frac{2}{3}}+a^{\frac{1}{3}}+1$, $1/(x^{\frac{1}{3}}-y^{\frac{1}{3}})$. 8. $p+p^{\frac{1}{2}}q^{\frac{1}{2}}-q$.

9. $a^{\frac{3}{2}}-3ab^{\frac{1}{3}}+3a^{\frac{1}{2}}b^{\frac{2}{3}}-b$.

10. $2a^{\frac{1}{2}}-3-a^{-\frac{1}{2}}$, $2xy^{-1}-3+4x^{-1}y$, $2x^{\frac{1}{3}}y^{-\frac{1}{4}}+3+5x^{-\frac{1}{3}}y^{\frac{1}{4}}$.

11. $(3a^{\frac{1}{2}}+1)/(a^{\frac{1}{2}}-1)$.

12. $2^{\frac{2}{3}}+2^{\frac{1}{3}}+1$, $\frac{1}{2}(3^{\frac{1}{3}}-1)$,
$a(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})/(b+c)$,
$x(y^{\frac{1}{4}}+z^{\frac{1}{4}})(y^{\frac{1}{2}}+z^{\frac{1}{2}})/(y-z)$,
$(a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}+c^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}-c^{\frac{1}{2}})/(a^2+b^2+c^2-2ab-2bc-2ca)$.

13. (1) $3x^2+9x+7$; (2) $a^{\frac{5}{6}}c^2/b^{\frac{1}{3}}$. 14. $x^2=az$. 15. $x^{0\cdot 7}-2x^{0\cdot 5}+3x^{0\cdot 3}$.

EXERCISES IX. Page 99.

1. (i) 7·75 ; (ii) 0·000692 ; (iii) 3·02 ; (iv) 0·760 ; (v) 0·0812.

2. (*a*) 303 ; (*b*) 2·67 ; (*c*) 17·5 ; (*d*) 8·50. 3. 1·61, 39·8.

4. 3·13. 5. 134, 1·34, 0·0134. 6. 2·06. 7. 8·26, 1·90. 8. 4·61.

9. 14·1. 10. 7·45, 0·0125, 5·68, 1550. 11. 1·72, 0·0198.

12. 3·34, 1·93, 1770000, 11·0. 13. 1·56, 195.

14. (i) 1·11 ; (ii) 1·46 ; (iii) 5·09. 15. 0·172.

16. 0·949. 17. 1·56, 1·69, 0·590.

EXERCISES X. Page 106.

1. 0·702. **2.** 1·35. **3.** 420. **4.** 27·6. **5.** 7,54. 127.

6. (i) 1·27 ; (ii) 0·29. **7.** (i) 8·16 ; (ii) 4570.

8. 7·48, 5·98, 5, 4·32. **9.** 320. **10.** $a=1{\cdot}1$, $b=3{\cdot}2$; 13·275.

11. $a=-2{\cdot}6$, $b=-36{\cdot}4$. **12.** 2920 ohms.

13. 2·10, 3·45, 4·41. **14.** 1·62. **15.** 957. **16.** 7480.

17. 0, −5·208, −5·063, −5·056, −5·046, −5·042, −5·013, −5·001 ; 10.

18. $s=2360$, $t=1{\cdot}37$. **19.** (*a*) 0·90 ; (*b*) 0·88. **20.** 0·23.

21. 118½ sq. ft. **22.** 5%. **23.** 14. **24.** £281.

25. £364. **26.** Pay £300 at once.

27. £111 at end of fourth year. **28.** One year.

29. The £1600 machine.

EXERCISES XI. Page 111.

2. 3, −2, −2. **3.** 2·58, 0·166, 1·43, 1·31, −0·994.

6. (i) 2·55; (ii) 2·71; (iii) 0, 0·431; (iv) 0·431, 0·683; (v) ±1·61.

7. 0·354, 1·49. **9.** 0·0536. **10.** 2, 3 ; 1·03.

11. $1{\cdot}94\times10^{5}$, $2{\cdot}35\times10^{15}$. **12.** 81·3.

13. 3·10 ; $A=-26{\cdot}2$, $m=-0{\cdot}481$, $C=36{\cdot}2$. **14.** Nearly 1800.

15. 1·65 cubic ft. **16.** 1·34. **17.** 81.

18. 1·01, 0·403. **19.** $x=2{\cdot}66$; $a=6{\cdot}43$; $b=1{\cdot}49$.

20. 1·03. **21.** $x=1{\cdot}20$, $y=16{\cdot}6$. **22.** 1·10.

23. 0·717, 0·144 ; $x=19{\cdot}6$. **24.** 6490.

25. 6×10^{-30}. **26.** $a=825$, $n=1{\cdot}30$. **27.** 41000, 0·0522.

EXERCISES XII. Page 118.

1. $y=1{\cdot}60+0{\cdot}40x^2$. **2.** $y=2{\cdot}85+1{\cdot}33/x$. **3.** $H=125+1{\cdot}26V^3$.

4. $V=41{\cdot}2+21{\cdot}8/A$. **5.** $V=2/\mu$ nearly. **6.** $p=3{\cdot}5+2{\cdot}8s^2$.

7. $\log n=1{\cdot}15-0{\cdot}63\log d$; 200.

8. $\log C=2{\cdot}64+0{\cdot}78\log A$; 9·1 amp. ; 0·092 sq. in.

9. $p=0{\cdot}08d+0{\cdot}04$. **10.** $\log t=0{\cdot}98+0{\cdot}76\log v$.

12. $1/u=0{\cdot}0045+0{\cdot}0023p$. **13.** $pv^{1{\cdot}13}=100$.

14. $y=17{\cdot}4e^{0{\cdot}273x}$. **15.** $w=5{\cdot}63+467/(p+50)$.

16. $1/u=0{\cdot}025+0{\cdot}0021p$. **17.** $U=H(1-0{\cdot}00000353H^2)$.

18. $P=32{\cdot}0v^{1{\cdot}76}$. **19.** $pv^{0{\cdot}855}=205$. **20.** $y=3{\cdot}65\sqrt{x}$.

EXERCISES XIII Page 128.

1.

θ	28°	57°	72°	80°
sin θ	0·47	0·84	0·95	0·98
cos θ	0·88	0·54	0·31	0·17
tan θ	0·53	1·54	3·08	5·67
cosec θ	2·13	1·19	1·05	1·02
sec θ	1·13	1·84	3·24	5·76
cot θ	1·88	0·65	0·32	0·18

2. 14° 30′, 30°, 48° 30′.
72° 30′, 60°, 37°.
26° 30′, 56° 30′, 63° 30′.
48°, 60°.
42°, 23° 30′.
63° 30′, 35° 30′, 26° 30′.

3. 1·81, 1·51 ; 0·004.

4. $1\frac{1}{3}$, 0·25.

5. $q/\sqrt{q^2-p^2}$, $p/\sqrt{q^2-p^2}$, $\sqrt{q^2-p^2}/p$.

8. $1/x$, $\sqrt{x^2-1}/x$, $1/\sqrt{x^2-1}$, $x/\sqrt{x^2-1}$, $\sqrt{x^2-1}$.

9. $1/\sqrt{1+y^2}$, $y/\sqrt{1+y^2}$, $\sqrt{1+y^2}/y$, $\sqrt{1+y^2}$, $1/y$.

11. (i) 2 ; (ii) 1 ; (iii) $\sec^2 B$; (iv) 0.

14. $\cos\theta = 2pq/(p^2+q^2)$, $\tan\theta = (p^2-q^2)/2pq$.

16. $2p/\sqrt{3}$. 17. 45° 14′.

18. (i) $BC = x \tan\theta$, $CA = x \sec\theta$; (ii) $AB = x \sin\theta$, $BC = x\cos\theta$;
(iii) $AB = x \cot\theta$, $AC = x \operatorname{cosec}\theta$;
(iv) $AB = x\sin\theta$, $BC = x\cos\theta$.

EXERCISES XIV. Page 134.

1. 98·7 ft. 2. 204 ft. 3. 30·6 in., 70·7 sq. in.
4. 4·08 in. 5. 7° 56′. 6. 39·1 sq. in. 7. 1·19 miles.
8. 320 ft. 9. 7·11 in., 3·48 in.
10. 57° 43′ N. of E. ; 5·98 miles. 11. 13·8 miles ; 53° 55′ W. of N.
12. 2·83 in. 13. $r\sqrt{(40-6\sqrt{3})/3}$. 14. 6·94, 11·2, 9·83 cm.
15. 14·2 in. 21. $B = 43°\ 41'$, $C = 74°\ 2'$, $c = 341$.
22. 103, 462 ft. 23. 92·2 ft., 62·2 ft.
24. $\frac{1}{2}na^2 \sin(360°/n)$, $na^2\tan(180°/n)$. 25. 1·39 miles.
26. 1·14 ft. 27. $2a\sin\theta - h \operatorname{cosec}\alpha \sin(\theta+\alpha)$.
28. 30·5 feet per sec., 35°.
29. 456 ft., S. 11° 41′ E. 30. $a \operatorname{cosec}(180°/n)$, $a\cot(180°/n)$.
31. 885 yards, 557 yards ; 1043 yards.

32. 9·54, 9·16 miles ; 13·2 miles ; 46° 9′ E. of N.

33. 294 ft. **34.** 7670 ft.

35. 110 ft., 128 ft., 42·1 or 174 ft. **39.** 38·5 ft. **40.** 5·87 miles.

EXERCISES XV. Page 147.

1. 35° 16′. **2.** 20° 42′.

3. 3600 ft. **4.** 65·7, 30·6 sq. ft. **5.** 293,000 cub. ft.

6. 3·54 sq. in. ; 7000 lb. per sq. in., 4950 lb. per sq. in. ; 4950 lb. per sq. in. each way.

7. 41° 24′ ; 55° 2′, 12° 41′. **8.** 1300 ft.

9. 280 ft. **10.** 27° 37′.

11. 4·69 ft. **12.** 70° 32′. **13.** 120°. **14.** 33·6 in.

15. $h\sqrt{\cot^2\beta-\cot^2\alpha}$; $h\cot\alpha\cot CBA$.

16. (i) 54° 28′ ; (ii) 80° 38′.

17. (i) 66° 9′ ; (ii) 77° 33′ ; (iii) 6·56 cm. ; (iv) 64° 32′ ; 64·7 c.c.

18. 92 ft. 1 in.

19. $\frac{1}{2}(\operatorname{cosec}\frac{1}{2}\alpha-1)\tan\beta$;
$d/(\cot\beta-\cot\gamma)$, $2d\cot\beta/(\cot\beta-\cot\gamma)(\operatorname{cosec}\frac{1}{2}\alpha-1)$.

20. 18·1 in., 1040 cub. in., 69° 11′. **21.** 5° 44′.

23. 36° 38′ ; 45°, 48° 1′ ; 2900 cub. ft.

24. 81° 57′ ; 2·62 cub. ft., 51° 15′, 66° 44′.

25. (i) 7·76 ft. ; (ii) 13° 57′, 40° 41′ ; (iii) 94·5 cub. ft.

26. 21 ft. horizontally from the deep end ; 41,000 cub. ft. ; $3\frac{1}{2}$ ft.

EXERCISES XVI. Page 163.

2. $2\pi/15$, $\frac{2}{3}\pi$, $\frac{3}{4}\pi$, $55\pi/36$, $-3\pi/40$.

3. 120°, 150°, 157° 30′, 270°, $1800^\circ/\pi$. **4.** 15·6″.

5. (i) 6·98 ft. ; (ii) 54·1 ft. **6.** 25·8 ft. **7.** About 1 mile.

8. 0·589 mile. **9.** $2\pi/9$, $3\pi/9$, $4\pi/9$. **10.** $5\pi/12$.

11. $\frac{1}{2}r^2(\alpha-\sin\alpha)$. **12.** 1420 yards. **13.** $\theta/2\pi$, 47° 10′.

14. 24,900 sq. miles. **15.** 17′ 22″. **16.** 198 ft.

17. 900 ft. **19.** 10·6 miles per min.

20. 24° 19′. **21.** $6\frac{1}{2}$ miles.

22. 52·9 miles ; 44·5 miles. **23.** 55 miles, 47′. **24.** 533 ft.

EXERCISES XVII. Page 174.

4. $\sin(3\pi/2+x)=-\cos x$, $\cos(3\pi/2+x)=\sin x$, $\tan(3\pi/2+x)=-\cot x$.

5. (1) $\sin x^\circ$; (2) $-\sin x^\circ$; (3) $\tan x^\circ$; (4) $-\sec x^\circ$.

6. $\sin(\frac{1}{2}\pi+x)=\cos x$, $\cos(\frac{1}{2}\pi+x)=-\sin x$, $\cot(\frac{1}{2}\pi+x)=-\tan x$.

7. (i) -320°, -220°, 40°, 140°; (ii) -310°, -50°, 50°, 310°; (iii) -300°, -120°, 60°, 240°.

8. $-59/15$. 9. $-129/52$.

10. (i) $n\,.\,180^\circ+(-1)^n\,30^\circ$, $k\,.\,180^\circ+(-1)^k\,14^\circ\,29'$;
(ii) $n\,.\,180^\circ+38^\circ\,40'$, $k\,.\,180^\circ+18^\circ\,26'$;
(iii) $n\,.\,180^\circ-(-1)^n\,30^\circ$; (iv) $n\,.\,180^\circ\pm60^\circ$; $(2k+1)\,45^\circ$;
(v) $n\,.\,180^\circ\pm60^\circ$, $k\,.\,180^\circ\pm30^\circ$;
(vi) $n\,.\,180^\circ+45^\circ$, $k\,.\,180^\circ+60^\circ$.

11. 7·90 sq. in. 12. (i) $n\,.\,360^\circ\pm74^\circ\,12'$; (ii) $k\,.\,360^\circ\pm70^\circ\,56'$.

13. $n.\,180^\circ+63^\circ\,26'$; (i) $n\,.\,180^\circ\pm60^\circ\,20'$; (ii) $(2k+1)\pi/2(n+1)$.

14. 79°. 15. $n\,.\,180^\circ\pm60^\circ$.

16. (i) 38·9 cm.; (ii) 633 sq. cm.; (iii) 140 sq. cm.

17. 52°, 101°, 180°, 259°, 308°.

18. $\theta=2n\pi/9$ or $2k\pi/5$. 19. Angle $PAB=42^\circ\,21'$.

20. $-0{\cdot}15$, $1{\cdot}29$. 21. -3, $-2{\cdot}73$, $-1{\cdot}27$, $0{\cdot}55$, $1{\cdot}79$, $2{\cdot}34$.

22. $0{\cdot}23\pi$, $0{\cdot}94\pi$. 23. 0·26, 1·63, 2·98. 24. 0·75, 2·67.

EXERCISES XVIII. Page 183.

1. (a) $y=3x-1$; (b) $2y=x+12$; (c) $3y+x+9=0$; (d) $3y+2x=12$.

2. (a) $2y=x+3$; (b) $y=x+1$; (c) $y+x+3=0$.

3. (a) $y=3x-1$, $3y+x=7$; (b) $2y+x=5$, $y-2x=0$; (c) $2x-3y+4=0$, $3x+2y=7$.

6. $(1,\ -1)$. 7. $2x-3y+5=0$. 8. $x+3y=1$.

9. $(0,\ 1)$. 10. $bx-ay=bx'-ay'$; $(2\frac{1}{2},\ 4\frac{1}{2})$.

11. $(nx_1+mx_2)/(n+m)$, $(ny_1+my_2)/(n+m)$; $(0,\ 0)$, $(4/3,\ 2/3)$, $(4,\ 6)$, $(8/3,\ 16/3)$.

12. $(3,\ 3)$, $(6,\ -6)$, $(0,\ 0)$; $3y+x=0$.

13. AC externally in the ratio 1 : 3; BD internally in the ratio 3 : 1.

14. $(-36/7,\ -45/7)$.

15. (1) $3x+4y=7$, $x+y=3$, $x+2y=5$; (2) $4x-3y=1$, $x-y=3$, $y-2x=5$; (3) $3y-4x=2$, $y-x=7$, $2x-y=12$; $(19,\ 26)$.

EXERCISES XIX. Page 189.

1. (i) $3x+2y=6$; (ii) $2y-3x=6$;
(iii) $7x+5y=35$; (iv) $2x+3y+12=0$.

2. (i) $4x+3y=12$; (ii) $5x+2y+10=0$.

3. $y-x=5$, $y+x=3$. **4.** $-c/a$, $-c/b$.

5. (i) 36° 52′, 1·2 ; (ii) −53° 8′, 1·2 ; (iii) 157° 23′, 1 ;
(iv) −112° 37′, 1.

6. $x/2h+y/2k=1$. **7.** $y-2x=0$. **8.** 118° 4′, 1.

9. $y^2=ax$. **10.** $y^2-2ax+a^2=0$.

11. $4(x^2+y^2)=c^2$; $9x^2+36y^2=4c^2$, $36x^2+9y^2=4c^2$.

12. $(a-b)y-ax+b^2=0$. **13.** $x(2-x)^2=y(2x-xy-4)$.

EXERCISES XX. Page 196.

1. (a) 1/5 ; (b) 1 ; (c) $23/\sqrt{194}$. **2.** 2/5, 0, $1/\sqrt{194}$.

4. 16/5. **5.** $5x^2+5y^2-10x-20y+9=0$.

8. $6x+8y=65$, $6x+8y+85=0$. **9.** $38y+19x=21$.

10. $x+9y-6=0$. **11.** $17x+15y+5=0$, $13x+21y+1=0$.

12. $3x+2y=7$. **13.** 1. **14.** 5·6.

15. $b(x^2+y^2)+2a^2y-a^2b=0$; $(0,\ -a^2/b)$.

16. $(lx_1+kx_2)/(l+k)$, $(ly_1+ky_2)/(l+k)$; $2xy=3x-6y$.

17. $(x-2a)(x^2+y^2)+a^2x=0$.

EXERCISES XXI. Page 204.

1. (i) $3x+4y=25$; (ii) $2x-3y=13$; (iii) $3x+2y=21$;
(iv) $x-2y=7$; (v) $3x+5y=1$.

2. $y-x\sqrt{3}=\pm 4$; $y+x\sqrt{3}=\pm 4$. **3.** $\frac{1}{2}(\pm 3\sqrt{5}-8)$. **5.** $2\sqrt{5}$.

7. $2x^2+2y^2+2x-5y=12$; $2x-y=4$, $2x+y+6=0$, $2x+11y=44$.

8. (i) $k=\pm a\sqrt{1+m^2}$; (ii) $k=mg-f\pm\sqrt{(1+m^2)(g^2+f^2-c)}$.

9. $my+x=0$. **10.** $my+x+mf+g=0$.

11. (i) $x^2+y^2-2x=0$; (ii) $x^2+y^2-2x-y=0$.

12. $x^2+y^2+6x-8y-40=0$. **13.** (i) $l/l'=m/m'$; (ii) $ll'+mm'=0$.

15. (29/55, 3/55).

17. $\{-(g+fm)/(1+m^2),\ -m(g+fm)/(1+m^2)\}$; (4/5, −8/5).

18. $c(1+m^2)=(g+fm)^2$; $x^2+y^2-4x+2y+4=0$, $4x+3y=0$.

19. $2x^2+2y^2-8x-5y+8=0$. **20.** $\frac{1}{2}\sqrt{A^2+B^2-4C}$; (3, 1).

21. $3x-2y=0,\ 2x+3y=0.$ 22. $7x+y=0,\ x-7y=0.$

23. $y=0,\ 12x=5y.$

24. $x^2+y^2=a^2+b^2$, the radii of the given circles being a and b.

26. $4x-7y=65,\ 8x+y=65$; $(4,\ -7),\ (8,\ 1)$.

27. $(6,\ 7),\ \{(6m^2-14m-6)/(1+m^2),\ (-7m^2-12m+7)/(1+m^2)\}$;
$6x+7y=85.$

28. $x^2+y^2-2x-4y+4=0$; $(-1,\ 0),\ (\frac{3}{2},\ \frac{5}{2})$.

29. $x^2+y^2-2x-4y=0.$

EXERCISES XXII. Page 211.

1. (i) $x+2y+1=0$; (ii) $7x-y-1=0$; (iii) $9x-6y-4=0$.

2. $(0,\ -3)$.

3. (i) $x^2+y^2-7x-13y+12=0$; (ii) $7x^2+7y^2-34x+30y-9=0$;
(iii) $x^2+y^2+3x-y=0$.

4. $x^2+y^2-x-2y=0.$

5. $3x^2+3y^2-16x+4y=0$; $\sqrt{68/3}$; $(8/3,\ -2/3)$; $4x-y=0$.

6. $4(x^2+y^2)-24x-12y-19=0,\ x^2+y^2+120x+60y+11=0.$

9. $4x-3y=0,\ 4x+3y=0$; $x=\pm 3$; $y=\pm 4$.

EXERCISES XXIII. Page 214.

1. (a) $3x-y+1=0,\ x+2y-1=0$; (b) $3x+y=0,\ x+3y=0$;
(c) $2x+y=0,\ 2x-y+1=0$; (d) $2x+3y+2=0,\ 5x-y+6=0$;
(e) $ax-by=0,\ bx-ay=0$.

3. $m=-8,\ n=3.$

EXERCISES XXIV. Page 216.

1. (i) $h=0$; (ii) $a=b$. 2. $9ab=8h^2$. 3. $h+3b=0$.

5. $ax^2+2hxy+by^2=0$. 7. $bx^2-2hxy+ay^2=0$.

EXERCISES XXV. Page 221.

1. (a) $x^2-4xy-y^2=0$; (b) $7x^2+2xy-7y^2=0$; (c) $x^2-y^2+14xy=0$.

5. $c(x^2+y^2)-2(ax+by)(px+qy)=0$; $ap+bq=c$.

6. $16x^2-24xy+9y^2=0$; the line $3x+4y=25$ touches the circle.

7. $(aa'-bb')^2+4(ha'+h'b)(hb'+h'a)=0$.

8. $(ap^2+2hpq+bq^2)/\sqrt{(a-b)^2+4h^2}$. 9. $4x^2-12xy+9y^2=0$.

10. (a) 12 ; (b) 1 ; (c) 24 ; (d) 11, $-196/15$.

EXERCISES XXVI. Page 226.

1. (i) 11; (ii) 1; (iii) $-35/27$; (iv) 0. **2.** -4.

3. (i) $(x+1)(x+2)(x+3)$; (ii) $(x-3)(x-2)(x-1)(x+1)$;
(iii) $(x-1)(x^2+x-1)$; (iv) $(x-4)(x^2+4x+1)$;
(v) $(x-2)(x-1)(x+1)^2(x+2)$.

4. (i) $-(a-b)(b-c)(c-a)$; (ii) $3(a-b)(b-c)(c-a)$;
(iii) $-(a-b)(b-c)(c-a)(ab+bc+ca)$;
(iv) $-(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)$;
(v) $(x+y)(y+z)(z+x)$.

7. (i) $-1, -1, 2$; (ii) $-2, 1, 5$; (iii) $-1\cdot5, 1\cdot2, 2$;
(iv) $1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{(4a-3b-c)/(b-c)}$.

8. $1, 2$. **9.** $(b-c)(x-b)(x-c)$. **10.** $(x+y)(2x-3y)(2x+7y)$.

11. $(x-2)\{(b-c)x-(a-b)\}$; $(b-c)(c+a)(a+b)$.

12. $-2, -3/5, 5/8$. **13.** $-1\frac{1}{2}, \frac{2}{3}, 3$. **14.** $-1\cdot3, -1, -0\cdot9$.

15. $-0\cdot5, -0\cdot5, 0\cdot5, 1\cdot2$. **16.** $-1\frac{1}{2}, \frac{3}{4}$.

17. (i) $a+b+c+d+e$; (ii) $a-b+c-d+e$; $a=1, b=9$.

18. $A=5, B=-9, C=5, D=-1$. **19.** $A=0, B=1, C=3, D=1$.

21. $(x+y-a+2b)(xy+\overline{2a+2b}\,.\,x-\overline{3a+2b}\,.\,y-10a^2-6ab-2b^2)$.

EXERCISES XXVII. Page 230.

1. $5/(x+1)-1/x$. **2.** $x^2-2x+2-2/(x+1)$.

3. $1/6(x+0\cdot1)+5/6(x-0\cdot5)$.

4. (i) $1/2(x-1)-1/2(x+1)$; (ii) $x-\frac{1}{2}+7/2(2x+1)$;
(iii) $1+1/(x-1)-1/(x+1)$; (iv) $3\cdot5/(x-1)+2\cdot7/(x+2)$;
(v) $5/6(x+2)-5/6(x+3\cdot2)$; (vi) $11/7(x-4)+3/7(x+3)$;
(vii) $11/7(x+4)+3/7(x-3)$; (viii) $1-17/6(a+4)+5/6(a-2)$;
(ix) $19/16(x-5)-3/16(x+3)$; (x) $1\cdot16/x+1\cdot84/(x-1\cdot5)$.

5. $1/(x+4)+2/(x-3)$. **6.** $1/(x-4)-1/(x-3)$.

7. $2/(x+3)-1/(x-5)$. **8.** $7/8x+41/24(x+4)+5/12(x-2)$.

9. $67/50(x+6)-12/25(x+1)-43/50(x-4)$.

10. $A=3, B=1, C=1$; $A=2, B=-10, C=9$.

11. $5/2(x-1)-7/10(x+1)-18/5(2x-3)$.

EXERCISES XXVIII. Page 232.

1. $-2, 1, -7, 0$. **2.** $25, -23, 2, -32$; $-9, 164, 264, -429$.

3. $abc+2fgh-af^2-bg^2-ch^2, 0, a^3+b^3+c^3-3abc$.

EXERCISES XXX. Page 240.

1. $4abc$. 2. 0. 3. -418. 4. $4xyz$.

7. (*a*) $11\frac{1}{2}$; (*b*) $-1\frac{1}{2}$; (*c*) -13 ; (*d*) -162.

8. (*a*) $x=1, y=1, z=1$; (*b*) $x=13/3, y=-125/57, z=52/57$;
(*c*) $p=2, q=4, r=3$.

10. $x=a/(a-b)(a-c), y=b/(b-c)(b-a), z=c/(c-a)(c-b)$.

11. $x=a, y=b, z=c$. 13. $Q\ (2, 1)$; $R\ (\frac{3}{4}, -1\frac{1}{2})$; $3\frac{3}{4}$.

14. $mx-ly=ma-lb$; 289/60. 15. 17.

16. Calling the given points A, B, C, D the rule on p. 240 shews that the formula gives $\triangle ADC - \triangle ABC$; this is zero since AC is parallel to DB.

17. $x=1, y=0, z=0$.

18. $x=l(l-b)(l-c)/a(a-b)(a-c)$; $y=l(l-c)(l-a)/b(b-c)(b-a)$;
$z=l(l-a)(l-b)/c(c-a)(c-b)$.

19. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$ 20. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

EXERCISES XXXI. Page 256.

2. 3, 7, 11. 5. $S/S'=(2n+1)/(n+1)$. 6. 0·010309.

7. 8 at least. 10. 18/55, 817/990, 4/7. 14. $pq/(p+q)$.

17. $\frac{1}{2}n(n^2+1)$. 18. £60,000. 19. 5% ; 3*s*. 6*d*.

20. £1620. 21. £21,900. ; $A=3550$. 22. 7·00 in.

23. $-17, 3$; $-\frac{4}{3}(1+2^{-19})$; $2(1-3^{-19})$; 22.

24. $\frac{4}{3}\{1-(-\frac{1}{2})^n\}$; $\frac{4}{3}$.

25. Blue, $n(3n-2)\pi$ sq. in. ; Red, $3n^2\pi$ sq. in. ; Yellow, $n(3n+2)\pi$ sq. in.

26. 11, $10\frac{1}{2}$, 10, ... 27. 202 sq. in. ; 32·5 sq. in. 28. 32.

30. 16. 31. $60+20\sqrt{3}, 60-20\sqrt{3}$ gallons per hour. 32. £118.

34. $NR^{-q}/(1-R^{-p})$ where $R=1+r/100$. If A receives his first £100 a year hence, he should pay B now £24·5.

35. 9 years. 36. £8720. 37. £401. 38. 21·85 years.

EXERCISES XXXII. Page 270.

1. (i) 120 ; (ii) 720. 2. (i) 15120 ; (ii) 59,049.

3. (i) 64 ; (ii) k^n. 4. 210 ; 30. 5. 2880.

6. 120. **7.** 72. **8.** 720. **9.** 480.
10. (i) 48; (ii) 68. **11.** 181,439. **12.** $|\underline{14}/24$; $|\underline{11}$; $|\underline{13}/24$; $|\underline{12}/24$.
13. $(n-2)(n-3)\,|\underline{n-2}$. **14.** 47, excluding 1 as a factor. **15.** 2880.
16. 246. **17.** 53. **18.** 7. **19.** 20. **20.** 54,054.
21. 342. **22.** 480; 252. **24.** 960. **25.** (*a*) 70; (*b*) 65.
26. $|\underline{8} \times |\underline{6}$. **27.** 1295. **28.** $|\underline{9} \times 14$.

EXERCISES XXXIII. Page 282.

1. $1+5x+10x^2+10x^3+5x^4+x^5$, $a^4-4a^3b+6a^2b^2-4ab^3+b^4$,
$x^7+7x^5+21x^3+35x+35/x+21/x^3+7/x^5+1/x^7$,
$243x^5-135x^4y+30x^3y^2-\frac{10}{3}x^2y^3+\frac{5}{27}xy^4-\frac{1}{243}y^5$, 98,
$a^{13}+13a^{12}b+78a^{11}b^2+286a^{10}b^3+715a^9b^4+1287a^8b^5+\dots$.
2. 1·030301, 0·59049, 0·998001, 84·28892481.
3. $70x^4$, $62720x^8$, $10\frac{1}{2}$, $-\frac{12920}{243}p^{14}a^7$, $252x^5$, $\frac{308}{243}x^6y^6$.
4. $126x^5$, $-126x^4$; $1716x^7y^{-6}$, $1716x^6y^{-7}$; $0{\cdot}028x^4$, $0{\cdot}56x^3$.
5. ${}_{11}C_5 \times 2^5$, ${}_{17}C_6 \times b^6$, $-{}_9C_3 \times 3^6$, ${}_5C_2 \times 2^3 . 3^2$, ${}_{17}C_6a^{11}b^6$.
6. 70, 29120. **7.** 378, −420, 135.
8. $32a^5+240a^4b+720a^3b^2+1080a^2b^3+810ab^4+243b^5$.
9. 28/27. **10.** $1-11x$, $1+5x$, 64.
12. 462, $41415\frac{1}{9}$, ${}_{25}C_{19} \times 3^{19}$. **13.** 86016, ${}_{13}C_4 10^4 . 20^9$.
14. $(3/2)^7$. **15.** 7/8.
20. $(4/3)^n-1$; $1-5x^2+\frac{5}{3}x^3+10x^4-\frac{20}{3}x^5-\frac{80}{9}x^6$.
22. ${}_nC_ra^{n-r}$; ${}_7C_3(1+x^2)^3 . (3/x)^4$. **23.** ${}_{15}C_2 100^{13} . 15^2$.
24. $(2x)^{17}-{}_{17}C_1(2x)^{16}(3y)+{}_{17}C_2(2x)^{15}(3y)^2-{}_{17}C_3(2x)^{14}(3y)^3$
$+{}_{17}C_4(2x)^{13}(3y)^4-\dots$; ${}_{17}C_3 3^{14}/2^{11}$; 12th term is $-{}_{17}C_6 2^{28}$,
13th term is ${}_{17}C_5 2^{29}$: these are numerically equal.
26. 0. **27.** 1·0564. **28.** $(1+x)^9$; 10^{18}, 10^{-18}.
29. (i) ${}_{17}C_7 2^7 . 3^{10}$; (ii) ${}_{17}C_3 2^3 . 3^{11}$; (iii) ${}_{17}C_4 3^4 . 2^5$; 0·984.
30. $17 \times 2^{18}/9$; 2·3076.
32. $-{}_nC_3(2x)^{n-3}(3a)^3$; $(-1)^{r-1}{}_nC_{r-1}(2x)^{n-r+1}(3a)^{r-1}$.
33. ${}_{20}C_8$; 0·97037. **34.** $1-12x+58x^2-144x^3+195x^4$; 1.
35. 55. **36.** 495; 0·4952. **39.** ${}_{2n}C_n$.

EXERCISES XXXIV. Page 290.

1. $\frac{1}{2}(n-1)n$, $\frac{1}{6}(n-1)n(2n-1)$, $\{\frac{1}{2}(n-1)n\}^2$.
2. (i) $\frac{1}{3}n(n+1)(n+2)$; (ii) $\frac{1}{3}n(16n^2-7)$;
(iii) $n(n+1)(n+2)(3n+1)/12$; (iv) $\frac{1}{3}n(4n^2-1)$.

5. $\frac{1}{2}n(n+1)(n^2+n-1)$.

6. (i) $\frac{9}{20}[1+(\frac{2}{3})^7]$, $\frac{9}{20}$; (ii) $\frac{1}{3}n(n+1)(4n+5)$.

7. (i) $n(n+1)(n+2)(3n+5)/12$; (ii) $n(n+1)(n+2)(3n+1)/12$.

8. $-n(2n+1)$. 9. $n(n+1)(6n^3+9n^2+n-1)/30$.

10. $a\{1-(n+1)r^n+nr^{n+1}\}/(1-r)^2$.

11. (i) $\frac{1}{4}n(13-14n)$; (ii) $12(2-\sqrt{2})$; (iii) $(\frac{4}{5})^n$.

12. $(\frac{5}{4})^7$. 13. $\frac{3}{4}\sqrt{2}$. 14. $\sqrt{\frac{2}{3}}$. 15. $(\frac{3}{2})^{\frac{1}{4}}$. 16. $[1-(\frac{1}{4})^{\frac{2}{3}}]$.

EXERCISES XXXV. Page 302.

1. $4{\cdot}41_{63^\circ}$; $9{\cdot}25_{41^\circ}$; $2{\cdot}3_{180^\circ}$; $5{\cdot}20_{121^\circ}$.

4. (i) $1{\cdot}14_{11^\circ}$; (ii) $6{\cdot}14_{351^\circ}$. 5. 6·19 miles N. 16° E.

6. 7·20 ft. per sec. E. 43° N.

7. (*a*) 36·1 lb. E. 56° N.; (*b*) 8·31 lb. E. 80° N.; (*c*) 11·2 tons W. 62° S.; (*d*) 8·44 kg. S. 54° W.

8. $7{\cdot}70_{308^\circ}$. 9. $7{\cdot}70_{128^\circ}$. 10. 5·17 tons, 2·63 tons.

11. 126°, 140°, 94°. 12. 3·18 lb. E. 36·5° N.

13. (i) $7{\cdot}09_{84^\circ}$; (ii) $5{\cdot}10_{129^\circ}$; (iii) $1{\cdot}13_{49^\circ}$; (iv) $2{\cdot}51_{322^\circ}$; (v) $1{\cdot}41_{209^\circ}$.

14. $8{\cdot}14_{338^\circ}$.

15. (i) $4{\cdot}78_{0^\circ}+3{\cdot}10_{90^\circ}$; (ii) $-10{\cdot}96_{0^\circ}+14{\cdot}03_{90^\circ}$;
(iii) $-43{\cdot}1_{0^\circ}-51{\cdot}3_{90^\circ}$; (iv) $88{\cdot}9_{0^\circ}-74{\cdot}6_{90^\circ}$;
(v) $0{\cdot}788_{0^\circ}+0{\cdot}368_{90^\circ}$; (vi) $0_{0^\circ}+1{\cdot}56_{90^\circ}$;
(vii) $-8{\cdot}45_{0^\circ}+0_{90^\circ}$; (viii) $0_{0^\circ}-47{\cdot}4_{90^\circ}$;
(ix) $-0{\cdot}081_{0^\circ}+0{\cdot}022_{90^\circ}$; (x) $331_{0^\circ}-709{\cdot}6_{90^\circ}$;
(xi) $438_{0^\circ}+649_{90^\circ}$; (xii) $-5{\cdot}13_{0^\circ}+5{\cdot}13_{90^\circ}$.

16. The magnitudes of the vectors are 14·9, 60·7, −0·78, −7·15, 0, 0, −15·6, 49, 664, 0·043.

17. The magnitudes of the vectors are 9·693, 28·31, 1·351. 3·18, 72·5, −84·5, 0, −105·1, 415, 0·0736.

19. The projections in the direction OZ are:

$2{\cdot}3\cos\alpha-1{\cdot}7\sin\alpha$, $-2{\cdot}5\cos\alpha-2{\cdot}4\sin\alpha$.
$-3{\cdot}1\cos\alpha+2{\cdot}0\sin\alpha$, $2{\cdot}1\cos\alpha+1{\cdot}8\sin\alpha$:

in the direction making +90° with OZ:

$2{\cdot}3\sin\alpha+1{\cdot}7\cos\alpha$, $-2{\cdot}5\sin\alpha+2{\cdot}4\cos\alpha$,
$-3{\cdot}1\sin\alpha-2{\cdot}0\cos\alpha$, $2{\cdot}1\sin\alpha-1{\cdot}8\cos\alpha$.

20. (4·12, 5·90); (0·243, 1·38); (−3·73, 4·44); (−4·04, −1·47); (0·958, −3·57).

EXERCISES XXXVI. Page 309.

1. 56/65, 33/65, 16/65, 63/65.

2. $-528/697$, $-455/697$, 672/697, 185/697.

3. $\cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin C \sin A - \cos C \sin A \sin B$.

4. $\dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

5. $(\sqrt{3}+1)/2\sqrt{2}$, $(\sqrt{3}-1)/2\sqrt{2}$.

6. $(\sqrt{3}+1)/2\sqrt{2}$, $(\sqrt{3}-1)/2\sqrt{2}$, $2-\sqrt{3}$.

9. (i) $\sin 6A$; (ii) $\cos \theta$; (iii) $\cos A \cos B$.

12. 13/9, $-1/3$. **14.** $\cos(\theta-\phi)\cos\phi$, $\sin(\theta+\phi)\cos\phi$.

16. (i) 90°, 337° 22′ ; (ii) 19° 20′, 303° 48′ ; (iii) 17° 35′, 229° 47′ ; (iv) 11° 47′, 145° 37′.

17. $r = 8{\cdot}062$, $\alpha = 150°\ 15'$; $\theta - 150°\ 15' = n\,.\,360° \pm 51°\ 40'$.

18. $r = 3{\cdot}606$, $\alpha = 303°\ 41'$; 9° 6′, 238° 16′. **19.** $-27°\ 18'$, $-78°\ 58'$.

20. $\pm(14\sqrt{6} \pm \sqrt{51})/50$. **22.** $R^2 + r^2 + c^2 + 2c(R\cos\alpha + r\sin\alpha)$.

23. (i) $\tan^{-1} 5{\cdot}5$; (ii) $\tan^{-1} 13$; (iii) $\tan^{-1} 1/2$; (iv) $\tan^{-1}\sqrt{73}/5$.

24. $\tan^{-1}(m-m')/(1+mm')$; (1) $m = m'$, (2) $1 + mm' = 0$.

25. $x^2 + y^2 \mp 2ay \cot\alpha = a^2$.

26. $x^2 + y^2 = 25$ and $x^2 + y^2 + 2x - 14y + 25 = 0$.

27. $\tan^{-1}(a_1b_2 - a_2b_1)/(a_1a_2 + b_1b_2)$. $x - y + 2 = 0$, $8x + 6y - 13 = 0$.

28. $5y - 3x = 0$, $3x + 5y - 15 = 0$, $5y + 6x = 30$; 2, 18/5 ; $\tan^{-1}(75/29)$.

30. $(y-k)(b + a\tan\phi) + (x-h)(a + b\tan\phi) = 0$,
$(y-k)(b - a\tan\phi) + (x-h)(a - b\tan\phi) = 0$.

31. Lines are real if $h^2 > ab$: $\tan^{-1}(\sqrt{5}-2)$, $-\tan^{-1}(\sqrt{5}+2)$.

EXERCISES XXXVII. Page 314.

1. $(\sqrt{3}-1)/2\sqrt{2}$, $(\sqrt{3}+1)/2\sqrt{2}$, $2-\sqrt{3}$; $\frac{1}{2}\sqrt{(2-\sqrt{2})}$, $\frac{1}{2}\sqrt{(2+\sqrt{2})}$, $\sqrt{2}-1$.

8. $8\cos^4\theta - 8\cos^2\theta + 1$.

9. $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$; $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

10. $\frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

EXERCISES XXXVIII. Page 315.

1. $\sin 3\theta + \sin \theta$. **2.** $\sin 7A - \sin 3A$. **3.** $\cos 9\phi + \cos 3\phi$.
4. $\cos 3x - \cos 7x$. **5.** $\frac{1}{2}(\sin 10a - \sin 4a)$. **6.** $\frac{1}{2}(\cos y - \cos 5y)$.
7. $\frac{1}{2}(\sin 7x - \sin 2x)$. **8.** $\frac{1}{2}\{\cos(3x-y) + \cos(x+y)\}$.
9. $\frac{1}{2}\{\cos(3\theta-\phi) - \cos(5\theta+\phi)\}$. **10.** $\frac{1}{2}\left(\sin\frac{\pi}{3} - \sin 2\theta\right)$.
11. $\frac{1}{2}(\sin 2a - 1)$. **12.** $-\frac{1}{2}\left(\sin 2\theta + \sin\frac{\pi}{3}\right)$.
13. $\frac{1}{2}\{\sin(4\theta-\phi) + \sin(2\theta+3\phi)\}$. **14.** $\frac{1}{2}(1 + \cos 2a)$.

EXERCISES XXXIX. Page 317.

1. $2 \sin 10x \cos 4x$. 2. $2 \cos 9\theta \sin 2\theta$. 3. $2 \cos 6x \cos x$.
4. $2 \sin 2a \sin 4a$. 5. $2 \sin 50° \cos 30°$. 6. $-2 \sin 5x \sin 2x$.
7. $-2 \sin 55° \sin 35°$. 8. $\cos A$. 9. $\sin x$.
10. $\sqrt{2} \sin \theta$. 11. 0. 12. $\cos(\alpha+\beta)\{1 + 2\cos\beta\}$.
13. $\sin(\alpha+\beta)\{1 + 2\cos\beta\}$. 14. $\sin 60° \, (1 + 2\cos 40°)$.

EXERCISES XL. Page 320.

1. (i) $x/\sqrt{1+x^2}$, $1/\sqrt{1+x^2}$. $2x/(1+x^2)$, $(1-x^2)/(1+x^2)$.
3. $3 \sin A - 4 \sin^3 A$, $4 \cos^3 A - 3 \cos A$.
5. $\tan \frac{1}{2}\theta = (x-y)/(x+y)$ or $-(x+y)/(x-y)$.
8. (i) $(2n+1)\frac{\pi}{2}$, $2k\pi$; (ii) $(2n+1)\pi$, $\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right)$; (iii) $(2n+1)\frac{\pi}{2}$, $2k\pi$, $2r\pi/7$; (iv) $(2n+1)\frac{\pi}{2}$, $k\pi \pm \frac{\pi}{3}$; (v) $n\pi/4$.
9. $\pi/4 < \theta < 3\pi/4$, $-3\pi/4 < \theta < -\pi/4$.
13. $\sin \theta = \pm 5/13$, tar $2\theta = 120/119$. 14. 14° 15′ and 59° 28′.
15. $(4x^3 - 3x)(1 - 2y)$; $(4x^3 - 3x)(1 + 2\sqrt{1-x^2})$; 0, $-\sqrt{3}/2$.
16. (i) $x = (2n+1)\frac{\pi}{2}$ or $3x = k\pi + (-1)^k\frac{\pi}{6}$; (ii) $3\theta = n\pi$ or $2\theta = 2r\pi \pm \frac{2\pi}{3}$; (iii) $x = 2n\pi$ or $3x = 2k\pi + \frac{\pi}{2}$.
26. $(p^2 - q^2)^2 = 16pq$.

EXERCISES XLI. Page 330.

1. 44° 24′. 4. 120°.
5. $\frac{1}{2}\sqrt{\{2b^2 + 2c^2 - a^2\}}$ and two similar expressions ;
$\sqrt{\{bc(b+c-a)(b+c+a)\}}/(b+c)$.

11. $B=59^\circ$, $C=74^\circ\ 31'$, $a=47{\cdot}6$. **14.** 26,000 ft.

18. $AE=19{\cdot}7$ ft., $BE=20{\cdot}1$ ft., $CE=20{\cdot}6$ ft.; $ADB=110^\circ\ 45'$, $ADC=134^\circ\ 37'$.

20. (i) 0·5; (ii) $\sqrt{3}/2$; (iii) $\sqrt{2}/2$.

EXERCISES XLII. Page 337.

1. (i) $C=77^\circ\ 25'$, $a=5{\cdot}05$, $b=7{\cdot}78$; (ii) $A=24^\circ$, $b=17{\cdot}4$, $c=8{\cdot}81$; (iii) $A=59^\circ\ 24'$, $a=24{\cdot}9$, $c=20{\cdot}5$.

2. (i) $A=78^\circ\ 36'$, $B=53^\circ\ 16'$, $C=48^\circ\ 8'$;
(ii) $A=29^\circ\ 24'$, $B=41^\circ\ 44'$, $C=108^\circ\ 52'$;
(iii) $A=98^\circ\ 18'$, $B=32^\circ\ 4'$, $C=49^\circ\ 38'$.

3. (i) $B=59^\circ\ 1'$, $C=71^\circ\ 29'$, $a=343$;
(ii) $A=42^\circ\ 6'$, $B=29^\circ\ 54'$, $c=107$;
(iii) $A=35^\circ\ 23'$, $C=99^\circ$, $b=42$.

4. (i) $A=6^\circ$, $13'$, $B=23^\circ\ 47'$, $a=2{\cdot}01$;
(ii) $A=91^\circ\ 50'$, $B=38^\circ\ 10'$, $a=12{\cdot}1$;
(iii) $A_1=58^\circ\ 12'$, $B_1=71^\circ\ 48'$, $a_1=8{\cdot}32$; $A_2=21^\circ\ 48'$, $B_2=108^\circ\ 12'$, $a_2=3{\cdot}64$;
(iv) no solution.

5. (i) $A=25^\circ\ 24'$, $B=44^\circ\ 36'$, $c=96{\cdot}4$;
(ii) $B=63^\circ\ 29'$, $a=41{\cdot}3$, $c=24{\cdot}1$.
(iii) $A=82^\circ\ 49'$, $B=55^\circ\ 46'$, $C=41^\circ\ 25'$;
(iv) $A=32^\circ\ 8'$, $B=40^\circ\ 52'$, $a=254$;
(v) $A=45^\circ\ 28'$, $B=43^\circ\ 28'$, $C=91^\circ\ 4'$;
(vi) $A=65^\circ\ 5'$, $B=42^\circ\ 41'$, $c=13{\cdot}3$.

6. $x=27^\circ$. **7.** $A=88^\circ$, $B=40^\circ$, $C=52^\circ$.

8. $A=30^\circ$, $C=75^\circ$, $b=c=10{\cdot}62$ cm.

9. 13,600 sq. yd.; 152 yd. **10.** $86^\circ\ 52'$, 24·8, 2·08.

11. $B=45^\circ\ 12'$, $C=59^\circ\ 34'$, $a=726$. **12.** 89°, 60°, 31°.

13. $B_1=113^\circ\ 10'$, $C_1=16^\circ\ 50'$, $c_1=9{\cdot}45$; $B_2=66^\circ\ 50'$, $C_2=63^\circ\ 10'$, $c_2=29{\cdot}1$.

14. 21·2 ft. **16.** 71 sq. units, $A=82^\circ\ 49'$.

17. $A=89^\circ$, $B=52^\circ$, $c=2280$.

EXERCISES XLIII. Page 343.

1. 177 ft. **2.** 18·9 ft.

4. (i) 81·4 cub. in.; (ii) $66^\circ\ 8'$; (iii) $99^\circ\ 26'$.

5. 8·5, 11·5 miles; $49^\circ\ 41'$. **6.** 47·5, $42^\circ\ 48'$.

7. 77 ft. **8.** 2420 ft. ; 41° 59′. **9.** 380 yards.
10. 1 minute. **12.** 5·24 miles. **13.** 6·53, 7·70 miles.
15. $\sin^{-1}(\sin\phi\cos\frac{1}{2}\theta)$. **16.** 2940 ft.
17. 17·3 ft. ; 22° 12′ ; 11° 19′.

MISCELLANEOUS EXERCISES. Page 346.

2. 109° 28′, 50° 28′, 20° 4′. **4.** 2·88 m.
5. $\theta+66°\,48'=n\,.\,360°\pm 48°\,58'$. **6.** $-40°\,12'$; 3·61.
7. 120° ; no solution.
9. $\sin(a+\theta+\phi)\sin(a+\theta-\phi)$. **10.** 79° 36′ ; 67·2 sq. in.
12. (i) No solution ; (ii) 270°, 150°, 30°. **14.** 0·995.
18. 98° 4′, 278° 4′. **19.** $\frac{5}{4}a^2\cot 36°$; 1·35 ft.
22. $\theta-50°\,12'=n\,.\,360°\pm 26°\,18'$.
23. $n\,.\,180°$, $(4k+1)30°$, $(4r-1)90°$.
27. $2\theta-60°=n\,.\,360°\pm 75°\,31'$. **28.** $\tan^2\beta$.
31. $\tan^{-1}\{(x+y+z-xyz)/(1-xy-yz-zx)\}$.
32. 25/24 : graphically $x=1\cdot 17$.
34. $A=86°\,49'$, $B=39°\,29'$, $c=314$.
36. (i) $\theta=n\,.\,90°$; (ii) $x+60°=n\,.\,360°\pm 45°$, $y+120°=n\,.\,360°\pm 45°$.

EXERCISES XLIV. Page 354.

4. $r=146$; $R=389$. **5.** $R=26\cdot 5$ in. ; $C=78°\,48'$, $B=54°\,12'$.
6. $\sqrt{bc(s-a)/s}$, etc. ; 14·3.
7. 12 in. ; $16\frac{2}{3}$ in., $a=19\cdot 2$ in., $b=27\cdot 2$ in.
8. (i) 993 sq. in. ; 13·0 in.
11. Triangle $BOC=\Delta\cot A\cot B$, etc.

EXERCISES XLV. Page 359.

1. If O is the pole and OX the initial line :—$X'OX$; OP and OP' where $XOP=a$ and $XOP'=\pi-a$; $P'OP$ where $XOP=-\tan^{-1}a/b$; if $QOX=\tan^{-1}\frac{4}{3}$, equation represents OP and OP' where each is inclined at $\pi/3$ to OQ.
2. $\sqrt{(5-2\sqrt{3})}$, $\sqrt{7}$, $\sqrt{10}$; $\frac{1}{2}(3\sqrt{3}-2)$.
3. $\sqrt{(20-8\sqrt{2})}$, $\sqrt{(5-2\sqrt{2})}$, $\sqrt{17}$; $\frac{1}{2}(5\sqrt{2}-4)$.
4. A straight line.

7. $\sin(\theta_1 - \theta_2)/r + \sin(\theta_2 - \theta)/r_1 + \sin(\theta - \theta_1)/r_2 = 0.$

9. $r \sin(\theta - a) = r_1 \sin(\theta_1 - a).$

11. $1/r = \cos(\theta - a)/a - \cos(\theta - \beta)/b.$ 12. $2a, \frac{1}{2}\pi.$

EXERCISES XLVI. Page 362.

2. (i) Circle ; centre at pole, radius a. (ii) Circle ; passing through pole, radius 2, centre at (2, 0°). (iii) Straight line perpendicular to initial line through (4, 0°). (iv) Two concentric centres, common centre at pole, radii 1 and 2. (v) Circle, centre at (1·5, 0°), radius 0·5.

3. $(a\sqrt{2}, \frac{1}{4}\pi), (a\sqrt{2}, \frac{3}{4}\pi).$

4. $r^2 - r\{r_1 \cos(\theta_1 - \theta) + r_2 \cos(\theta_2 - \theta)\} + r_1 r_2 \cos(\theta_1 - \theta_2) = 0.$

5. $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

6. Straight line parallel to initial line through $(a, \frac{1}{2}\pi)$; circle passing through pole, diameter a, centre at $(\frac{1}{2}a, \frac{1}{2}\pi)$; circle passing through pole, centre at $(\frac{1}{2}\sqrt{a^2 + b^2}, \tan^{-1} b/a).$

9. $(2\sqrt{2}, \frac{1}{4}\pi)$ and $(2\sqrt{2}, 3\pi/4).$

12. (i) A circle, $r \sec \theta = \text{constant}$; (ii) straight line, $r \cos \theta = \text{constant}.$

13. (i) $k^2 r^2 - 2ckr \cos(\theta - a) + c^2 - a^2 = 0$;
(ii) $(c^2 - a^2)r^2 - 2cl^2 r \cos(\theta - a) + l^4 = 0.$

15. $c^2 + c'^2 - 2cc' \cos(a - a') = a^2 + a'^2.$

EXERCISES XLVII. Page 378.

1. 16, 16·36, 16·3216, 16·032016 ; 33·6, 32·16, 32·016 ft. per sec. ; 32 ft. per sec.

2. 12 ft. per sec. ; $t = 4\frac{6}{7}$. 3. 10.

4. $0{\cdot}5 + 2{\cdot}3x^2 + 4{\cdot}6x\,\Delta x + 2{\cdot}3(\Delta x)^2$; $4{\cdot}6x + 2{\cdot}3\,\Delta x$; $4{\cdot}6x$.

5. 10·864 ; 0·752, 0·07376, 0·0073616 ; 7·52, 7·376, 7·3616 ; 7·36.

6. 0·12 sq. in. per sec. 7. $U/\pi r^2$ ft. per sec.

8. (i) 2 ; (ii) $9x^2 + 4$; (iii) $\frac{2}{3}x^{-\frac{2}{3}} + \frac{2}{5}x^{-\frac{3}{5}}$; (iv) $-2x^{-3}$; (v) $\frac{3}{2}x^{-\frac{1}{2}} - 5x^{-2}$.

9. (i) $x^2 + 4x + 5$; (ii) $4(2y + 1)$; (iii) $-\frac{1}{2}z^{-\frac{3}{2}}$; (iv) $3a(at + b)^2$;
(v) $3av^2 + \frac{1}{2}bv^{-\frac{1}{2}} + cv^{-2}$.

10. (i) $y - 6x + 2 = 0,\ 6y + x - 25 = 0$;
(ii) $3y - 161x + 318 = 0,\ 161y + 3x - 8864 = 0$;
(iii) $ty - x = at^2,\ y + tx = 2at + at^3$;
(iv) $y - 3ah^2x + 2ah^3 = 0,\ 3ah^2y + x = 3a^2h^5 + h$.

11. $6x^2-12x+6$; $(1,\ -2)$; 0·06 in each case, a point of inflexion.

12. $3x^2-2$; $y-10x+13=0$. 13. $21x^2+2$.

14. (i) $-1/x^2$; (ii) $-\frac{1}{2}x^{-\frac{3}{2}}$; (iii) $\frac{1}{3}x^{-\frac{2}{3}}$.

15. (i) $3x^2$; (ii) $-2x\ ^{3}$; (iii) $1/(x+1)^2$; (iv) $(ad-bc)/(cx+d)^2$;
(v) $2(x+1)$; (vi) $2a(ax+b)$; (vii) $1/2\sqrt{x+1}$; (viii) $\frac{1}{3}x^{-\frac{2}{3}}$.

EXERCISES XLVIII. Page 386.

1. (*a*) (1, 8); max. (*b*) (0, -1); min. (*c*) $(\frac{1}{4}, \frac{7}{8})$; min.

2. $\sqrt{3}/3$. 3. $2a/5,\ 3a/5$. 4. $\frac{1}{2}p,\ \frac{1}{2}p$.

5. 6 in. 6. (1, -2) min., (-1, 2) max. 7. $\sqrt[3]{1/4}$.

9. 2 cub. ft. 10. 15·2 knots; 1·07% greater, 0·91% greater.

11. 17·0 knots. 12. £2·57. 13. $x=4\frac{1}{6},\ y=10$.

14. $V/2R$. 15. 2·55 cub. ft.

16. (*a*) (2, 9) min., (1, 10) max., (1·5, 9·5) point of inflexion;
(*b*) (2, -15) min., (-2, 17) max., (0, 1) point of inflexion;
(*c*) (0, 1) min., (1, 2), $(\frac{1}{3}, \frac{38}{27})$ points of inflexion;
(*d*) (1, 0) min., (-1, 4) max., (0, 2), $(1/\sqrt{2},\ 2-7\sqrt{2}/8)$,
$(-1/\sqrt{2},\ 2+7\sqrt{2}/8)$ points of inflexion;
(*e*) (1, 27) max., (0, 8), (-2, 0) points of inflexion.

EXERCISES XLIX. Page 392.

1. x^2+a. 2. $\frac{1}{2}z^2+z+a$. 3. $\frac{1}{9}x^3+x^2+a$.

4. t^3-t^2+t+a. 5. $\frac{2}{3}u^{\frac{3}{2}}+a$. 6. $-1/1{\cdot}4y^{1{\cdot}4}+a$.

7. $-1/2x^2+a$. 8. $t^2+\log_e t+a$.

9. $3\log_e x+4/x-1/2x^2+a$. 10. $y=x^2+2x-1$.

11. $3y=x^{\frac{3}{2}}$. 12. $y=2(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$.

13. $v=u+t^2$; $s=ut+\frac{1}{3}t^3$.

14. $v=u+4t+\frac{2}{3}t^3$; $s=ut+2t^2+\frac{1}{6}t^4$. 15. $y+\log_e x+1=0$.

16. $y=\frac{1}{2}x^2+2\log_e\frac{1}{2}x$. 17. $y-ax+a=0$; $ay+x-1=0$.

EXERCISES L. Page 400.

1. (*a*) 1; (*b*) 4; (*c*) $\frac{1}{6}$; (*d*) $\frac{1}{2}$; (*e*) 2·04; (*f*) 1·92; (*g*) $-9\frac{1}{6}$; (*h*) $1\frac{3}{4}$.

2. (*a*) 36; (*b*) $-\frac{3}{4}$; (*c*) 56·5. 3. Displacement; velocity.

4. $\frac{1}{80}T^2$ miles. 5. 88·4 sq. cm.; 86·8, 1·9%.

6. $1{\cdot}75\times10^5$ cub. ft.; 5810 cub. ft. 7. 42·2.

8. 245,000 ft. lb. 9. 0·56.
10. 16,200 ft. lb., 27,500 ft. lb. 11. 0·48.
12. 37·2, 37·5 lb. per sq. in.

EXERCISES LI. Page 417.

1. If the original is a straight line parallel to the axis of X, the integrals are straight lines.

2. $\int_0^b ax^{0\cdot5}\,dx$. 3. $\int_0^c \left(\frac{y}{a}\right)^2 dy$. 4. $\int_0^a 4\pi r^2 dr$.

5. Integral curve is graph of $\sin\theta$; area $\doteqdot 1$.
6. A velocity-time curve. 7. 0·694 ; 0·6931 ; 1·253.
8. 17·5. 9. 6·68. 10. 1·18.
11. 2·54. 12. Total area $\doteqdot 0\cdot571$. **14.** Total area $\doteqdot 424$.
15. 12,600. 16. Total distance $\doteqdot 7\cdot56$ miles.
17. Total area $\doteqdot 6\cdot78$. 18. 23·7 ft. per sec. **19.** $\frac{1}{3}\pi m^2h^3$. **20.** $\frac{4}{3}\pi a^3$.
21. $\pi[a^2(h_2-h_1)+ab(h_2{}^2-h_1{}^2)+\frac{1}{3}(b^2+2ac)(h_2{}^3-h_1{}^3)$
$+\frac{1}{2}bc(h_2{}^4-h_1{}^4)+\frac{1}{5}c^2(h_2{}^5-h_1{}^5)]$.

22. $\frac{4}{5}\pi a^{\frac{1}{2}}h^{\frac{5}{2}}$. 23. (i) $\pi c^3(k-h)/hk$; (ii) $\pi c^3(h-k)$.
24. 12·5 in.-lb. 25. $\frac{1}{3}Ml^2$. **26.** $Ma^2/2$.
27. $2Ma^2/5$. 28. $\frac{1}{3}l$. **30.** $\bar{x}=\frac{2}{3}h$, $\bar{y}=0$.

EXERCISES LII. Page 428.

1. $-x(2-x^2)^{-\frac{1}{2}}$. 2. $2x(2x^2-1)^{-\frac{1}{2}}$. 3. $x(x^2+a^2)^{-\frac{1}{2}}$.
4. $(ax+b)(ax^2+2bx+c)^{-\frac{1}{2}}$. 5. $-x(1+x^2)^{-\frac{3}{2}}$.
6. $x^2(x^3-1)^{-\frac{2}{3}}$. 7. $-(ax^2+2bx+c)(ax^3+3bx^2+3cx+d)^{-\frac{4}{3}}$.
8. $2(x-1)$. 9. $2x(2x^2+1)$.
10. $(ax+b)^2(cx+d)^3\{7acx+3ad+4bc\}$.
11. $(2x^2+1)(x^2+1)^{-\frac{1}{2}}$. **12.** $(4x^3+x^2-2)(x^3-1)^{-\frac{2}{3}}$.
13. $\frac{1}{2}(3acx+2ad+bc)(cx+d)^{-\frac{1}{2}}$. **14.** $-(x-2)^{-2}$.
15. $-(1+x^2)(x^2-1)^{-2}$. **16.** $(ps-rq)(rx+s)^{-2}$.
17. $(5x+2)(32-5x)(3-x)^{-2}$. **18.** $-(x+3)(x+1)^{-\frac{1}{2}}(x-1)^{-2}$.
19. $-(x+1)^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}}$. **20.** $-2x(x^2+1)^{-\frac{1}{2}}(x^2-1)^{-\frac{3}{2}}$.
21. $a/(ax+b)$. **22.** $1/\sqrt{x^2+1}$. **23.** $U/4\pi r^2$ ft. per sec.
24. $\frac{1}{6}\sqrt{3}$ ft. per min. **25.** $ty-x=at^2$; $y+tx=2at+at^3$.
26. $2x+3y=5$; $2x-3y=5$.

EXERCISES LIII. Page 435.

1. $3x^2, 6x, 6$; $6x-2, 6, 0$; $-x^{-2}, 2x^{-3}, -6x^{-4}$; $-2(x+1)^{-3}, 6(x+1)^{-4}, -24(x+1)^{-5}$; $(ad-bc)(cx+d)^{-2}$, $-2c(ad-bc)(cx+d)^{-3}$, $6c^2(ad-bc)(cx+d)^{-4}$; $\frac{1}{2}(x+1)^{-\frac{1}{2}}$, $-\frac{1}{4}(x+1)^{-\frac{3}{2}}$, $\frac{3}{8}(x+1)^{-\frac{5}{2}}$; $x(2-3x)$, $2(1-3x)$, -6 ; $(x+2)^2(x-1)(5x+1)$, $2(x+2)(10x^2+4x-5)$, $6(10x^2+16x+1)$.

2. (i) Concave at both; (ii) convex at (1, 2), concave at (1, −2); (iii) concave at (1, 3), convex at (−1, −3); (iv) concave at (1, 2), convex at $(-3, \frac{2}{3})$.

3. $x = 1/\sqrt{3}$.

4. Maximum when number is −1, minimum when +1.

5. Square. 6. $\frac{1}{6}a \times \frac{2}{3}a \times \frac{2}{3}a$. 7. 5·71 miles from X.

8. (i) 1 ft. square by 2 ft. (ii) 1·2 ft.; vol. = 1·728 cub. ft.
(iii) radius = 0·724 ft.; vol. = 1·592 cub. ft.
(iv) radius = 0·64 ft., length = 2 ft., vol. = 2·55 cub. ft.

10. When $x = \frac{1}{2}a$. 11. At point dividing PQ in ratio $4 : \sqrt[3]{100}$.

13. (i) $y = 2 - 1/x$; (ii) $y = cx + \frac{1}{2}ax^2 + \frac{1}{6}bx^3$; (iii) $\frac{w}{24}(l-x)\{(l-x)^3 - 4l^3\}$.

14. $y = d + \frac{1}{2}ax^2 + \frac{1}{6}bx^3 + \frac{1}{12}cx^4$. 15. $y = x^3 - x + 2$.

16. $y = \frac{Wx^2}{6EI}\{3l - x\}$. 17. $y = \frac{wx^2}{24EI}\{6l^2 - 4lx + x^2\}$; $\frac{wl^4}{8EI}$.

18. $\frac{wx^2}{24EI}\{x - 2l\}^2$; $\frac{wl^4}{24EI}$.

EXERCISES LIV. Page 439.

1. $\frac{1}{2}\log_e(x^2+1)$. 2. $-\frac{1}{2}(x^2+a^2)^{-1}$. 3. $\sqrt{ax^2+2bx+c}$.

4. $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}$. 5. $-\frac{1}{2}(2x^2-5x+8)^{-2}$. 6. $2\sqrt{x^2-3x+1}$.

7. $\frac{2}{9}(x^3+1)^{\frac{3}{2}}$. 8. $\frac{1}{8}(x^6+2)^{\frac{4}{3}}$.

9. $\frac{2}{15}\sqrt{x+1}\{3x^2-4x+8\}$. 10. $16\frac{1}{3}$.

11. $\frac{1}{15}$. 12. $12\frac{2}{3}$. 13. $\frac{1}{6}\log_e(\frac{35}{11})$.

14. $\frac{1}{4}\log_e(\frac{189}{5})$. 15. $\log_e(\frac{9}{2})$. 16. $\frac{1}{3}$.

17. $\frac{1}{3}b^3$. 18. $\frac{1}{a}(1 - 1/\sqrt{2})$.

EXERCISES LV. Page 442.

1. $a\cos(ax+b)$. 2. $-a\sin(ax-b)$. 3. $p\sec^2(px+q)$.

4. $-2\operatorname{cosec}^2 2x$. 5. $3\sec(3x+4)\tan(3x+4)$.

6. $-\frac{1}{2}\operatorname{cosec}\frac{1}{2}x\cot\frac{1}{2}x$. 7. $\sin 2x$. 8. $-a\sin(2ax+2b)$.

9. $2l\tan(lx-m)\sec^2(lx-m)$. 10. $2\cos 2x\cos x-\sin 2x\sin x$.

11. $-a\sin ax\cos bx-b\sin bx\cos ax$.

12. $ax^2\cos(ax+b)+2x\sin(ax+b)$. 13. $(-x\sin x-\cos x)/x^2$.

14. $\{a\sin x-(ax+b)\cos x\}/\sin^2 x$. 15. $(-2\cos x)/(1+\sin x)^2$.

16. $-a/\sqrt{1-(ax+b)^2}$. 17. $1/(1+x^2)$. 18. $1/x\sqrt{x^2-1}$.

19. $-\sin x$. 20. $-\cos x$. 21. $2\sec^2 x\tan x$.

22. $2\cos 2x$. 23. $-a\sin x-b\cos x$. 24. $-\operatorname{cosec}^2 x$.

25. $\sec x\tan x$. 26. $-\operatorname{cosec} x\cot x$. 27. $-2\operatorname{cosec} x\cot x$.

29. $-\frac{1}{2}\cos 2x$. 30. $\frac{1}{a}\sin(ax+b)$. 31. $\frac{1}{p}\tan(px+q)$.

32. $\frac{1}{2}x-\frac{1}{4}\sin 2x$. 33. $\frac{1}{2}x-\frac{1}{4a}\sin(2ax+2b)$. 34. $-\frac{2}{3}\cos^3 x$.

35. $-\frac{1}{6}\cos 3x+\frac{1}{2}\cos x$.

36. $-\frac{1}{2}\left\{\frac{\cos(a+b)x}{a+b}+\frac{\cos(a-b)x}{a-b}\right\}$.

37. $\frac{1}{2}\left\{\frac{\sin(a+b)x}{a+b}+\frac{\sin(a-b)x}{a-b}\right\}$.

38. $\frac{1}{3}\sin^3 x$. 39. $-\frac{1}{4}\cos^4 x$. 40. $-\log_e\cos x$.

41. $\frac{1}{a}\sin^{-1}\left(\frac{ax}{b}\right)$. 42. $\frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)$. 43. $\sin^{-1}\left(\frac{x-2}{2}\right)$.

44. $\frac{1}{8}\tan^{-1}\left(\frac{2x+1}{4}\right)$. 45. $\frac{\pi}{4}$. 46. $\frac{\pi}{12}$.

47. $\tan^{-1}2-\frac{\pi}{4}$. 48. $\frac{\pi}{3}$.

MISCELLANEOUS EXERCISES. Page 449.

1. $4x^3+5$; $\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{x+\Delta x+1}{x+\Delta x+2}-\frac{x+1}{x+2}\right]$, $\frac{1}{(x+2)^2}$.

2. $\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{1}{x+\Delta x}-\frac{1}{x}\right]$, $-\frac{1}{x^2}$; $x^2-x^{-\frac{1}{2}}$; $-8(2x+3)^5$.

3. (i) $-\frac{2}{3}x^{-\frac{5}{3}}$; (ii) $\frac{6x+7}{\sqrt{4x+7}}$; (iii) $\frac{x\cos x-3\sin x}{x^4}$;

Tangent, $y-9x+14=0$; $\frac{14}{9}$, -14.

4. $\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}[\sin(x+\Delta x)-\sin x]$, $\cos x$;

$\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}[\tan(x+\Delta x+1)-\tan(x+1)]$, $\sec^2(x+1)$;

$-\frac{1}{\sqrt{(x+1)(x-1)^3}}$, $-\frac{1}{x^2+1}$, $-\frac{x^2+4x-1}{(x^2-x-1)^2}$.

5. $y+3x-10=0$.

6. (i) $\dfrac{d^2y}{dx^2}=4x(x^2-1)$, $y=\frac{1}{5}x^5-\frac{2}{3}x^3+x+c$;

(ii) $\dfrac{dy}{dx}=(a-b)\sin 2x$, $\dfrac{d^2y}{dx^2}=2(a-b)\cos 2x$.

7. Max. when $x=2$; min. when $x=4$. $a=20$ or 16.

8. $\dfrac{16000\sqrt{3}\pi}{27}$ cub. in.

9. Increases as x increases from $\frac{1}{2}\pi$ to $\frac{2}{3}\pi$: stationary when $x=0$. $\dfrac{\pi}{2}$, $\dfrac{2\pi}{3}$ and π.

11. $57{\cdot}6$.

12. $\frac{1}{3}(2x+1)^{\frac{3}{2}}$; $4\frac{1}{8}$; $5{\cdot}1$ lb.

13. $\dfrac{243}{40}\pi a^3$.

14. $10\frac{2}{3}$; $\dfrac{512}{15}\pi$.

15. $-\frac{1}{8}(6x-5)^{-2}$, $\frac{1}{3}(2x-3)^{\frac{3}{2}}$, $\dfrac{x}{2}-\dfrac{\sin 2x}{4}$; 16π.

16. $7{\cdot}85\times 10^7$ sq. mi.

17. $10(\sqrt{2}-1)$ in.

18. $140\frac{5}{8}$.

19. $10\frac{2}{3}$; $\dfrac{4096\pi}{105}$.

20. $y=x^5-2x^4+5x^3-11x^2+15x$.

21. (i) Impulse or Momentum; (ii) Work or Energy.

22. (*a*) 36·2 ft. per sec.; (*b*) 23 ft. per sec.

24. Tangent is $\dfrac{x}{a}\sec\theta+\dfrac{y}{b}\operatorname{cosec}\theta=1$, Normal is $ax\cos\theta-by\sin\theta$ $=a^2\cos^4\theta-b^2\sin^4\theta$.

26. $\cos\theta=\dfrac{r^2+a^2-c^2}{2ar}$, $\dfrac{dr}{d\theta}=\dfrac{2ar^2\sin\theta}{r^2+c^2-a^2}$.

27. $\frac{1}{2}a$.

30. $(x+2)(x-2)^3$.

32. (i) $\frac{2}{3}b$; (ii) (1) $\frac{1}{2}h$, (2) $\frac{3}{4}h$; (iii) $\frac{5}{4}r$.

33. (i) $\dfrac{2}{3}\cdot\dfrac{b^2+3bc+3c^2}{b+2c}$; (ii) (1) $\dfrac{1}{2}\cdot\dfrac{h^2+4ch+6c^2}{h+3c}$, (2) $\dfrac{1}{2}\cdot\dfrac{3h^2+8ch+6c^2}{2h+3c}$.

34. (i) $M\dfrac{b^2}{12}$, (ii) $M\dfrac{a^2}{3}$, (iii) $\dfrac{3}{2}Mr^2$, (iv) $\dfrac{Mr^2}{4}$, where M is the mass of the body in each case.

35. $M\left(a^2+\dfrac{r^2}{4}\right)$.

36. $M\left(\dfrac{r^2}{4}+\dfrac{l^2}{3}\right)$.

40. $\dfrac{41\pi}{2}$.

41. $\dfrac{8}{15}$.

42. $\dfrac{\pi}{4}$.

43. $2-\frac{1}{2}\pi$; $\pi\left[2\log_e 2-\dfrac{4}{3}\right]$.

45. $\tan^{-1}\left(\dfrac{b}{a}\right)^{\frac{1}{3}}$; $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

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